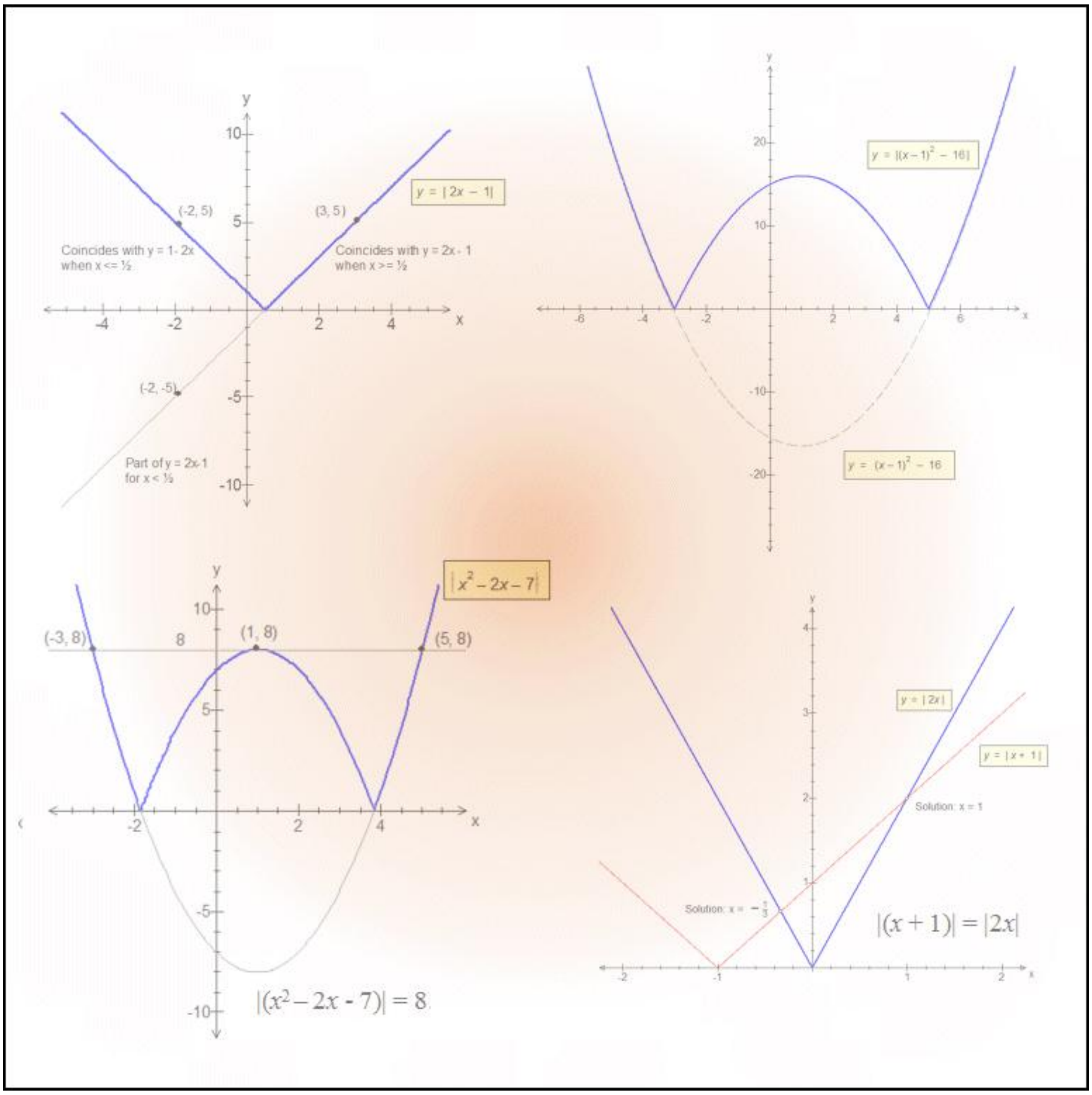
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

THE MODULUS FUNCTION
$|\,x\,|$



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THE MODULUS FUNCTION

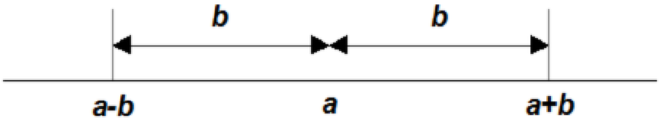
The modulus function, | x |.

The modulus of x, $|x|$ is defined as follows:

$|x| = x$ for $x \geq 0$, e.g. $|5| = 5$. (i.e. $|x| = x$ for positive x)
$|x| = -x$ for $x < 0$, e.g. $|-2| = 2$. (i.e. $|x| = -x$ for negative x)

It can also be described as the magnitude of a number, disregarding the sign, and it never takes a negative value.

The expression $|x-a|$ can be interpreted as the distance between two numbers x and a on the number line.

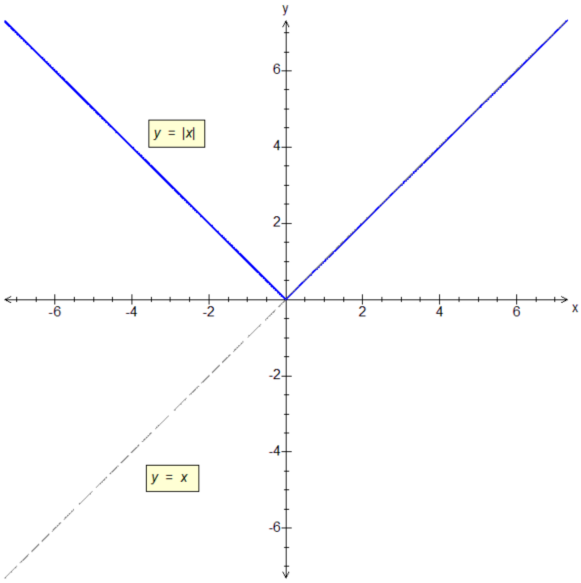


The statement $|x-a| < b$ is another way of saying that the distance between x and a is less than b.

x must lie between the vertical lines, in other words $a - b < x < a + b$.

The graph of $y = |x|$ is therefore identical to the graph of $y = x$ when x is positive. (The illustrated graph of $y = x$ is offset slightly to show the relationship.)

When x is negative, the graph of $y = |x|$ is a reflection of the graph of $y = x$ in the x-axis.



Modulus of a function, | *f* (*x*) |.

Example (1): Sketch the graph of $y = (x-1)^2 - 16$, and from it sketch the graph of $y = | (x-1)^2 - 16 |$.

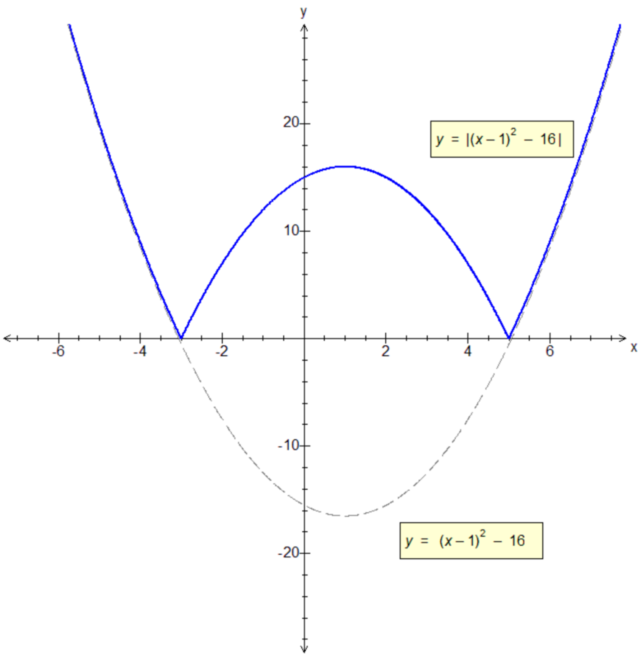
The function $y = (x-1)^2 - 16$ is a quadratic with roots of –3 and 5, and a minimum point of (1, -16). It is therefore below the x–axis for $-3 < x < 5$.

When we take the modulus of this function, all positive values of y remain unchanged, but negative values are multiplied by –1.

The two graphs show this clearly.

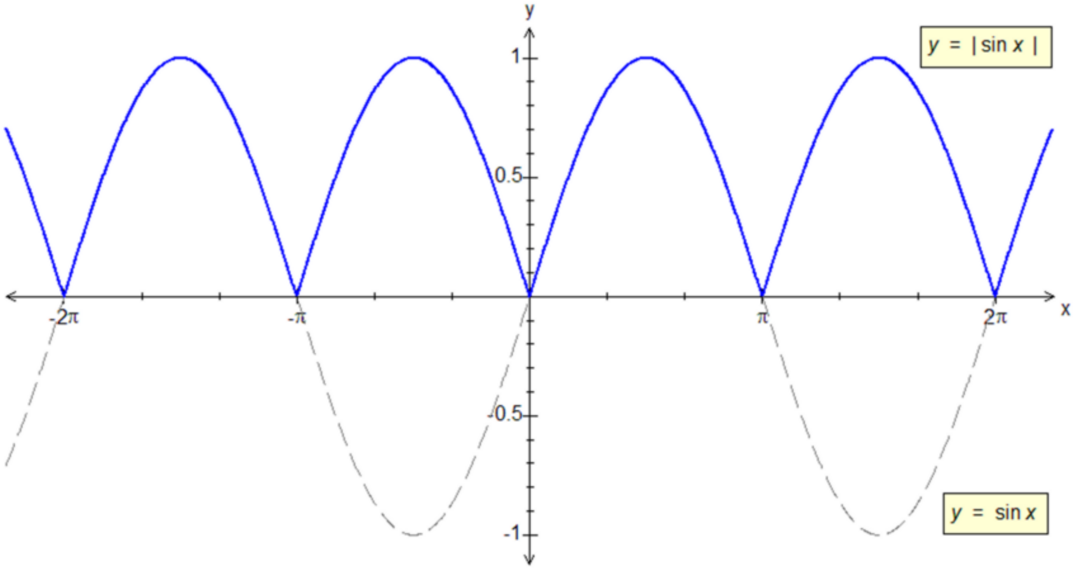
Where the original function $(x-1)^2 - 16$ takes a positive value, the graphs of the two functions coincide.

However, where the original function , $(x-1)^2 - 16$, takes a negative value, then the corresponding part of the graph of $| (x-1)^2 - 16 |$ is a reflection of the original in the x-axis.

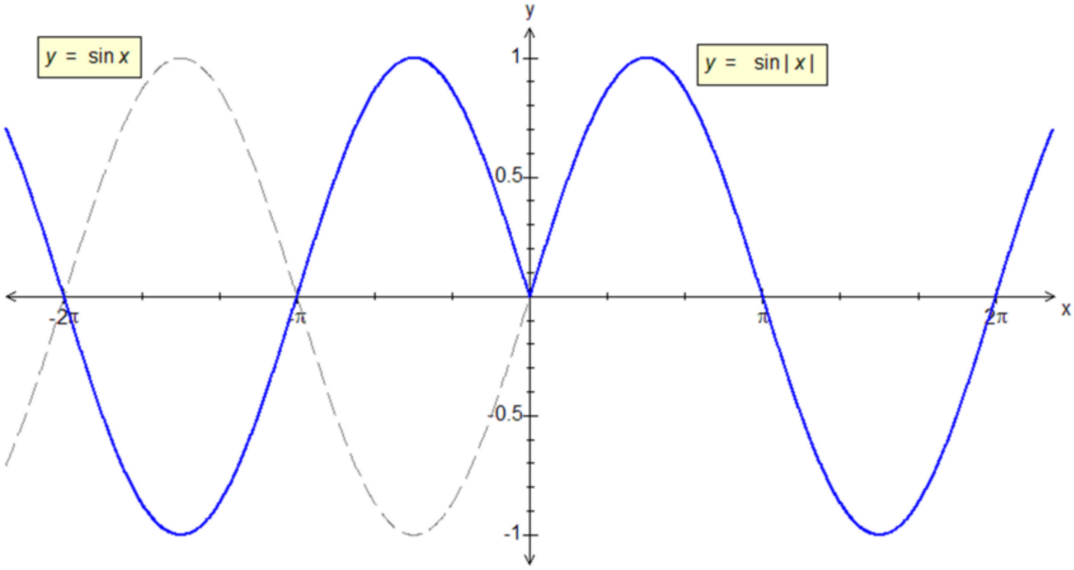


The graph is accurate here, but examination questions would generally only require a sketch with the key points clearly shown. In this case they would be (-3, 0), (5, 0), (1, 16) and (1, -16).

Example (2): Sketch the graph of $y = \sin x°$ for $-2\pi \le x \le 2\pi$, and from it sketch the graphs of $y = |\sin x|$ and $y = \sin(|x|)$. What can you say about the two resulting graphs ?



The graph of $y = |\sin x|$ coincides with that of $y = \sin x$ whenever $\sin x$ is positive, but is a reflection of $y = \sin x$ in the x-axis whenever $\sin x$ is negative.



The graph of $y = \sin |x|$ coincides with that of $y = \sin x$ whenever x is positive, but is a reflection of $y = \sin x$ in the y-axis whenever x is negative.

It can be seen that the the graphs of $y = |\sin x|$ and $y = \sin(|x|)$ are different.

This holds true for most functions, i.e $|f(x)| \neq f(|x|)$.

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Example (3):

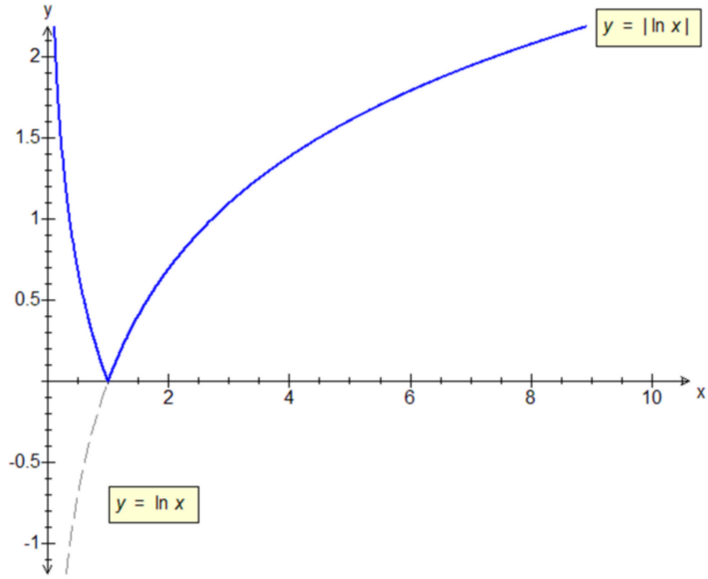
i) Sketch the graph $y = | \ln x |$.
ii) State the domain and range of $f(x) = \ln | x |$.

i) Recall the characteristics of the graph of $y = \ln x$.

The x – intercept is at (1, 0), there is an asymptote at $x = 0$ and the function is undefined for $x \le 0$.

The graph of $y = | \ln x |$ coincides with that of $y = \ln x$ for $x \ge 1$, but is a reflection of $y = \ln x$ in the y-axis for $0 < x < 1$.

ii) The domain of $f(x) = \ln | x |$ consists of all the non-zero real numbers, and its range consists of the entire set of real numbers.



The function $\ln(|x|)$ and related functions such as $\log_{10}(|x|)$ have important applications in calculus, and can also be used as a 'workaround' to solve certain equations involving logarithms.

Example (4): Solve $\log(a + 10) = 2\log(|a - 10|)$.

This is almost the same as an example from an earlier section, but we are dealing with logarithms of the **modulus** of the number a - 10.

The base of the logarithm is immaterial here !

$\log(a + 10) = 2\log(|a - 10|) \Rightarrow \log(a + 10) = \log((a - 10)^2)$ using log laws. Note that there is no need to put a modulus around the squared term, since the square of any real number is positive.

Hence $a + 10 = (a - 10)^2$ (taking antilogs)

This rearranges into a standard quadratic:

$a^2 - 20a + 100 - (a + 10) = 0$

$\Rightarrow a^2 - 21a + 90 = 0$
$\Rightarrow (a - 15)(a - 6) = 0$
$\therefore a = 15$ or 6.

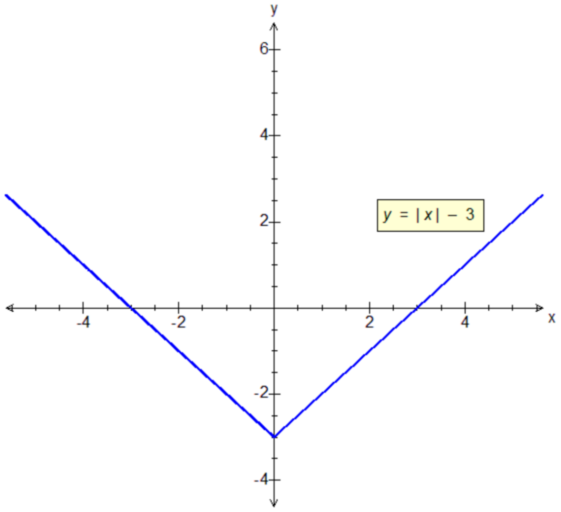
The equation has solutions of $a = 15$ and $a = 6$.

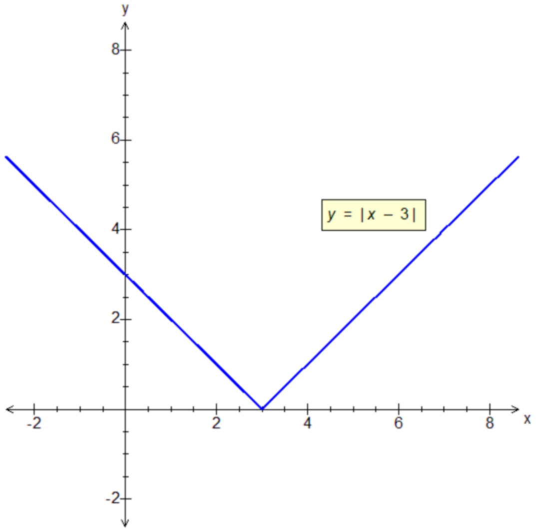
Substituting $a = 15$ into the original would give log 25 = 2 log 5.
With $a = 6$, we have log 16 = 2 log (|-4|) or log 16 = 2 log 4 , which is also allowable.

(Had the question been about solving $\log(a + 10) = 2\log(a - 10)$, without the modulus sign, $a = 6$ would not have been a solution, as the expression would have become log 16 = 2 log (-4), and there is no logarithm of a negative number.)

The graphs of $|x|$ and related functions can be transformed in the same way as those of other functions.

Example (5): Sketch (on separate diagrams), the graphs of $y = |x| - 3$, $y = |x - 3|$ and $y = |2x|$.





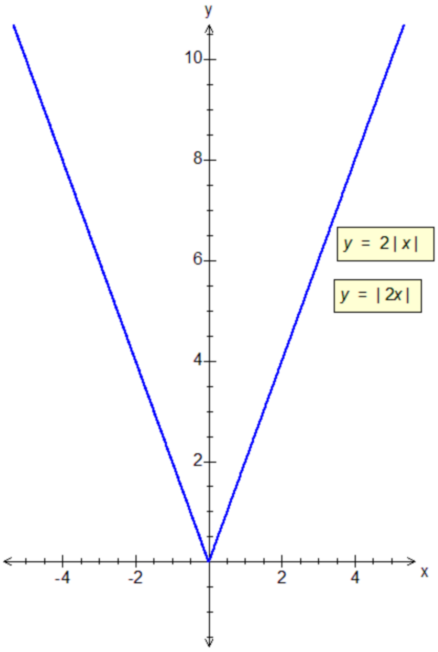
Here, the graph of $y = |x| - 3$ is that of $y = |x|$ translated by the vector $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$.

The graph of $y = |x - 3|$ is that of $y = |x|$ translated by the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

The two graphs are therefore quite different.

The graph of $y = |2x|$ is that of $y = |x|$, but stretched by a factor of ½ in the x-direction.

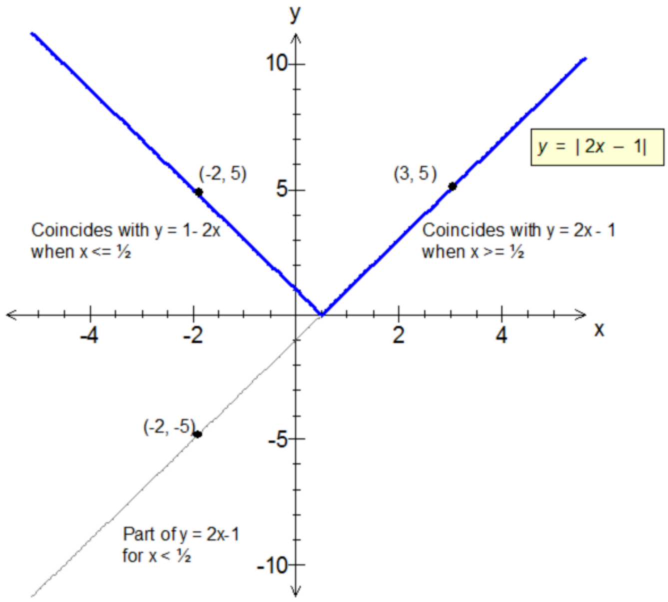
Note that the graph of $y = |2x|$ is the same as that of $y = 2|x|$, but this assumption is generally false for most functions.



Solving equations involving the modulus function.

Example (6): Find the solutions of the equation $|(2x-1)| = 5$.

Looking at the graphs, we can see that $|(2x-1)|$ coincides with $2x - 1$ whenever $2x - 1 \geq 0$, or $x \geq ½$.



When $2x - 1 \geq 0$, i.e. when $x < ½$,
the graph of $|(2x-1)|$ coincides with that of $2x - 1$ reflected in the x-axis.

Reflection in the x-axis is equivalent to multiplying the original function by a factor of –1, and so that part of the graph of $|(2x-1)|$ coincides with that of $-(2x-1)$, or $1 - 2x$.

There are thus two solutions of $|(2x-1)| = 5$.

The first is the 'obvious' one satisfying $2x - 1 = 5$, or $x = 3$.

The second is the one satisfying $1 - 2x = 5$, or $x = -2$.

Looking at the graphs, though, suggests another way of finding the second solution.
The point (-2, 5) on the graph of $|(2x-1)|$ corresponds to the point (-2, -5) on the graph of $2x - 1$.

Instead of multiplying the LHS by –1 to give $1 - 2x = 5$, we could multiply the RHS by –1 to give $2x - 1 = -5$, again leading to $x = -2$.

From this example, we can deduce that the solution(s) of the equation $|f(x)| = k$
can be found by solving two separate equations:

- The 'obvious' one(s) of $\boldsymbol{f(x) = k}$
- The 'alternative' one(s) of $\boldsymbol{-f(x) = k}$, which can in turn be rewritten as $\boldsymbol{f(x) = -k}$.

Example (7): Find the solutions of the equation $|(4x+3)| = 11$.

The first solution is the "$f(x) = k$" form, namely $4x + 3 = 11 \Rightarrow 4x = 8 \Rightarrow x = 2$.

The second solution can be found either by solving

$4x + 3 = -11 \Rightarrow 4x = -14 \Rightarrow x = 3½$, multiplying RHS by – 1 ("$f(x) = -k$" form),

or

$-(4x+3) = 11 \Rightarrow -4x - 3 = 11 \Rightarrow -4x = 14 \Rightarrow x = -3½$, multiplying LHS by –1 ("$-f(x) = k$" form).

The first method is easier to use.

Example (8): Find the solutions of the equation $|(x^2 - 2x - 7)| = 8$.

This is a quadratic, but the same method can be used as for linear examples.

The first solution set can be found by solving $x^2 - 2x - 7 = 8$ or $x^2 - 2x - 15 = 0$, which in turn factorises to $(x + 3)(x - 5) = 0$, giving solutions of $\mathbf{x = 5, x = -3}$.

The second solution set can be found by solving either or $x^2 - 2x - 7 = -8$ or $-(x^2 - 2x - 7) = 8$. Both methods give the same result.

$\mathbf{x^2 - 2x - 7 = -8}$

$x^2 - 2x - 7 = -8 \Rightarrow x^2 - 2x + 1 = 0$

$\Rightarrow (x - 1)^2 = 0$ (factorising), giving a solution of $\mathbf{x = 1}$.

$\mathbf{-(x^2 - 2x - 7) = 8}$

$-(x^2 - 2x - 7) = 8 \Rightarrow -x^2 + 2x + 7 = 8 \Rightarrow -x^2 + 2x - 1 = 0$

$\Rightarrow x^2 - 2x + 1 = 0$ (multiplying both sides by –1 to make x^2 term positive)

$\Rightarrow (x - 1)^2 = 0$ (factorising), giving a solution of $\mathbf{x = 1}$.

The first method is better, as the algebra works out much simpler.

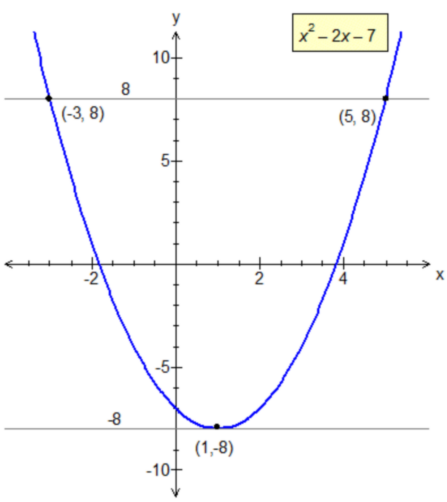
$\therefore$ **The solutions of $|(x^2 - 2x - 7)| = 8$ are $x = 1$, $x = 5$ and $x = -3$.**

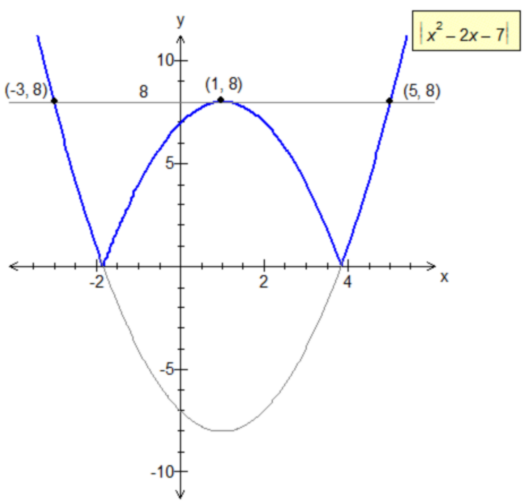
The solutions to the equation can be illustrated graphically.

The first method (left) shows the graph of the function $x^2 - 2x - 7$.
Its modulus is equal to 8 when its value is either 8 or –8.
The parabola meets the line $y = 8$ when $x = -3$ or $x = 5$, and meets the line $y = -8$ when $x = 1$.

The second method (right) shows the graph of the function $|(x^2 - 2x - 7)|$.
It is coincident with the graph of $x^2 - 2x - 7$ when $x^2 - 2x - 7 \geq 0$.
However, when $x^2 - 2x - 7 < 0$, the graph of $|(x^2 - 2x - 7)|$ coincides with the graph of $-(x^2 - 2x - 7)$.
(The negative part of the original graph of $x^2 - 2x - 7$ has been included for reference).

The graph of $|(x^2 - 2x - 7)|$ meets the line $y = 8$ when $x = -3$, 1 or 5.





Example (9): Find the solutions of the equation $|(x^2 – 5x - 1)| = 5$.

The first solution set can be found by solving $x^2 – 5x - 1 = 5 \Rightarrow x^2 – 5x - 6 = 0$, which in turn factorises to $(x + 1)(x – 6) = 0$, giving solutions of $x = 6$, $x = -1$.

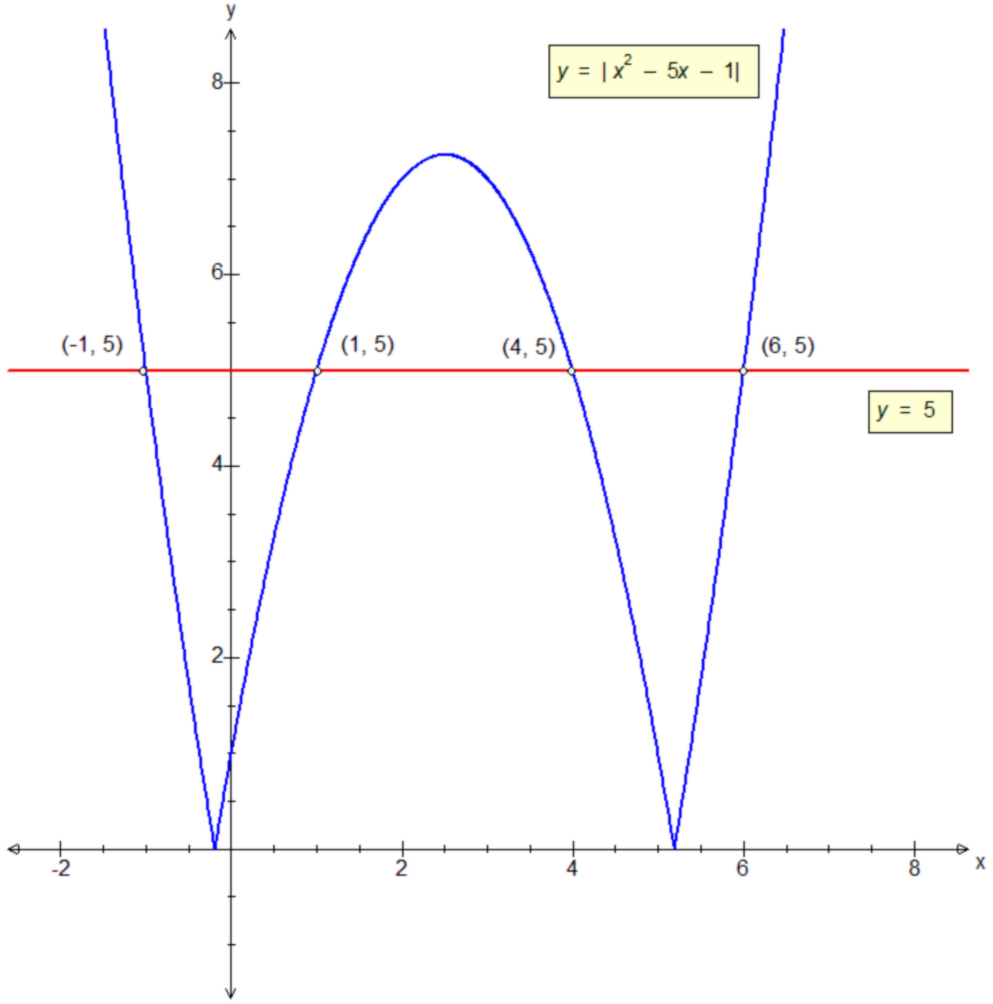
The second solution set can be found by solving $x^2 – 5x - 1 = -5$ (easier option).

$x^2 – 5x – 1 = -5$

$x^2 – 5x – 1 = -5 \Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x - 4)(x -1) = 0$ (factorising), giving solutions of $x = 4$, $x = 1$.

$\therefore$ **The solutions of $|(x^2 – 5x - 1)| = 5$ are $x = -1$, $x = 1$, $x = 4$ and $x = 6$.**



Example (10): Find the solutions of the equation $|(x+1)| = |2x|$.

This example is different, because we have a modulus function of x on both sides of the equation.

Nevertheless, we can still solve the equation in a similar way to those of the form $|f(x)| = k$.

The solution(s) of the equation $|f(x)| = |g(x)|$ can be found by solving two separate equations:

- The 'obvious' one(s) of $\boldsymbol{f(x) = g(x)}$
- The 'alternative' one(s) of $\boldsymbol{-f(x) = g(x)}$, which can in turn be rewritten as $\boldsymbol{f(x) = -g(x)}$.

The first solution is that of
$x + 1 = 2x \Rightarrow x = 1$.

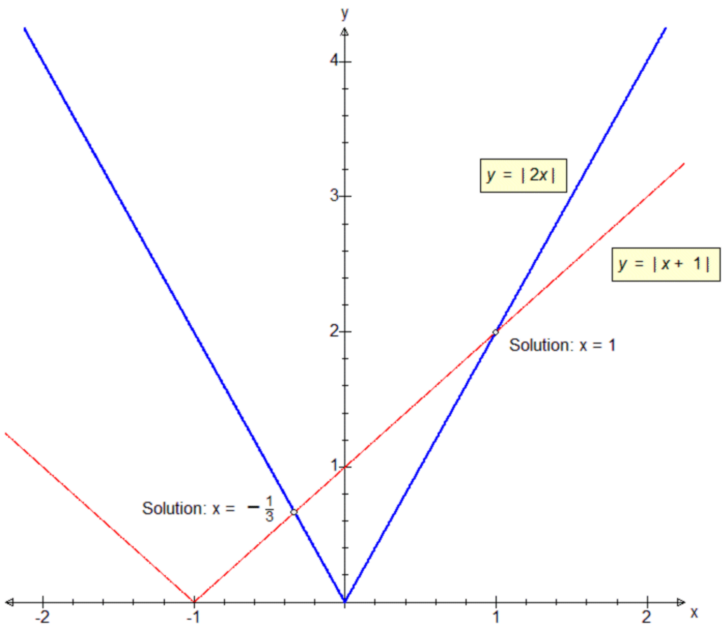
The second solution can be found either by solving

$\boldsymbol{-(x+1) = 2x}$ **;**
$\Rightarrow -x - 1 = 2x$
$\Rightarrow -1 = 3x \Rightarrow x = -\frac{1}{3}$

or

$\boldsymbol{x + 1 = -2x}$ **;**
$x + 1 = -2x$
$\Rightarrow 3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$.

(The second form is easier).



Another method would be to square both sides of the equation and solve as follows;

$|(x+1)| = |2x| \Rightarrow (x+1)^2 = (2x)^2$.

(Note that a squared quantity is always positive, so the modulus sign can be removed).
$(x+1)^2 = (2x)^2 \Rightarrow x^2 + 2x + 1 = 4x^2$.
$\Rightarrow 0 = 3x^2 - 2x - 1$.

Factorising the quadratic gives $3x^2 - 2x - 1 = 0 \Rightarrow (3x+1)(x-1) = 0$.
The roots, and thus the solutions of $|(x+1)| = |2x|$, are $x = 1$ and $x = -\frac{1}{3}$.

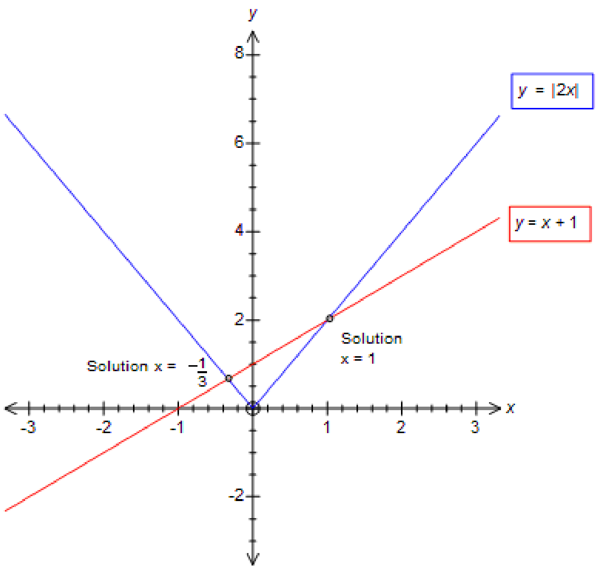
Care is required if we have a modulus function on one side of the equation, but a non-modulus function on the other, as the next two examples will show.

Example (10a): Find the solutions of the equation $(x + 1) = |2x|$.

This is very similar to example (10), but this time we have a non-modulus function of x on one side of the equation, and a modulus function on the other.

We will try the method of solving separate equations again:

- The 'obvious' one(s) of $\boldsymbol{f(x) = g(x)}$
- The 'alternative' one(s) of $\boldsymbol{-f(x) = g(x)}$, which can in turn be rewritten as $\boldsymbol{f(x) = -g(x)}$.



Again, the first solution is that of
$x + 1 = 2x \Rightarrow x = 1$.

The second solution can be found by solving

$\boldsymbol{x + 1 = -2x}$ **;**
$x + 1 = -2x$
$\Rightarrow 3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$.

There seems to be no difference between the solution to this example and that of Example 10.

Or we can square both sides;

$|(x + 1)| = |2x| \Rightarrow (x + 1)^2 = (2x)^2$.

(Note that a squared quantity is always positive, so the modulus sign can be removed).
$(x + 1)^2 = (2x)^2 \Rightarrow x^2 + 2x + 1 = 4x^2$
$\Rightarrow 0 = 3x^2 - 2x - 1$.

Factorising the quadratic gives $3x^2 - 2x - 1 = 0 \Rightarrow (3x + 1)(x - 1) = 0$.
The roots, and thus the solutions of $(x + 1) = |2x|$, are $x = 1$ and $x = -\frac{1}{3}$.

Check: $x = 1$; $x + 1 = 2$, and $|2x| = |2| = 2$.
Also: $x = -\frac{1}{3}$; $x + 1 = \frac{2}{3}$, and $|2x| = |-\frac{2}{3}| = \frac{2}{3}$.

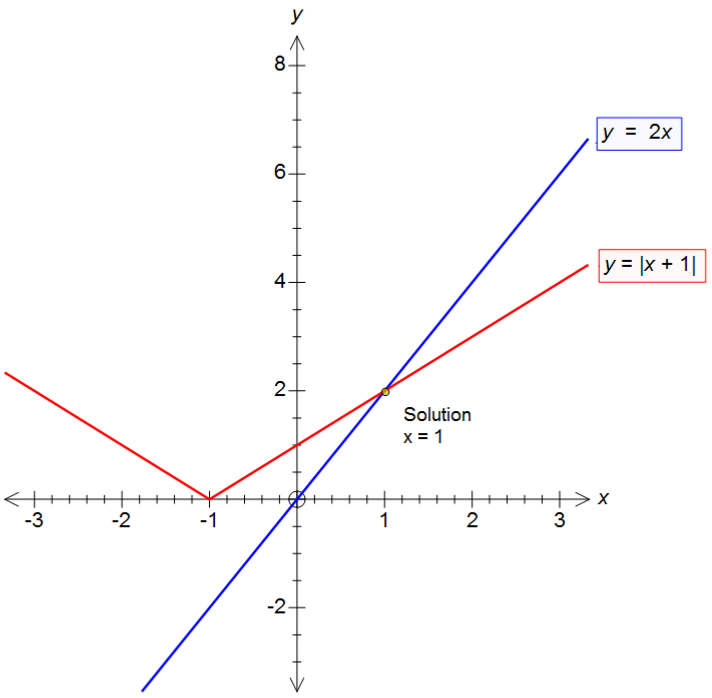
The next example is similar, but there is an important difference in the final result.

Example (10b): Find the solutions of the equation $|(x + 1)| = 2x$.

It might be thought that if we followed the same technique as we did in Example 10a, then the solutions of $|(x + 1)| = 2x$ would be $x = 1$ and $x = -\frac{1}{3}$.

Checking the results gives:
$x = 1$; $|(x + 1)| = 2$, and $2x = 2$.
$x = -\frac{1}{3}$; $|(x + 1)| = \frac{2}{3}$, and $2x = -\frac{2}{3}$.

The second 'solution' seems to be incorrect here – if we were to plot the graphs, they will only intersect at the one point (1, 2), giving $x = 1$ as the only solution.



Because the modulus function by definition is positive, then a 'solution' found using the earlier methods is only valid if substituting for x in the non-modulus function also gives a positive result. Hence the non-solution of $x = -\frac{1}{3}$; $|(x + 1)| = \frac{2}{3}$, and $2x = -\frac{2}{3}$.

In Example 10a, the non-modulus function of $(x + 1)$ returned a positive value for both values of x, so the two graphs met at two points, giving two solutions.

Inequalities involving the modulus function where one side is a number.

Inequalities involving the modulus function are solved in a similar way to the corresponding equations, although care is needed with sign reversals.

Example (11): Find the solutions of the inequality $|(4x + 3)| < 11$.

The first solution set is the 'obvious' one of $4x + 3 < 11 \Rightarrow 4x < 8 \Rightarrow x < 2$.

The second solution set can be found either multiplying the LHS by – 1 or the RHS by -1.

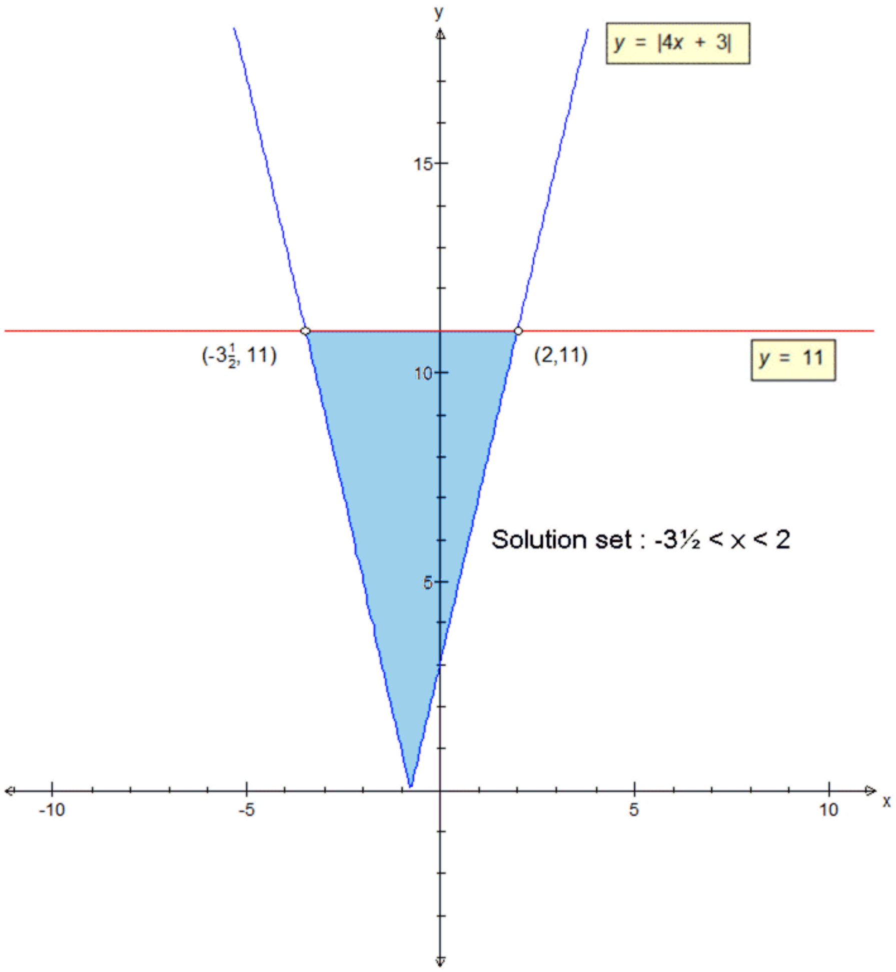
Multiplying LHS by -1:
$-(4x + 3) < 11 \Rightarrow -4x - 3 < 11 \Rightarrow -4x < 14 \Rightarrow x > -3\frac{1}{2}$. (We had to reverse the sign in the last step, when we divided by -4).

Multiplying RHS by -1 **plus an immediate inequality sign reversal**:

$4x + 3 > -11 \Rightarrow 4x > -14 \Rightarrow x > -3\frac{1}{2}$.

The two solution sets can be combined to give $\mathbf{-3\frac{1}{2} < x < 2}$.

Whenever the second solution set is found by reversing the *quantity* on the opposite side of the inequality sign, then the *direction* of the inequality sign must also be reversed.



Example (12): Find the solutions of the inequality $|(x^2 - 5x - 1)| \geq 5$. (This is a modification of Example (9)).

The first solution set can be found by solving $x^2 - 5x - 1 \geq 5 \Rightarrow x^2 - 5x - 6 \geq 0$, which in turn factorises to $(x + 1)(x - 6) \geq 0$, giving two solution sets of $x \geq 6$, $x \leq -1$.

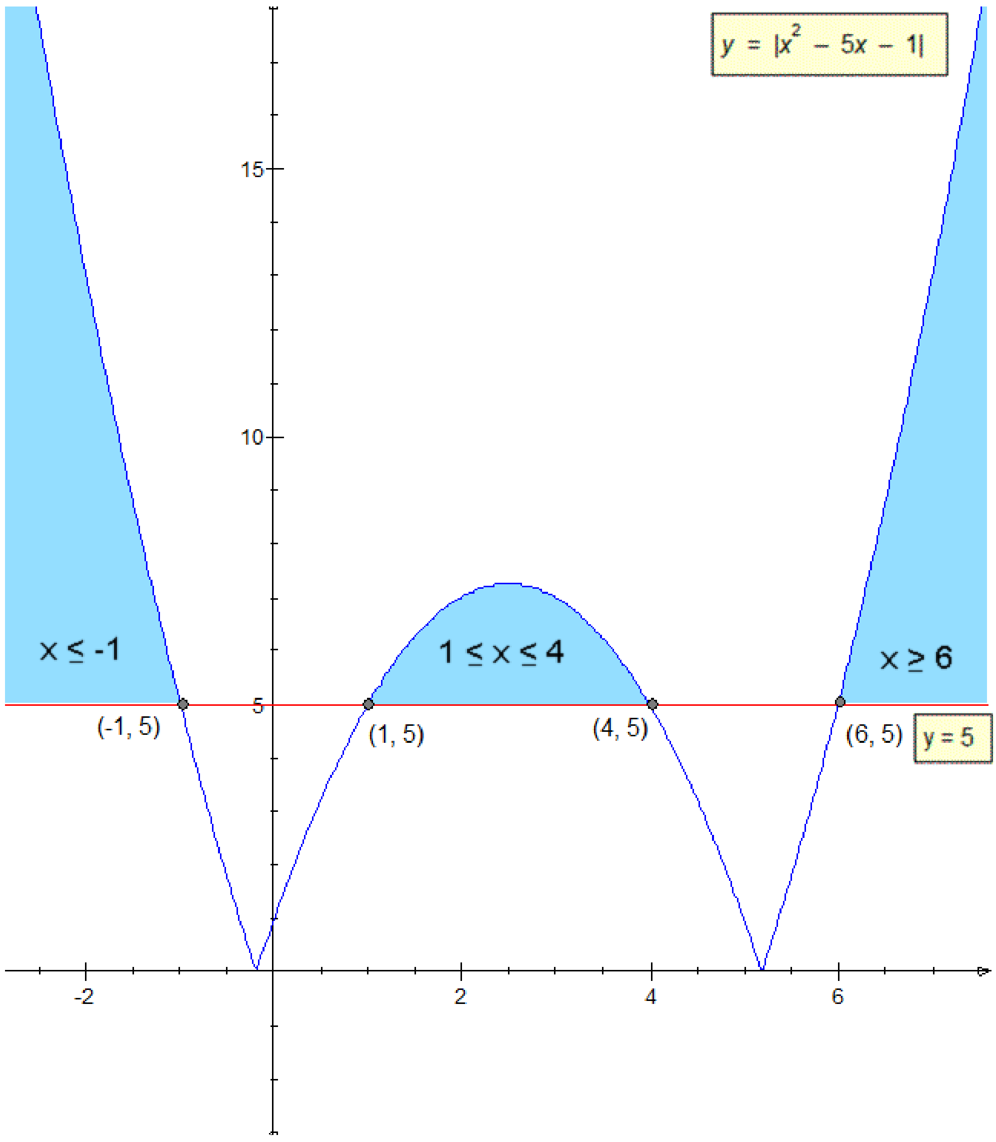
The second solution set can be found by solving $x^2 - 5x - 1 \leq -5$. Again, as we have reversed the sign of the quantity on the RHS, the inequality sign also had to be reversed.

$\mathbf{x^2 - 5x - 1 \leq -5}$

$x^2 - 5x - 1 \leq -5 \Rightarrow x^2 - 5x + 4 \leq 0$

$\Rightarrow (x - 4)(x - 1) \leq 0$ (factorising), giving the solution set of $1 \leq x \leq 4$.

$\therefore$ **The solution sets of $|(x^2 - 5x - 1)| \geq 5$ are $x \leq -1$, $1 \leq x \leq 4$ and $x \geq 6$.**



Inequalities involving the modulus function where there are modulus expressions on both sides.

Example (13): Find the solutions of the inequality $|(x + 1)| > |2x|$. (Modification of Example (10)).

This time we have an algebraic expression on both sides of the inequality. We can therefore either:

i) solve the corresponding equation, sketch the graphs of the two functions and find where the graph of $|(x + 1)|$ lies above the graph of $|2x|$, or

ii) square both sides, solve the related quadratic equation, plot its graph, and from there solve the inequality.

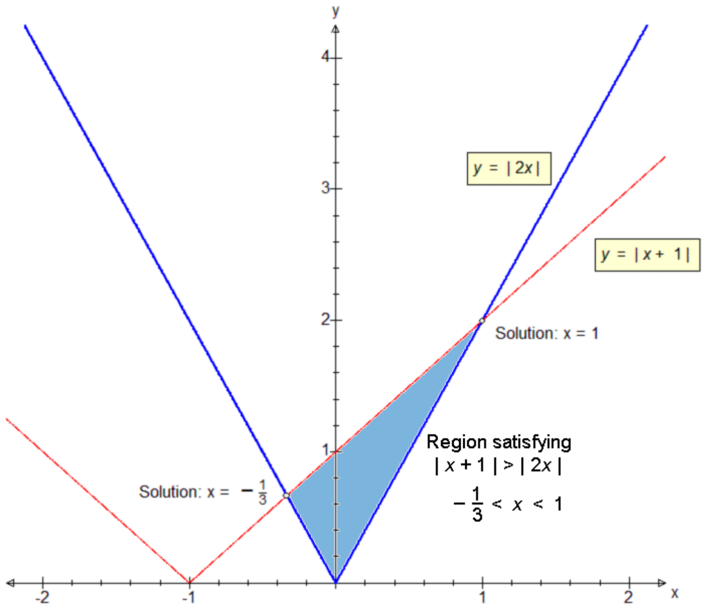
Method (i):

The solutions to the corresponding equation $| x + 1| = | 2x |$ are

$x = -\frac{1}{3}$ and $x = 1$.

The graph of $| x + 1|$ is above the graph of $| 2x |$ for the solution set of

$-\frac{1}{3} < x < 1$.



Method (ii) – squaring both sides

$|(x + 1)| > |2x| \Rightarrow (x + 1)^2 > (2x)^2$.

$(x + 1)^2 > (2x)^2 \Rightarrow x^2 + 2x + 1 > 4x^2 \Rightarrow$

$0 > 3x^2 - 2x - 1 \Rightarrow 3x^2 - 2x - 1 < 0.$

Factorising the corresponding quadratic gives $3x^2 - 2x - 1 = 0 \Rightarrow (3x + 1)(x – 1) = 0$.
The roots, and thus the solutions of $|(x + 1)| = |2x|$, are $x = 1$ and $x = -\frac{1}{3}$.

The sketch right shows the solution set of the inequality $3x^2 - 2x - 1 < 0$,

or $-\frac{1}{3} < x < 1$.