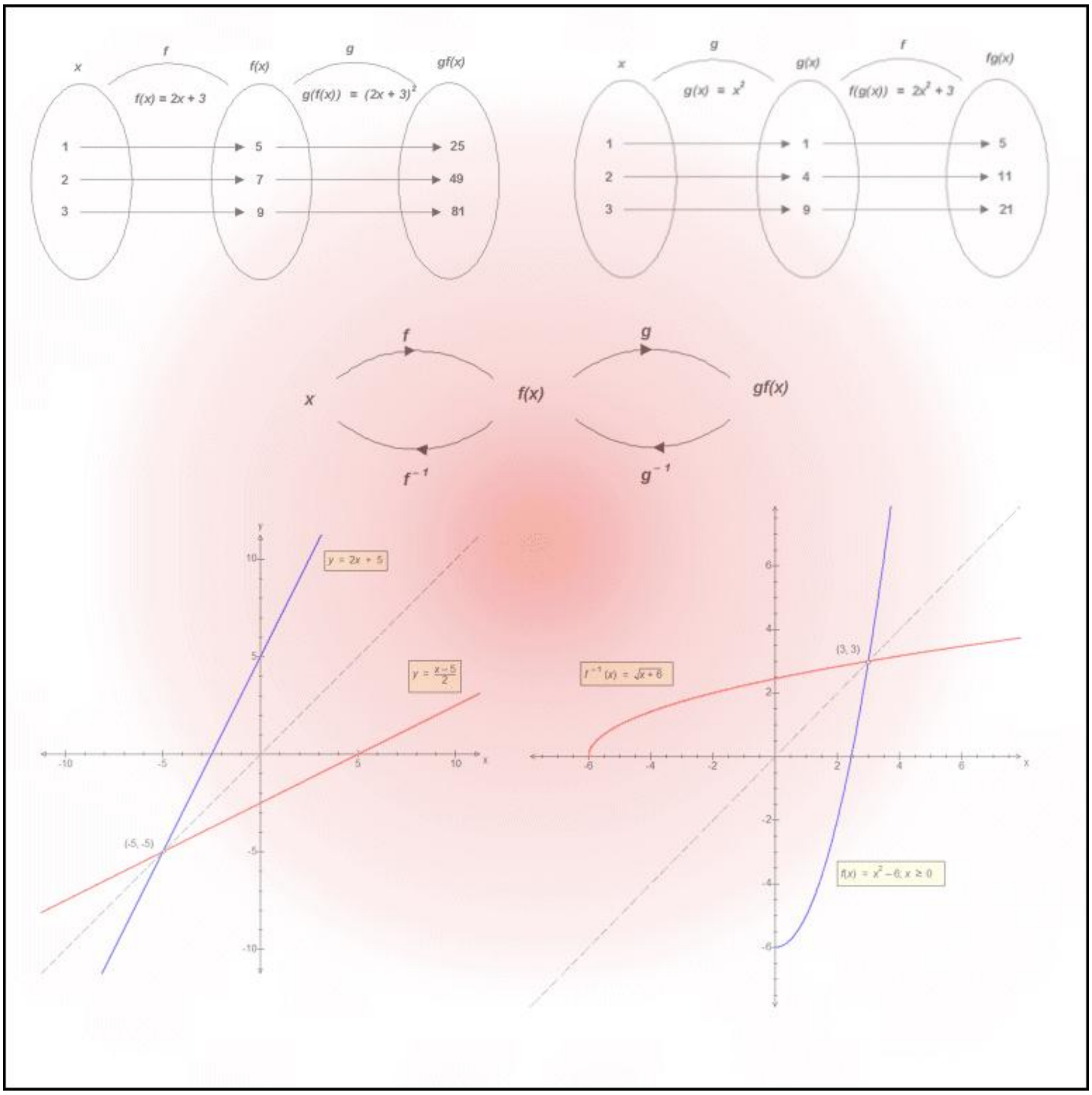
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

FUNCTIONS



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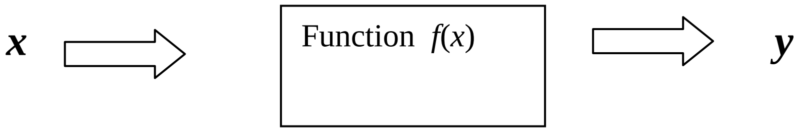
FUNCTIONS

A function can be thought of as being a rule which takes each member x of a set of values, and then assigns it to a value y.

In short, a function **maps** each value of x to an **image** value of y.



Functions are usually denoted by the letters f, g or h.

A function f which squares a number x and then adds 5 to the result can therefore be expressed in various styles:

$y = x^2 + 5$ (if the result is assigned the variable y)

$f(x) = x^2 + 5$

$f: x \mapsto x^2 + 5$

Here, f maps 4 to 21, and therefore we say that $f(4) = 21$.

Example (1) : A function g maps x to y by the following rule; multiply x by 3 and subtract 2 from the result. Express this in function notation, and write out $g(3)$.

The function can be denoted either by

$g(x) = 3x - 2$ or $g: x \mapsto 3x - 2$

$g(3) = 7$ (multiply 3 by 3 and subtract 2)

Functions usually act on the set of real numbers, rational numbers or integers.

Note: The following symbols are used for the subsets of the set of numbers.

$\mathbb{N}$ for the set of natural numbers (positive whole numbers)

$\mathbb{Z}$ for the set of integers (all positive and negative whole numbers, plus zero)

$\mathbb{Q}$ for the set of rational numbers (all integers, plus all fractions)

$\mathbb{R}$ for the set of real numbers (all rational numbers, plus irrational numbers like $\sqrt{2}$ and π)

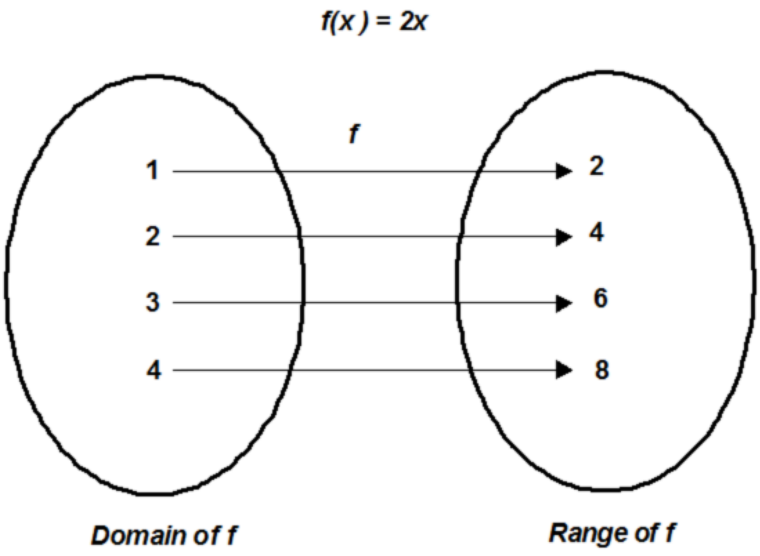
Domain and range of a function.

The **domain** of a function is the set of values on which it acts. It can be finite or infinite.
The **range** of a function is the corresponding set of image values.

Example (2): The function $f(x) = 2x$ acts on the set of numbers x where $1 \le x \le 4$, $x \in \mathbb{Z}$.

Find its domain and range.

The domain of the function is the set of integers {1, 2, 3, 4}.
The resulting range is {2, 4, 6, 8}.



Most functions have as their domain the set of real numbers, $\mathbb{R}$, but in many cases certain values must be omitted for the function to be valid.

Example (3): What are the domains of the following functions:

i) $f(x) = 3x - 4$; ii) $g(x) = \dfrac{2}{1-x}$; iii) $h(x) = 5 \ln x$

Function i) acts on all real numbers without restriction. The domain of $f(x)$ is therefore $x \in \mathbb{R}$.

Function ii) acts on all real numbers except $x = 1$, because $g(x)$ is undefined for that value of x.

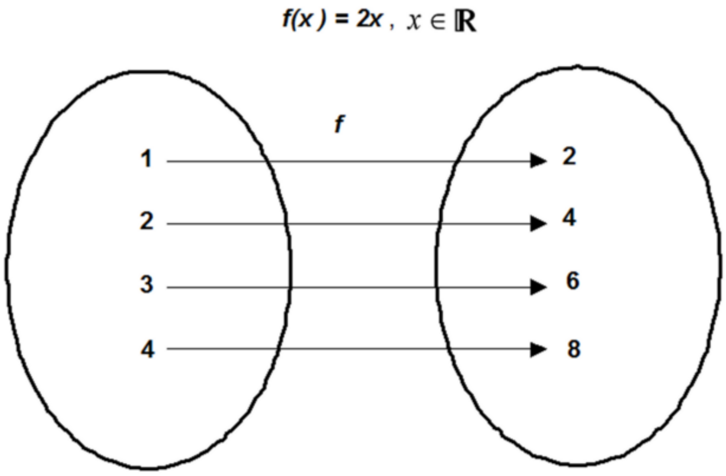
Hence the domain of $g(x)$ is $x \in \mathbb{R}$, $x \neq 1$.

Function iii) acts only on positive real numbers greater than 0, as the natural logarithm function is only defined for those numbers. The domain of $h(x)$ is thus $x \in \mathbb{R}$, $x > 0$.

Every member of the domain of a function can only have one image, but it is possible for different values of the domain to have the same image.

The function $f(x) = 2x$ has for its domain the entire set of real numbers, though only four examples are shown on the diagram. Each value of y in the range has a unique value of x for which $f(x) = y$.

$f(x) = 2x$ is therefore an example of a **one-to-one** function: each member of the domain has exactly one value in the image set.



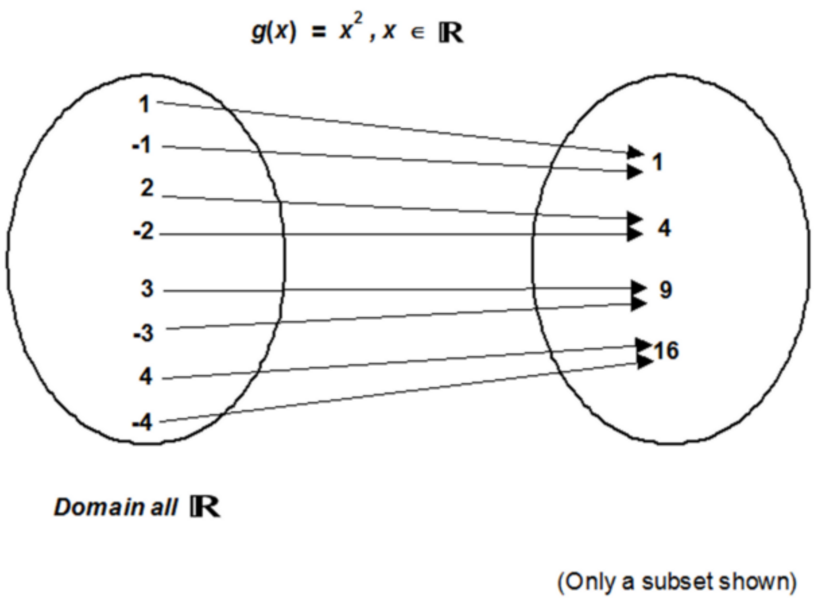
The function $g(x) = x^2$ has for its domain the entire set of real numbers, though only eight examples are shown on the diagram.

This time, each illustrated value of y has two distinct values of x for which $g(x) = y$.

(Only $y = 0$ has a unique value of x with it.)

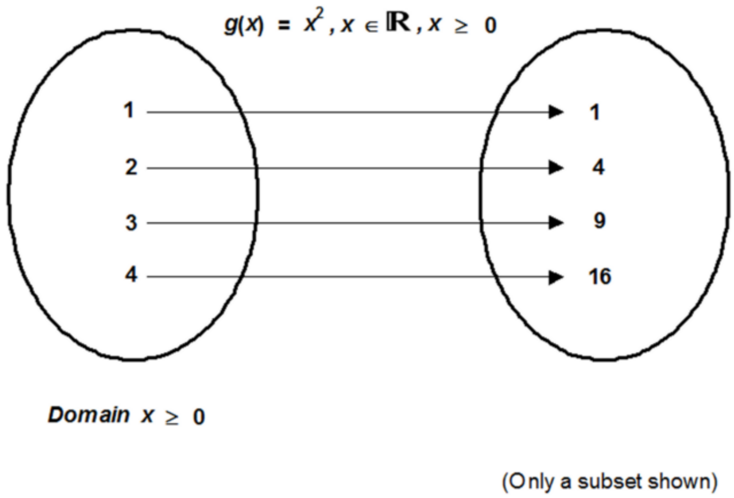
Also, the range of $g(x)$ is restricted to real numbers **greater than or equal to zero**, as no real number can have a negative square.

$g(x) = x^2$ is therefore an example of a **many-to-one** function.

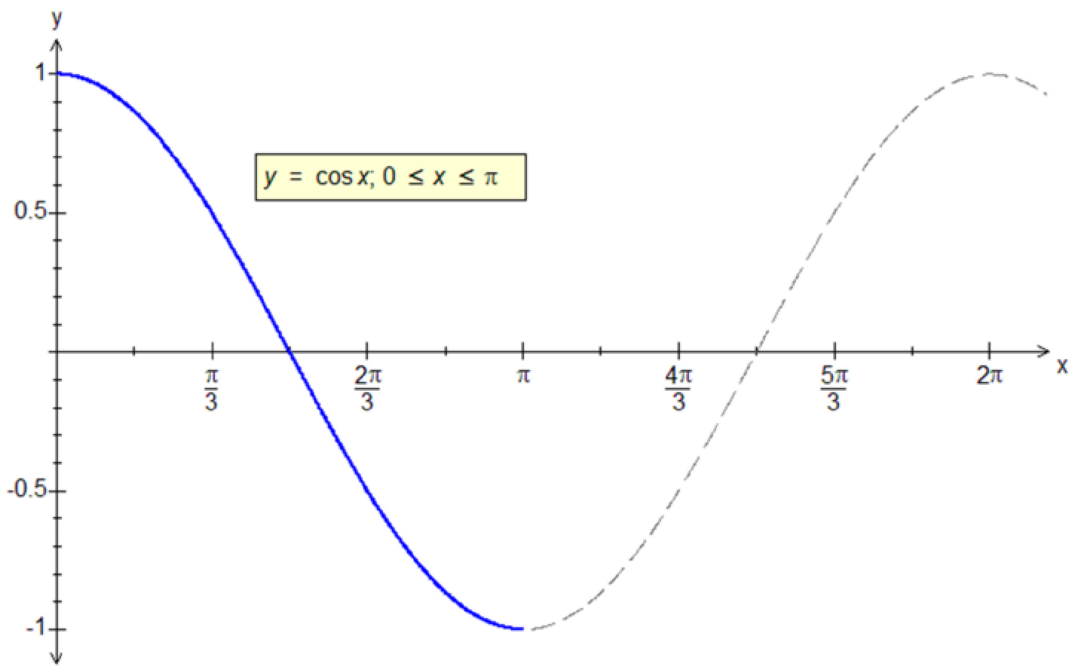


Sometimes a function may be many-to-one, or one-to-one, depending on the restrictions (if any) placed on the domain.

The function $g(x) = x^2$ above is a many-to-one function over the whole domain, but when the domain is restricted to non-negative numbers, it becomes a one-to-one function.



Example (4): The function $f(x) = \cos x$ is a many-to-one function with a repeating period of 2π . What are its domain and range ? How can it be restricted to become a one-to-one function ?



The domain of $y = f(x)$ is $x \in \mathbb{R}$ and its range is $-1 \le y \le 1$.

The function is many-to-one, but it can be defined as one-to-one if the domain is restricted to

$x \in \mathbb{R}$, $0 \le x \le \pi$. (This portion of the graph is shown in bold.)

Transformations of functions with restricted domains.

If a function has a restricted range or domain, then it is possible to use the ideas of transformations of graphs to find the ranges and domains of transformed basic functions. This idea had been seen in the section on trigonometric graph transformations in C1 / C2.

The next examples are illustrated by an arbitrary function $f(x)$, whose graph is shown on the right.

The domain of $f(x)$ is

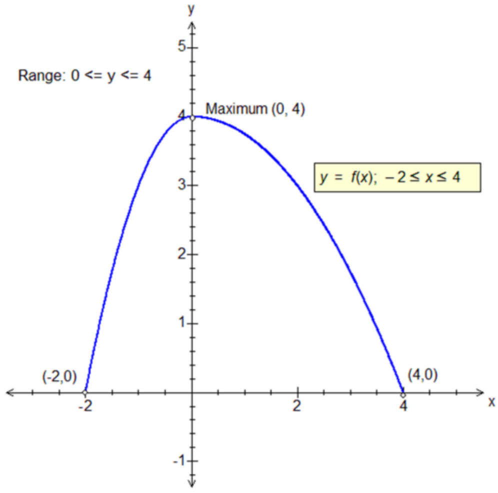
$x \in \mathbb{R}$, $-2 \leq x \leq 4$.

The range of $y = f(x)$ is

$y \in \mathbb{R}$, $0 \leq y \leq 4$.

This graph is the key to the next series of examples.

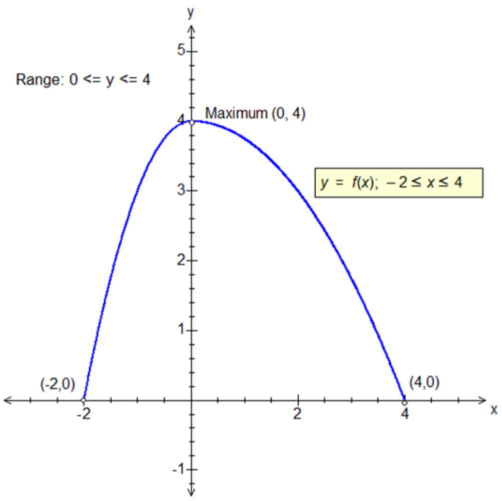
The domain end points are (-2, 0) and (4, 0), and the maximum point is at (0, 4).

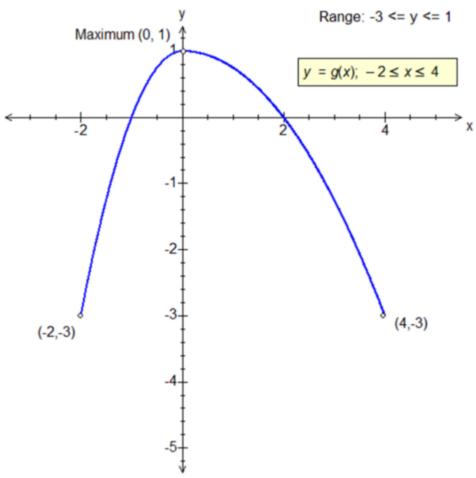


Examples (5): Describe and sketch the following transformations of the function $f(x)$ to $g(x)$, giving details of any changes to the range and domain in each case:

i) $f(x)$ to $g(x) = f(x) - 3$; ii) $f(x)$ to $g(x) = f(x - 2)$; iii) $f(x)$ to $g(x) = 2f(x)$;
iv) $f(x)$ to $g(x) = - f(x)$; v) $f(x)$ to $g(x) = f(2x)$; vi) $f(x)$ to $g(x) = f(-x)$

<u>**i) $f(x)$ to $g(x) = f(x) - 3$**</u>

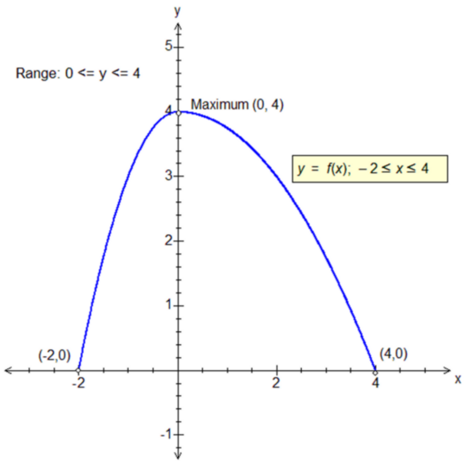


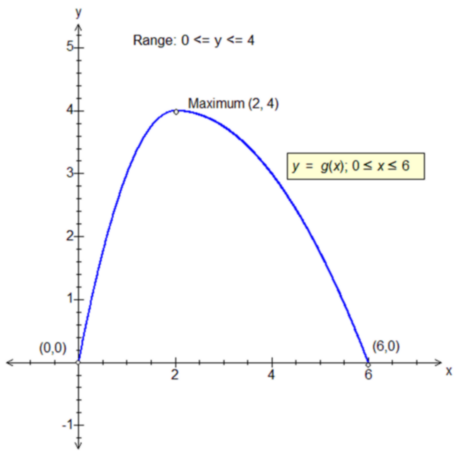


The graph of $g(x)$ is a translation of that of $f(x)$ by the vector $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$. The domain of g is still $-2 \leq x \leq 4$, but the range of $y = g(x)$ is translated along with the graph, to $-3 \leq y \leq 1$.

If a graph is translated in the *y*-direction, then the range is correspondingly transformed.

ii) $f(x)$ to $g(x) = f(x - 2)$

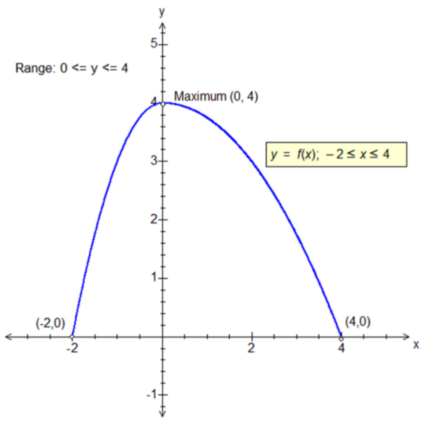


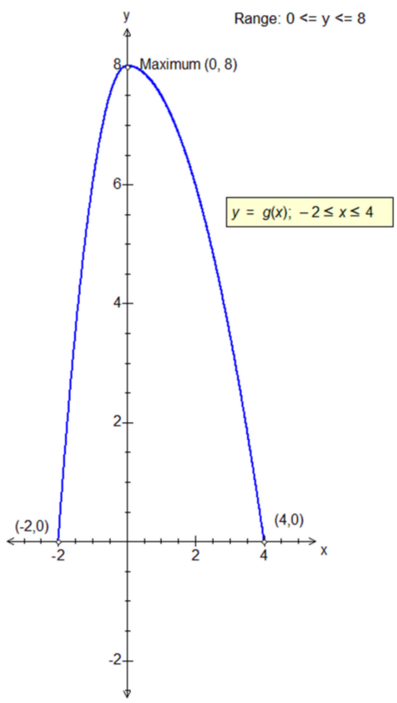


The graph of $g(x)$ is a translation of that of $f(x)$ by the vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$. The domain of g is now translated with the graph, to $0 \le x \le 6$, but the range of $y = g(x)$ is unchanged at $0 \le y \le 4$.

If a graph is translated in the x-direction, then the domain is correspondingly transformed.

iii) $f(x)$ to $g(x) = 2f(x)$



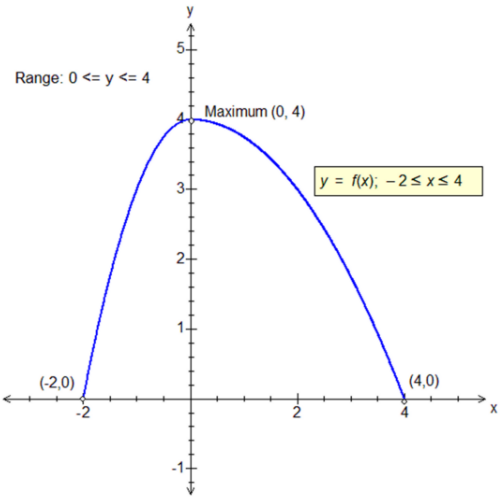


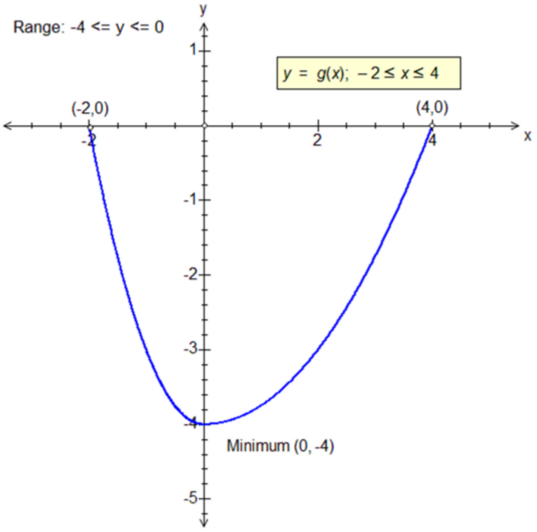
The graph of $g(x)$ is a y-stretch of that of $f(x)$ with scale factor 2.

The domain of g is still $-2 \le x \le 4$, but the range of $y = g(x)$ is scaled by the y-stretch factor of 2, i.e. to $0 \le y \le 8$.

If a graph is stretched in the y-direction, then the range is correspondingly transformed.

iv) $f(x)$ to $g(x) = - f(x)$





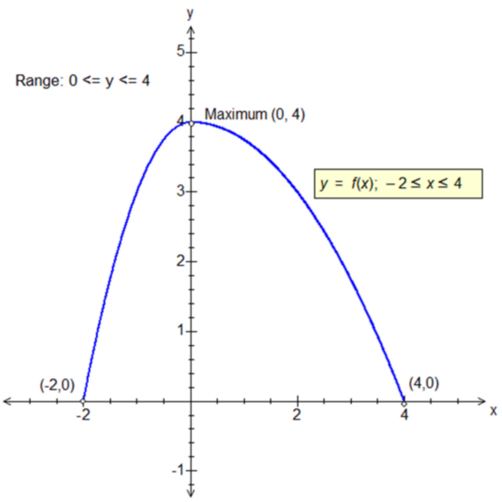
The graph of $g(x)$ is a reflection of that of $f(x)$ in the x-axis, or a special case of a y-stretch with scale factor of -1.

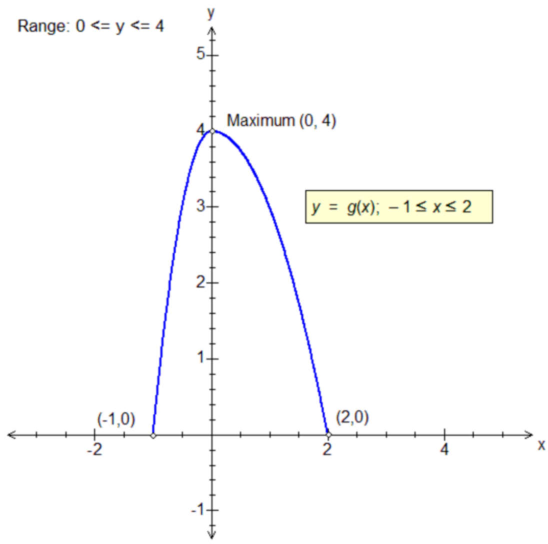
The domain of g is still $-2 \le x \le 4$, but the range of $y = g(x)$ is multiplied by -1, or has its signs reversed , to $0 \ge y \ge -4$ or $-4 \le y \le 0$.

In addition, the maximum point on the graph of $f(x)$ is transformed into a minimum point on the graph of $g(x)$.

If the graph of $f(x)$ is transformed to the graph of - $f(x)$, then the range is reversed in sign.

v) $f(x)$ to $g(x) = f(2x)$

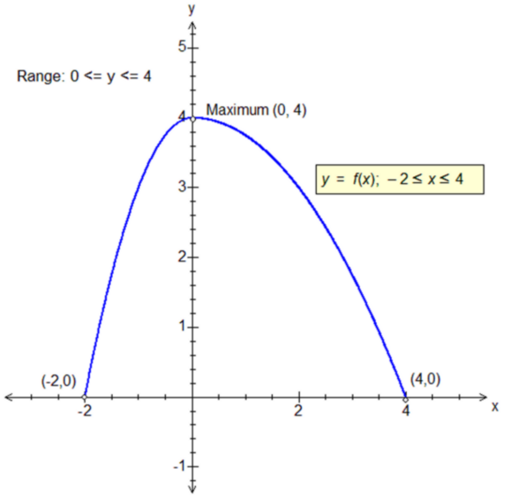


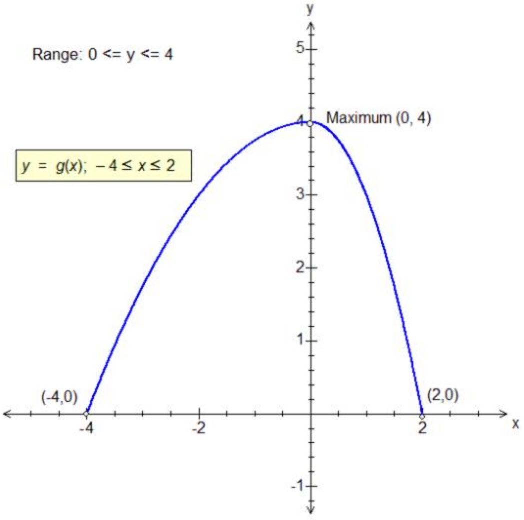


The graph of $g(x)$ is an x-stretch translation of the graph of $f(x)$ with a scale factor of ½. The domain of g is transformed with the graph to $-1 \le x \le 2$, but the range of $y = g(x)$ is unchanged at $0 \le y \le 4$.

If a graph is stretched in the x-direction, then the domain is correspondingly transformed.

vi) $f(x)$ to $g(x) = f(-x)$





The graph of $g(x)$ is a reflection of that of $f(x)$ in the y-axis, or a special case of an x-stretch with scale factor of -1.

The range of $y = g(x)$ is still $0 \le y \le 4$, but the domain of g is multiplied by -1, or has its signs reversed , to $2 \ge y \ge -4$ or $-4 \le y \le 2$.

If the graph of $f(x)$ is transformed to the graph of $f(-x)$, then the domain is reversed in sign.

Examples (6a): Describe the following transformations, and state the ranges and domains of the functions before and after transformation.

i) $f(x) = x^2$ to $g(x) = x^2 + 5$; ii) $f(x) = \ln x$ to $g(x) = \ln(x + 4)$
iii) $f(x) = \sin x$ to $g(x) = 5 \sin x$; iv) $f(x) = \cos x$, $0 \le x \le 2\pi$ to $g(x) = \cos 2x$

i) The graph of $g(x)$ is a translation of that of $f(x)$ by the vector $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$.

Given that the range of $f(x) = x^2$ is the set of all real numbers greater than zero, then the range of $g(x) = x^2 + 5$ is the set of all real numbers greater than 5.
The domain of g is the same as the domain of f.

ii) The graph of $g(x)$ is a translation of that of $f(x)$ by the vector $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$.

Given that the domain of $f(x) = \ln x$ is the set of all real numbers greater than zero, then the domain of $g(x) = \ln(x + 4)$ is the set of all real numbers greater than -4.
The ranges of g and f are both equal.

iii) The graph of $g(x)$ is a y-stretch of that of $f(x)$, with a scale factor of 5.
Given that the range of $f(x) = \sin x$ is the set of all real numbers from -1 to 1 inclusive, then the range of $g(x) = 5 \sin x$ is the set of all real numbers from -5 to 5 inclusive.

The domains of g and f are the same, i.e. the set of real numbers.

iv) The graph of $g(x)$ is an x-stretch of that of $f(x)$, with a scale factor of ½.
The function $f(x) = \cos x$ has a the restricted domain of $0 \le x \le 2\pi$.
Transforming its graph to that of $g(x) = \cos 2x$ also transforms the domain to $0 \le x \le \pi$.

Example (6b): Find the range of $h(x) = 1\text{-}2x^2$.

We begin with the function x^2, whose range is all real numbers greater than or equal to zero.
Next, we see the y-stretch with scale factor -2 which maps x^2 to $-2x^2$, which means transforming the range to all the real numbers *less than or equal to zero.* This is due to the negative stretch factor.

Secondly, the function $-2x^2$ is mapped to $h(x) = 1\text{-}2x^2$ by a translation with the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This has the effect of transforming the range by +1, i.e. to all real numbers less than or equal to 1.

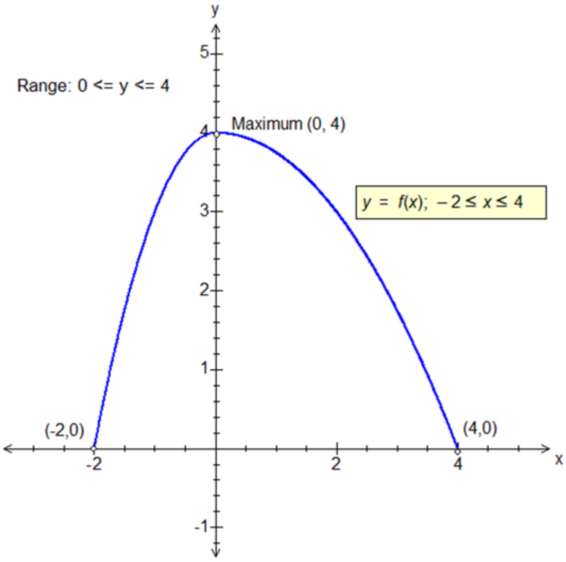
(We could have also reasoned that, since the minimum value of the squared term is zero, subtracting that from the 1 would give a maximum value of 1 in the range.)

Composite transformations of functions with restricted domain.

Examples (7): Describe and sketch **t**he following transformations of the function $f(x)$ to $g(x)$, giving details of any changes to the range and domain in each case, using the graph of $f(x)$ from Examples 5:

i) $f(x)$ to $g(x) = f(2x) + 3$
ii) $f(x)$ to $g(x) = 2f(x) + 3$
iii) $f(x)$ to $g(x) = f(2x + 3)$

Case iii) is the most difficult one to work out – see the corresponding section of the document on "Transforming Graphs".



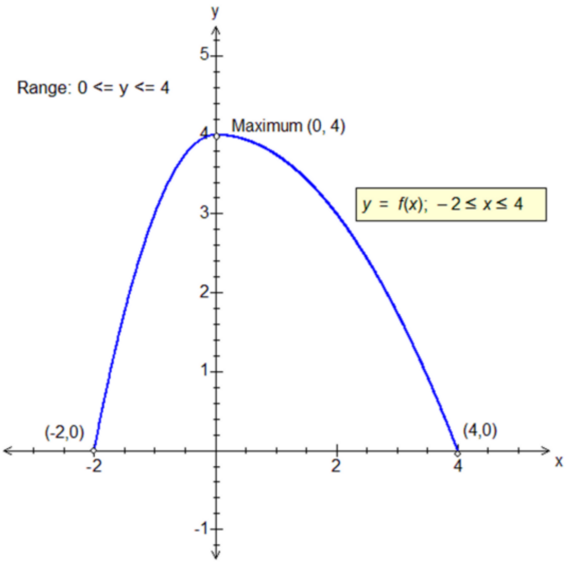
(The intermediate stages of each composite transformation are not shown here.)

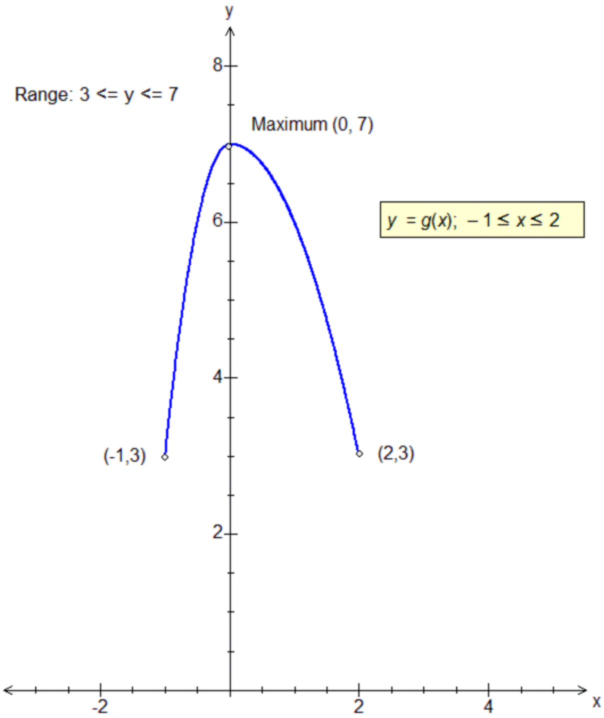
i) $f(x)$ to $g(x) = f(2x) + 3$

We begin with an x-stretch with scale factor ½ to transform $y = f(x)$ to $y = f(2x)$.
This transforms the domain from $-2 \le x \le 4$ to $-1 \le x \le 2$.

Next, a translation by vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ transforms the range from $0 \le y \le 4$ to $3 \le y \le 7$.

The maximum has been transformed to (0, 7), the domain start to (-1, 3) and the domain end to (2, 3).





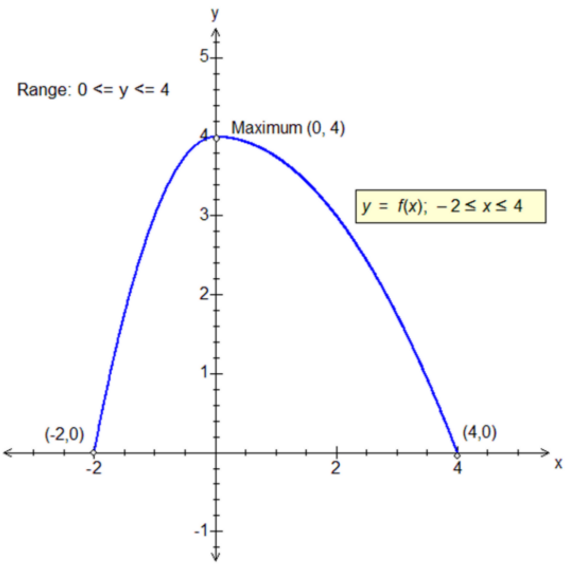
ii) $f(x)$ to $g(x) = 2f(x) + 3$

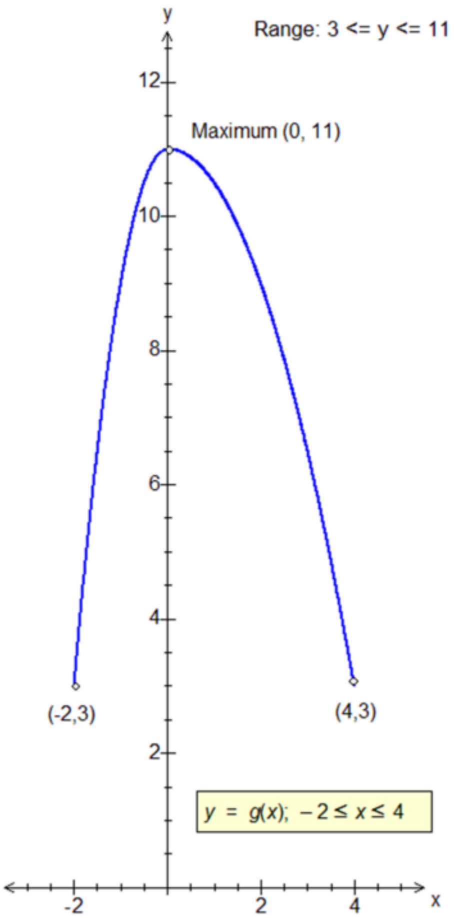
We begin with a y-stretch with scale factor 2 to transform $y = f(x)$ to $y = 2f(x)$.

This transforms the range from $0 \le y \le 4$ to $0 \le y \le 8$.

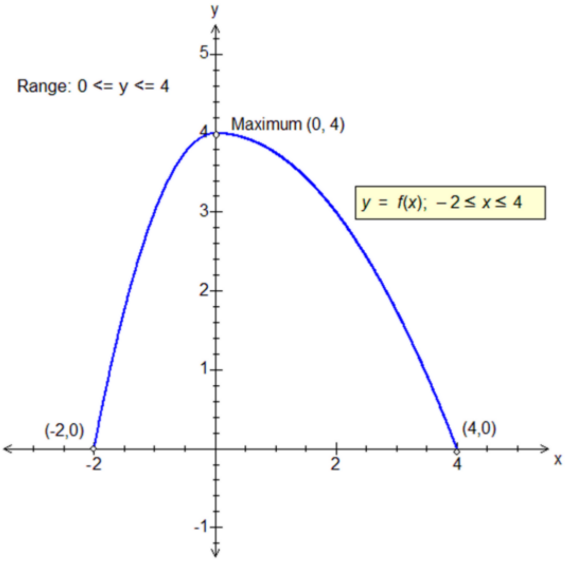
Next, a translation by vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ transforms the range from $0 \le y \le 8$ to $3 \le y \le 11$.

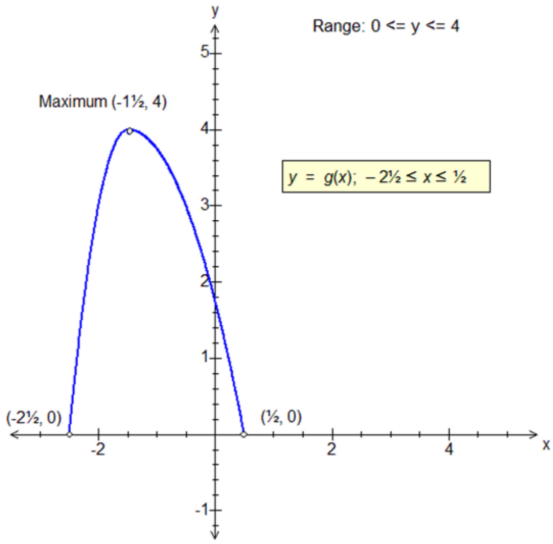
The maximum has been transformed to (0, 11), the domain start to (-2, 3) and the domain end to (4, 3).





iii) the trickiest - translation by vector $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ to transform to $f(x+3)$ and then an x-stretch by scale factor ½ to transform to $g(x) = f(2x + 3)$.





Both transformations leave the range unchanged. Firstly, the translation transforms the maximum point to (-3, 4) and the domain to $-5 \le x \le 1$.

Secondly, the x-stretch transforms the maximum to (-1½, 4) and the domain to $-2½ \le x \le ½$.

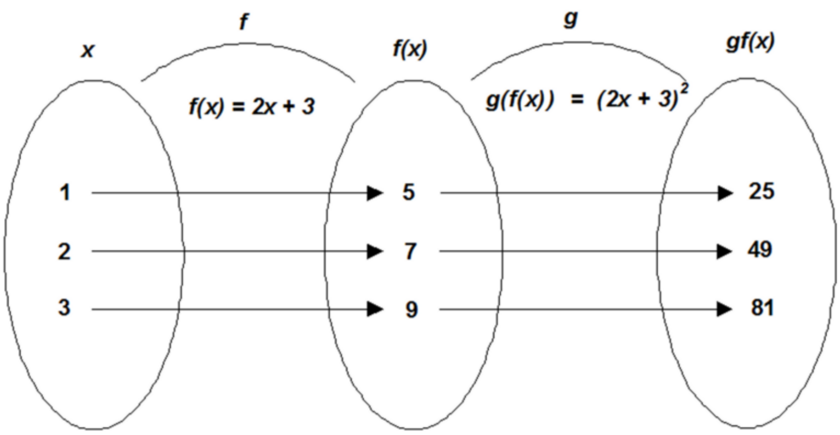
When the function $f(x)$ is transformed to $f(ax + b)$ we can always check the new limits of the domain by solving $ax + b = x_1$ and $ax + b = x_2$ where x_1 and x_2 are the limits of the domain of f.

Here we solve $2x + 3 = -2 \Rightarrow 2x = -5 \Rightarrow x = -2½$ and $2x + 3 = 4 \Rightarrow 2x = 1 \Rightarrow x = ½$.

Composition of functions.

This is closely related to composite transformations.

Example (8): Take two functions $f(x) = 2x + 3$ and $g(x) = x^2$.

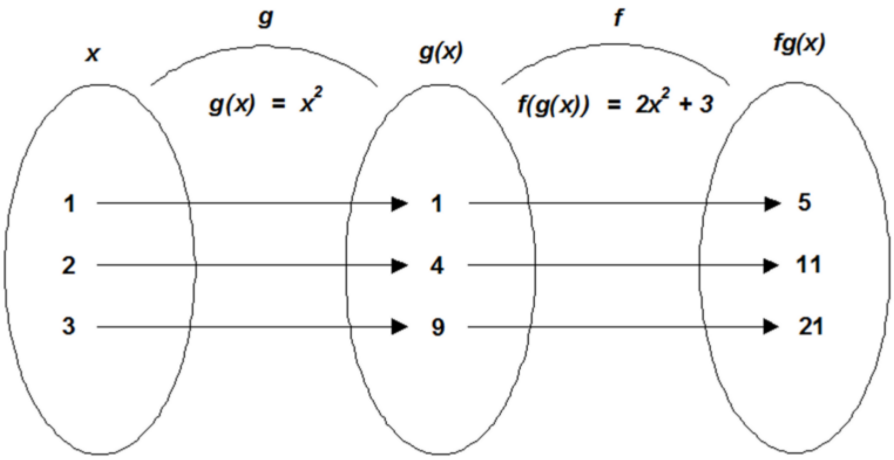


Here the function *f* is applied to *x* which doubles *x* and then adds 3. Next, the function *g* is applied to square that result.

This is written as $gf(x)$ or $g(f(x))$, and this new function is equivalent to $(2x + 3)^2$.

Note how, in the written expression, the function closest to the *x* is performed first.

The order in which the functions are applied is important ! The diagram below shows what happens when *g* is performed first. This time, *g* is applied to *x* first, squaring it. Then *f* doubles the result and adds 3 to it. This is written as $fg(x)$ or $f(g(x))$, and this new function is equivalent to $2x^2 + 3$.



Generally, for any two functions *f* and *g*, the composite functions *fg* and *gf* will be different.

Functions can also be combined with themselves; thus $ff = 2(2x + 3) + 3$, i.e. $4x + 9$,
and *gg* is the same as $(x^2)^2$, i.e. x^4.

Thus $ff(x) = 13, 17, 21$ for $x = 1, 2, 3$, and $gg(x) = 1, 16, 81$ for $x = 1, 2, 3$.

Example (9): Let $f(x) = x - 5$ and $g(x) = x^3$.

a) Find expressions for: i) $fg(x)$; ii) $gf(x)$; iii) $ff(x)$; iv) $gg(x)$; v) $gfg(x)$; vi) $fgf(x)$

b) Use the results in part (a) to find the values of:
i) $fg(4)$, ii) $gf(4)$, iii) $ff(3)$, iv) $gg(2)$, v) $gfg(2)$, vi) $fgf(8)$.

Part a):

i) We apply g (cubing) first, then f (subtracting 5), so $fg(x) = x^3 – 5$.
This is the same as substituting x^3 for x in the expression for $f(x)$.

ii) This time we apply f first (subtract 5), then apply g, so $gf(x) = (x- 5)^3$.
This is the same as substituting x-5 for x in the expression for $g(x)$.

iii) We substitute x-5 for x in the expression for $f(x)$ to obtain $ff(x) = (x – 5) – 5$, i.e. $ff(x) = x – 10$.

iv) Substituting x^3 for x in the expression for $g(x)$, we have $gg(x) = (x^3)^3$, i.e. $gg(x) = x^9$.

v) From the result in i), we apply g again, i.e. substitute $x^3 – 5$ for x in the expression for $g(x)$:
i.e. $gfg(x) = (x^3-5)^3$.

vi) From the result in ii), we apply f again, i.e. substitute $(x- 5)^3$ for x in the expression for $f(x)$:
i.e. $fgf(x) = (x- 5)^3-5$.

Part b);

i) $fg(4) = 4^3 – 5 = 59$.

ii) $gf (4) = (4- 5)^3 = -1$.

iii) $ff(3) = 3 – 10 = -7$.

iv) $gg(2) = 2^9 = 512$.

v) $gfg(2) = (2^3-5)^3 = 3^3 = 27$.

vi) $fgf(8) = (8- 5)^3-5 = 3^3 – 5 = 22$.

Example (10): Two functions are defined as follows :

$f(x) = x^2 + 4$; $g(x) = 1 - 2x$

i) Find expressions for $fg(x)$ and $gf(x)$, and hence work out $fg(3)$ and $gf(3)$.
ii) Solve $gf(x) = -15$.
iii) Solve $fg(x) = 53$.

iv) Find expressions for $ff(x)$ and $gg(x)$, and hence work out $ff(2)$ and $gg(2)$.
v) Solve $ff(x) = 29$.
vi) Solve $gg(x) = 23$.

i) $fg(x) = f(1 - 2x) = (1 - 2x)^2 + 4$, or $4x^2 - 4x + 5$. Substituting $x = 3$, $fg(3) = 29$.

$gf(x) = 1 - 2f(x) = 1 - 2(x^2+4)$, or $-2x^2 - 7$. Substituting $x = 3$, $gf(3) = -25$.

ii) To solve $gf(x) = -15$, write as $-2x^2 - 7 = -15$ and rearrange as
$8 - 2x^2 = 0 \rightarrow 4 - x^2 = 0 \rightarrow (2 + x)(2 - x) = 0 \rightarrow$ $\boldsymbol{x = \pm 2.}$

iii) We solve $fg(x) = 53 \rightarrow 4x^2 - 4x + 5 = 53 \rightarrow 4x^2 - 4x - 48 = 0$
$\rightarrow x^2 - x - 12 = 0 \rightarrow (x + 3)(x - 4) = 0 \rightarrow$ $\boldsymbol{x = 4 , x = -3.}$

iv) $ff(x) = f(x^2 + 4) = (x^2 + 4)^2 + 4$. Also, $gg(x) = g(1 - 2x) = 1 - 2(1 - 2x) = 4x - 1$.
Hence $ff(2) = 68$ and $gg(2) = 7$.

v) As $ff(x) = (x^2 + 4)^2 + 4$ we solve $(x^2 + 4)^2 + 4 = 29 \rightarrow (x^2 + 4)^2 = 25 \rightarrow x^2 + 4 = 5$
$\rightarrow x^2 = 1 \rightarrow$ $\boldsymbol{x = \pm 1.}$

(We excluded the case $x^2 + 4 = -5 \rightarrow x^2 = -9$ when taking the square root, because this equation has no solution.)

vi) As $gg(x) = 4x - 1$, we solve $4x - 1 = 23 \rightarrow 4x = 24 \rightarrow$ $\boldsymbol{x = 6.}$

Composition of functions – restricted domains.

Care must be taken when any of the individual functions making up a composite one have restricted domains, as the next example will show.

Example (11): Let $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{1-x}$.

i) State the domains and ranges of $f(x)$ and $g(x)$.

ii) Find an expression for $fg(x)$. What is the value of $fg(-3)$, and why cannot $fg(1)$ and $fg(2)$ be found ?

iii) Find the domain and range of $fg(x)$.

iv) Find an expression for $gf(x)$. What is $gf(4)$, and why cannot $gf(1)$ and $gf(-1)$ be evaluated ?

v) Find the domain and range of $gf(x)$.

i) Since only positive numbers and zero have a square root, the domain of $f(x)$ is $x \in \mathbb{R}$, $x \geq 0$.

With $g(x)$ the only restriction to the domain is that x does not equal to 1, otherwise we would have division by zero. The domain of $y = g(x)$ is therefore $x \in \mathbb{R}$, $x \neq 1$.

The range of $y = f(x)$ is all the non-negative real numbers , i.e. $y \in \mathbb{R}$, $y \geq 0$. This is because the square root function is defined as the *positive* square root.

Similarly, $g(x)$ can take any value except zero, as it is a reciprocal function.

The range of $y = g(x)$ is thus, $y \in \mathbb{R}$, $y \neq 0$.

ii) We carry out g first and then take the square root of the result, so $fg(x) = \sqrt{\frac{1}{1-x}}$.
Here, $g(-3) = \frac{1}{1-(-3)}$ or $\frac{1}{4}$, and $f(\frac{1}{4}) = \sqrt{(\frac{1}{4})} = \frac{1}{2}$. Hence $fg(-3) = \frac{1}{2}$.
We cannot find $fg(1)$ since $x = 1$ is omitted from the domain of g. (Cannot divide by zero).

Similarly we cannot find $fg(2)$, since although $g(2)$ can be worked out as $\frac{1}{1-2}$ or -1, we then run into the impossibility of trying to find $\sqrt{(-1)}$ as no number has a negative square root.

iii) The domain of $fg(x)$ therefore cannot include a value of 1, and neither could it take any values of x which would give a value of $g(x) < 0$, or $\frac{1}{1-x} < 0$ This corresponds to all $x > 1$, so all those values must also be omitted from the domain.

Combining these two restrictions gives the domain of $fg(x)$ as $x \in \mathbb{R}$, $x < 1$.

The range of $g(x)$ is all the real numbers except zero, therefore the range of $fg(x)$ is the same as that of $f(x)$, but with $f(0)$ or 0 omitted.

The range of $y = fg(x)$ is therefore $y \in \mathbb{R}$, $y > 0$.

Let $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{1-x}$.

iv) We carry out f first and then g, so we replace the x with $\sqrt{x}$ in the expression for $g(x)$, i.e. $gf(x) = \frac{1}{1-\sqrt{x}}$. Letting $x = 4$, $gf(4) = \frac{1}{1-\sqrt{4}} = -1$.

We cannot find $gf(1)$, because although $f(1) = \sqrt{1} = 1$ is allowable, the value of $x = 1$ is omitted from the domain of g as we cannot divide by zero.

We cannot find $gf(-1)$, because there is no real value of $\sqrt{(-1)}$.

v) The domain of $gf(x)$ therefore cannot include any negative numbers since no negative number can have a square root. It also cannot include 1 since $f(1)$, or 1, is omitted from the domain of g.

Combining the restrictions, the domain of $gf(x)$ is $x \in \mathbb{R}$, $x > 0$ and $x \neq 1$.

The range of $y = gf(x)$ is slightly trickier to find. No reciprocal function can include zero in its range, so we must exclude zero from the set of values. Secondly, we know that the range of $\sqrt{x}$ is all the positive numbers plus zero.

Two transformations map $\sqrt{x}$ to $1-\sqrt{x}$:

Firstly, a y-stretch of scale factor -1 maps $\sqrt{x}$ to $-\sqrt{x}$, which in turn transforms the range to all the negative numbers plus zero.

Then, a y-translation with vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ maps $-\sqrt{x}$ to $1-\sqrt{x}$, which finally transforms the range to all the real numbers less than or equal to 1.

Finally, since the positive numbers in the range of $1-\sqrt{x}$ cannot exceed 1, the positive reciprocal of $1-\sqrt{x}$ cannot be less than 1.

The range of $y = gf(x)$ is therefore $y \in \mathbb{R}$, $y < 0$ or $y > 1$.

The inverse of a function.

The inverse of a function *f* is another function that reverses whatever *f* does. This inverse function is usually written f^{-1}.

It therefore follows that for a function $f(x)$ with an inverse (see later), $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

Example (12): Find the inverses of the following functions:

i) $f(x) = x + 4$; ii) $g(x) = 3x$; iii) $h(x) = 2x + 5$; iv) $fg(x)$ (using the examples in i) and ii); v) $gf(x)$.

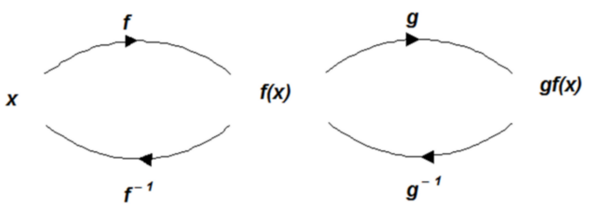
In i), the function *f* adds 4 to *x* ; the inverse function f^{-1} subtracts 4 from it,
$\therefore \quad f^{-1}(x) = x - 4$.

In ii), the function *g* multiplies *x* by 3 ; the inverse function g^{-1} divides it by 3,
$\therefore \quad g^{-1}(x) = \frac{x}{3}$.

In iii), the function *h* can be seen as a two-step process; first double *x* and then add 5 to the result. The inverse function must undo the processes, but in the reverse order to the original function.

To undo *h*, we must subtract 5 first and then halve the result.
$\therefore \quad h^{-1}(x) = \frac{x-5}{2}$.

The result from iii) can be generalised in the diagram below: **the inverse of gf, $(gf)^{-1}$, is $f^{-1}g^{-1}$.**

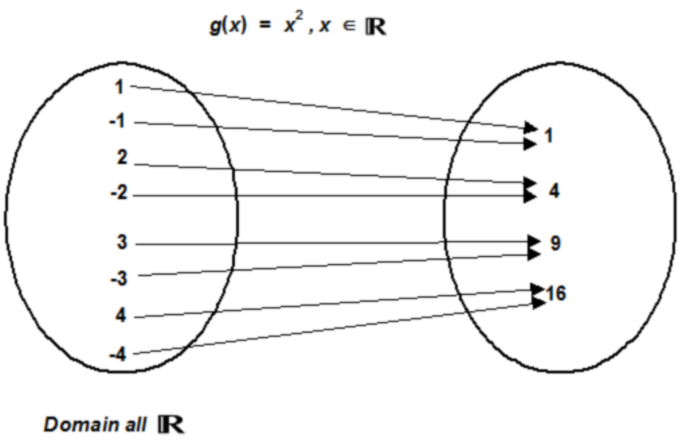


For iv), we already have the inverses of *f* and *g*. The inverse of $fg(x)$ is thus $g^{-1}f^{-1}(x)$ or $\frac{x-4}{3}$.

In v), the inverse of $gf(x)$ is thus $f^{-1}g^{-1}(x)$ or $\frac{x}{3} - 4$.

For a function to have an inverse, it must be **one-to-one**.

Take the function $g(x) = x^2$, whose domain is the entire set of real numbers, though only eight examples are shown on the diagram. This is a many-to-one function as it stands – for example, 16 is the square of both 4 and –4.

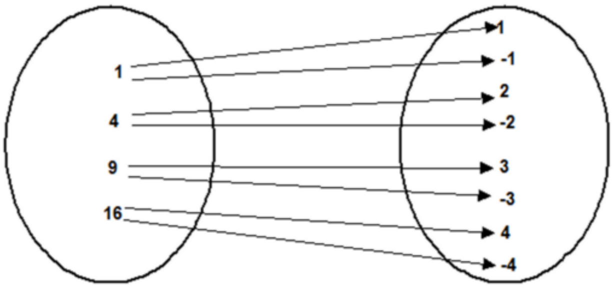


(Only a subset shown)

If we were to reverse the diagram, we would run into a problem at once. For example, the square root of 16 is 4, but we could have also obtained 16 by squaring the number -4.

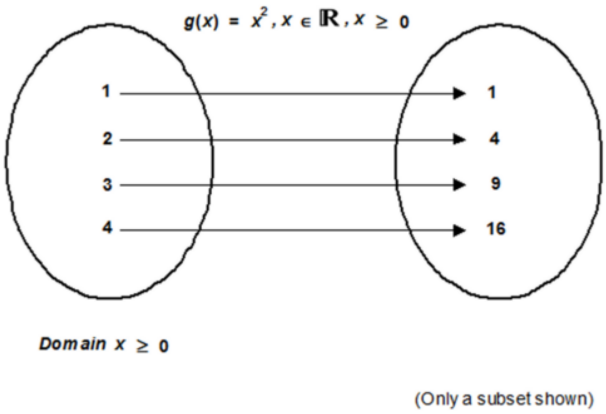
The resulting mapping is not a function, because it is one-to-many. Every member of the domain of a function can only have one image !

∴ A many-to-one function does not have an inverse.



However, a many-to-one function *can* be changed to a one-to-one function by suitably restricting its domain. By restricting the domain of $g(x) = x^2$ to the set of positive real numbers (plus zero), we now have a one-to-one function.

Both the domain and range of g are now the set of non-negative real numbers.

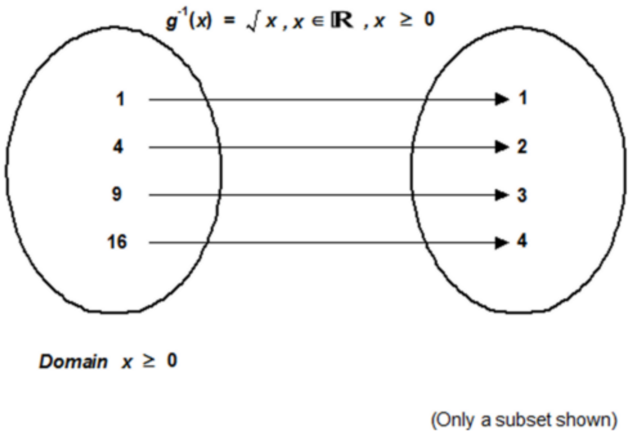


The inverse of $g(x)$ or $g^{-1}(x)$ is the square root function, $\sqrt{x}$, taken to be the *positive* square root of x by convention.

Because negative real numbers are no longer in the domain of g, they are likewise absent from the range of g^{-1}.

The domain of an inverse function is the range of the original , and vice versa.

The function is now one-to-one.



The graph shows the relationship between the functions $g(x)$ and $g^{-1}(x)$.

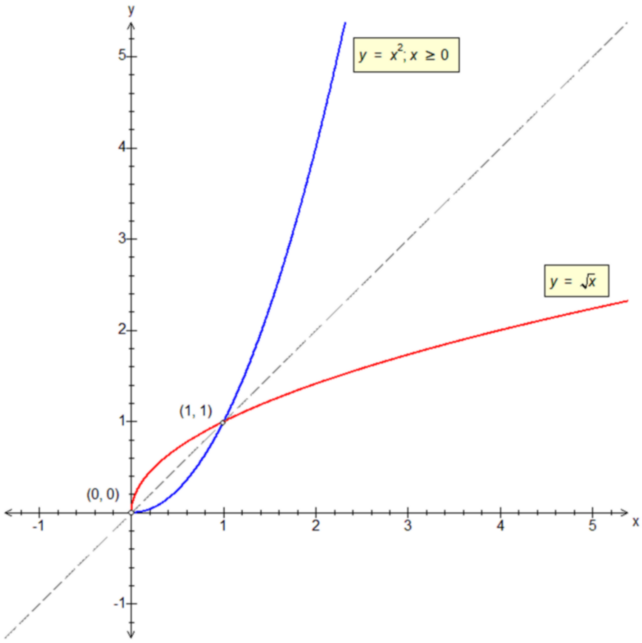
The graph of $g(x)$ is that of $y = x^2$, but with the domain restricted to $x \geq 0$.

The graph of $g^{-1}(x)$ is that of $y = \sqrt{x}$, again with the same domain restriction, $x \geq 0$.

Each graph is a reflection of the other in the line $y = x$.

Some functions are self-inverses, for example the reciprocal function $1/x$.

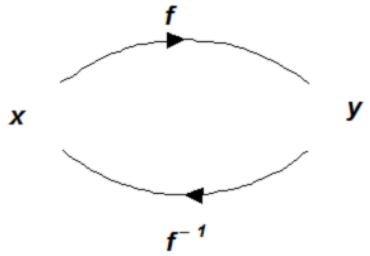
Note also how the graphs of $g(x)$ and $g^{-1}(x)$ cross at the point (1, 1). This is because $1^2 = \sqrt{1} = 1$.



In general, the graphs of $y = f(x)$ and $y = f^{-1}(x)$ will cross each other (and the line $y = x$) whenever a solution can be found of $f(x) = x$, domain restrictions notwithstanding.

Finding the inverse of a one-to-one function (Harder Example).

In Example (12), we found the inverses of various linear functions mentally, but the method below is to be recommended for more complicated examples.
The easiest way of finding the inverse of a function is to write it in the form $y = f(x)$ and then manipulate it to put x on the left-hand side, i.e. to express it in the form $x = f^{-1}(y)$.
The inverse function is then re-written in terms of x by exchanging the x and y.



The domain of f^{-1} is the range of f and vice-versa.

Example (13): Find the inverse of the function (using the method above) $f(x) = 2x + 5$, $x \in \mathbb{R}$.
Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same diagram, showing their point of intersection.

Write the function as $y = 2x + 5$.

Turn it round to bring x to the left-hand side:

$$2x + 5 = y$$

Make x the subject by algebraic manipulation:

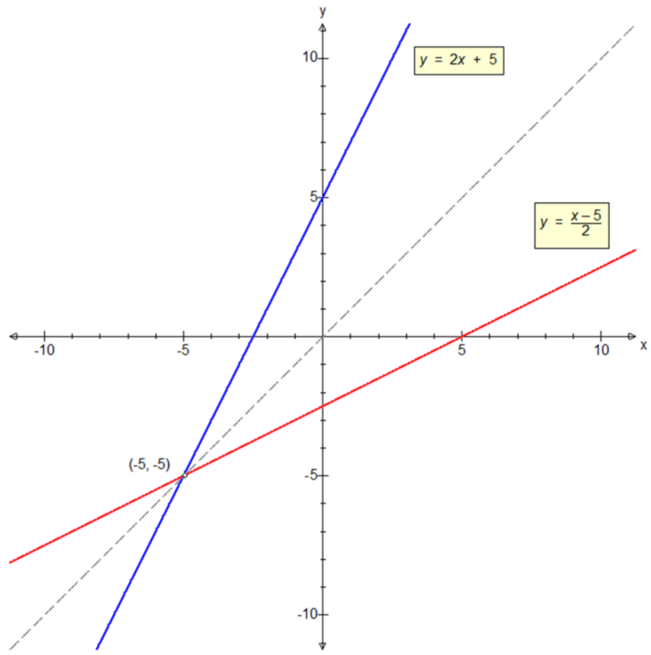
$$2x = y - 5$$

$$\Rightarrow x = \frac{y-5}{2}$$

As $x = f^{-1}(y)$, the inverse function is here defined in terms of y.

The final step is to exchange x and y in order to express the inverse function $y = f^{-1}(x)$.

Here, $y = \dfrac{x-5}{2}, x \in \mathbb{R}$.



(Each graph is a reflection of the other in the dotted line $y = x$).

The point of intersection can be found by solving $f(x) = x$, because it would also lie on the line $y = x$.
So $2x + 5 = x \Rightarrow x + 5 = 0 \Rightarrow x = y = -5$.

$\therefore$ the graphs of $f(x)$ and $f^{-1}(x)$ intersect at (-5, -5).

It may also be noticed that the gradient of $f(x) = 2$ and that of $f^{-1}(x) = ½$, giving a product of 1.
This product rule is true for all linear function pairs $f(x)$ and $f^{-1}(x)$.

(Do not confuse this with the rule for perpendicular lines, whose gradients have a product of –1).

The document "The Chain, Product and Quotient Rules" includes a further non-linear example of this product rule.

Example (14): Find the inverse of the function

$$f(x) = \left(\frac{1+x}{1-x}\right),\ x \in \mathbb{R}\ ,\ x \neq 1.$$

Define the function as $y = \left(\frac{1+x}{1-x}\right)$

Multiply out brackets and turn expression round:

$$\Rightarrow 1+x = y(1-x)$$
$$\Rightarrow 1+x = y - xy$$

Collect x terms to one side:

$$\Rightarrow\ 1+xy+x = y$$
$$\Rightarrow\ 1+x(y+1) = y$$

Isolate x as subject:

$$\Rightarrow\ x(y+1) = y-1$$

$$x = \left(\frac{y-1}{y+1}\right)$$

As $x = f^{-1}(y)$, the inverse function is here defined in terms of y.

We therefore exchange x and y to obtain the inverse function $y = f^{-1}(x)$:

$$y = \left(\frac{x-1}{x+1}\right),\ x \in \mathbb{R}\ ,\ x \neq -1.$$

(Remember that the inverse function is defined for all real numbers except x = -1.)

Example(15):

i) Find the inverse of the function (using the method above) $f(x) = x^2 - 6$, $x \geq 0$, $x \in \mathbb{R}$,stating the domain of the inverse function $f^{-1}(x)$.

ii) Solve the equation $f^{-1}(x) = f(x)$. Show the result on a sketch graph of $f^{-1}(x)$ and $f(x)$.

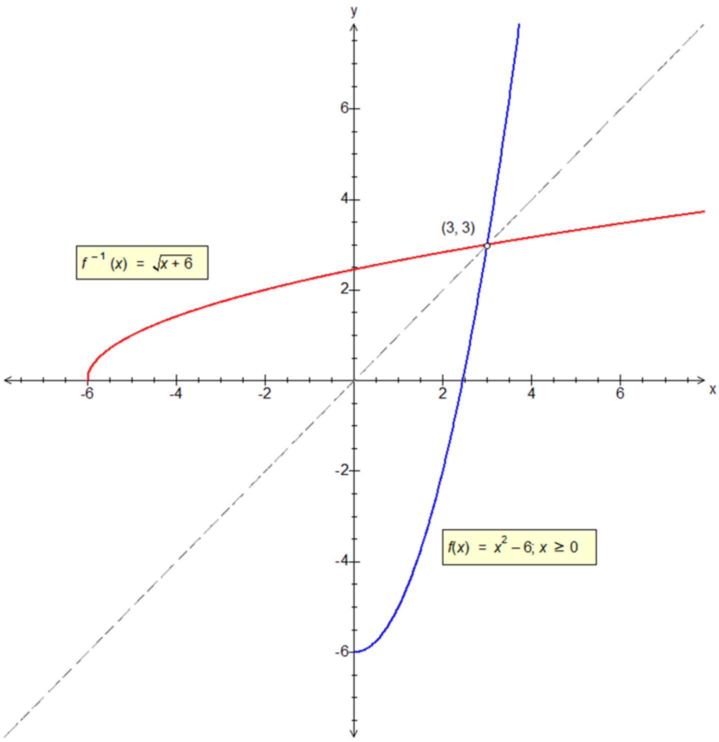
i) Begin by writing the function as $y = x^2 - 6$, followed by making x the subject.

$$y = x^2 - 6 \;\Rightarrow\; x^2 = y + 6 \Rightarrow\; x = \sqrt{y+6}\,.$$

Exchanging x and y we have
$y = \sqrt{x+6}$, and therefore
$f^{1}(x) = \sqrt{x+6}$.

Since the range of $f(x)$ is all the real numbers greater than or equal to -6, the domain of $f^{1}(x)$ is the same, namely

$x \in \mathbb{R}$, $x \geq -6$.



ii) To solve the equation $f^{-1}(x) = f(x)$, we deduce that any solution of this equation would, by symmetry, also satisfy $f(x) = x$ and, for that matter, $f^{-1}(x) = x$.

Solving $x^2 - 6 = x$, we continue
$x^2 - x - 6 = 0 \Rightarrow (x - 3)(x + 2) = 0$
$\Rightarrow x = 3, x = -2$.

We must reject $x = -2$ since it is not in the domain of $f(x)$, and therefore $x = 3$ is the only solution of $f^{-1}(x) = f(x)$.

The two graphs therefore meet at (3, 3).

Note: We do *not* try solving the equation as

$x^2 - 6 = \sqrt{x+6}$ and then

squaring both sides to obtain

$(x^2 - 6)^2 = x + 6$ followed by

$x^4 - 12x^2 + 36 = x + 6$ and

$x^4 - 12x^2 - x + 30 = 0$.

(Although this is also correct, the algebra is more awkward, although we could try substituting various values for x until we get $x^4 - 12x^2 - x + 30 = 0$. When $x = 3$, $81 - 108 - 3 + 30 = 0$).

Example (16):

i) Find the inverse of the function (using the method above) $f(x) = x^2 + 2$, $x \geq 0$, $x \in \mathbb{R}$,stating the domain of the inverse function $f^{-1}(x)$.

ii) Show, without a sketch graph, that there is no solution of the equation $f^{1}(x) = f(x)$.

i) Letting $y = x^2 + 2 \Rightarrow x^2 = y - 2 \Rightarrow x = \sqrt{y-2}$.

Exchanging x and y we have $y = \sqrt{x-2}$, and therefore $f^{1}(x) = \sqrt{x-2}$.

Since the range of $f(x)$ is all the real numbers greater than or equal to 2, the domain of $f^{1}(x)$ is the same, namely $x \in \mathbb{R}$, $x \geq 2$.

ii) Any solution of $f^{-1}(x) = f(x)$ would satisfy $f(x) = x$.

In solving $x^2 + 2 = x$, we continue to $x^2 - x + 2 = 0$.

The discriminant of this quadratic, $b^2 - 4ac = -7$ (i.e. < 0) , so the equation $f^{-1}(x) = f(x)$ has no solution.