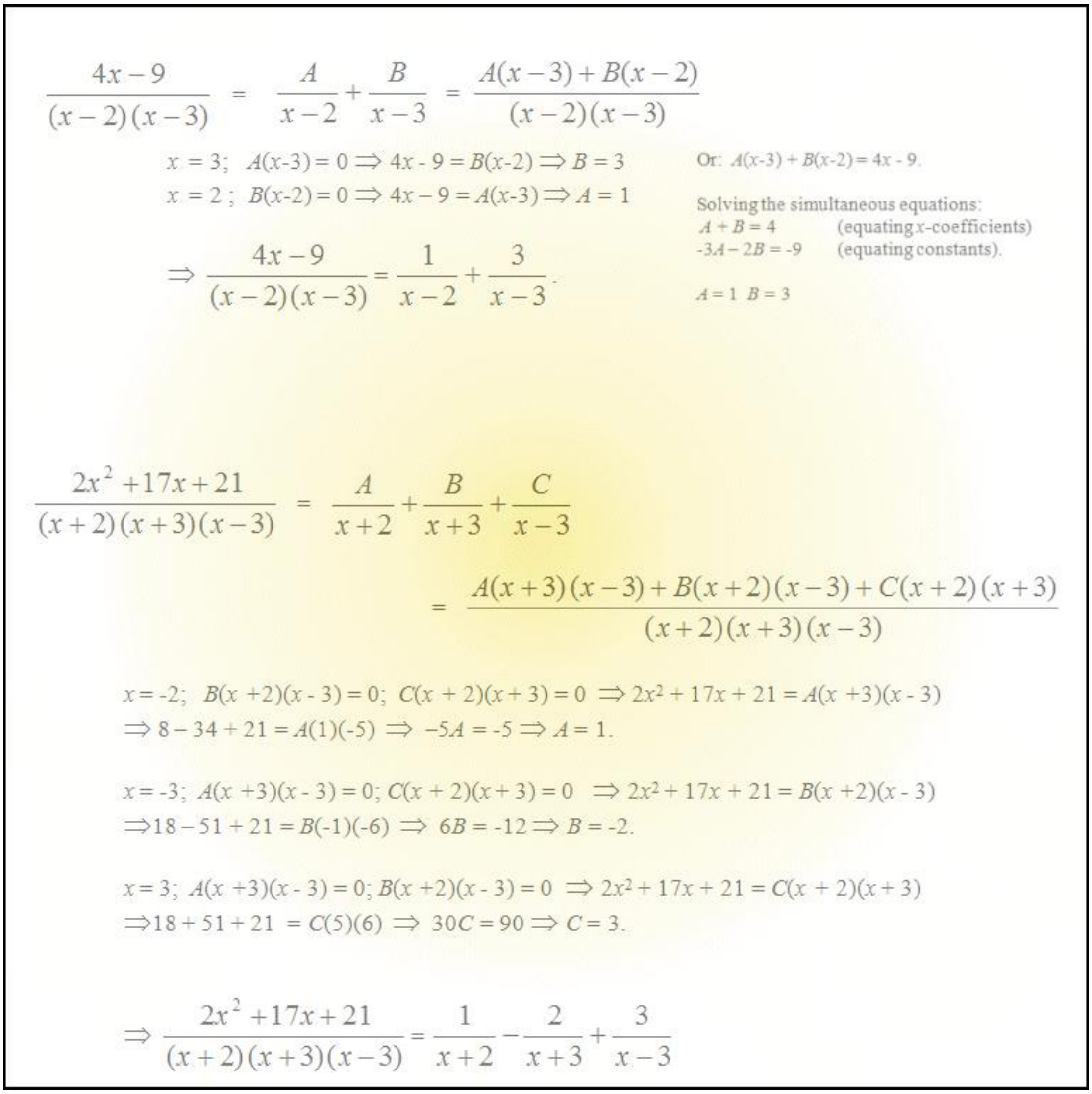
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

PARTIAL FRACTIONS



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Partial Fractions.

The section on rational expressions included methods of addition and subtraction of algebraic fractions. Thus an expression like $\frac{3}{x-2}-\frac{2}{x+1}$ can be converted to a single fraction as follows:

$\frac{3(x+1)}{(x-2)(x+1)}-\frac{2(x-2)}{(x-2)(x+1)}$, to $\frac{3x+3-2x+4}{(x-2)(x+1)}$ and finally $\frac{x+7}{(x-2)(x+1)}$.

Sometimes it might be useful to carry out the process in reverse, where we would convert an expression like $\frac{x+7}{(x-2)(x+1)}$ into the form $\frac{A}{x-2}+\frac{B}{x+1}$. This process is known as **decomposing the expression into partial fractions**.

Expanding $\frac{A}{x-2}+\frac{B}{x+1}$ gives $\frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$.

To find the values of A and B, we substitute convenient values for x to make each term on the top line equal to zero.

Let $x = -1$ (to make $A(x+1) = 0$) , then $x + 7 = B(x-2)$, or $6 = -3B \rightarrow B = -2$.
Let $x = 2$, (to make $B(x-2) = 0$), then $x + 7 = A(x+1)$, or $9 = 3A \rightarrow A = 3$.

Hence $\frac{x+7}{(x-2)(x+1)} = \frac{3}{x-2}-\frac{2}{x+1}$.

This method is also often illustrated by the **cover-up rule**. Here we cover up each term in the denominator in turn and substitute the value of x that makes the covered-up term equal to zero.

$\frac{x+7}{(.......)(x+1)}$ Covering up $(x – 2)$ and substituting $x = 2$ gives $A = \frac{9}{3}$ or 3.

$\frac{x+7}{(x-2)(.......)}$ Covering up $(x + 1)$ and substituting $x = -1$ gives $B = \frac{6}{-3}$ or -2.

It must be stressed that the substitution method and the cover-up rule can only be used exclusively when we have **distinct linear** factors in the denominator. In fact, when there are **repeated** or quadratic factors in the denominator, the cover-up method is 'dodgy' and should not be used (see later) !

Another method of finding A and B is to **equate the coefficients**.

Inspection of the numerators in the expanded expression $\frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$ gives $A(x+1) + B(x-2) = x + 7$.

From this, we can solve the simultaneous equations:
$A + B = 1$ (equating x-coefficients)
$A – 2B = 7$ (equating constants).

Subtracting the second equation from the first gives $3B = -6$ and thus $B = -2$.
Substituting in the first equation gives $A – 2 = 1$ and thus $A = 3$.

Most examination questions on partial fractions cover the following cases:

Two distinct linear factors in the denominator
Three distinct linear factors in the denominator
Three linear factors in the denominator, but one repeated

Two distinct linear factors in the denominator.
This case was covered in the previous example – here are two others.

Example (1): Express $\frac{4x-9}{(x-2)(x-3)}$ in partial fractions.

(Copyright OUP, *Understanding Pure Mathematics*, Sadler & Thorning, ISBN 9780199142590, Exercise 18A, Q.1)

The partial fraction will be in the form $\frac{A}{x-2}+\frac{B}{x-3}$.

(Long working)

Expand the partial fraction to $\frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$, and substitute values for x to make each term in the top line zero in turn.

When $x = 3$, the term in A becomes zero, and thus $4x - 9 = B(x-2)$, or $3 = B \Rightarrow B = 3$.
When $x = 2$, the term in B becomes zero, and thus $4x – 9 = A(x-3)$, or $-1 = -1(A) \Rightarrow A = 1$.

Hence $\frac{4x-9}{(x-2)(x-3)} = \frac{1}{x-2}+\frac{3}{x-3}$.

(Cover-up rule working)

$\frac{4x-9}{(.......)(x-3)}$ To find A, we cover up $(x – 2)$ and substitute $x = 2$ to give $A = \frac{-1}{-1}$ or 1.

$\frac{4x-9}{(x-2)(.......)}$ To find B, we cover up $(x - 3)$ and substitute $x = 3$ to give $B = \frac{3}{1}$ or 3.

(Method of equating coefficients)

The top line of the expanded expression $\frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$ is equivalent to $4x – 9$.

Therefore $A(x-3) + B(x-2) = 4x - 9$.

Solving the simultaneous equations:

$A + B = 4$ (equating x-coefficients)
$-3A – 2B = -9$ (equating constants).

Hence $\mathbf{2A + 2B = 8}$; $\mathbf{-3A – 2B = -9}$.

Adding the last two equations gives $-A = -1$ and thus $A = 1$.
Substituting in the first equation gives $1 + B = 4$ and thus $B = 3$.

Example (2): Express $\frac{4x+5}{x^2-x-12}$ in partial factions.

Firstly, we need to factorise the denominator to $(x-4)(x+3)$, giving a resulting partial fraction of the form $\frac{A}{x-4}+\frac{B}{x+3}$,

(Long working)

Expand the partial fraction to $\frac{A(x+3)+B(x-4)}{(x-4)(x+3)}$ and substitute values for x as in the last example.
Substitute values of 3 and 2 for x, so that the corresponding term in the top line equals zero.

When $x = -3$, the term in A becomes zero, and thus $4x + 5 = B(x-4)$, or $-7 = -7(B) \Rightarrow B = 1$.
When $x = 4$, the term in B becomes zero, and thus $4x + 5 = A(x+3)$, or $21 = 7(A) \Rightarrow A = 3$.

Hence $\frac{4x+5}{x^2-x-12} = \frac{3}{x-4}+\frac{1}{x+3}$.

(Cover-up rule working)

$\frac{4x+5}{(.......)(x+3)}$ To find A, we cover up $(x-4)$ and substitute $x = 4$ to give $A = \frac{21}{7}$ or 3.

$\frac{4x+5}{(x-4)(.......)}$ To find B, we cover up $(x+3)$ and substitute $x = -3$ to give $B = \frac{-7}{-7}$ or 1.

(Method of equating coefficients)

The top line of the expanded expression, $A(x+3) + B(x-4) = 4x + 5$.

Solving the simultaneous equations:

$A + B = 4$ (equating x-coefficients)
$3A - 4B = 5$ (equating constants).

Hence $\mathbf{4A + 4B = 16}$; $\mathbf{3A - 4B = 5}$.

Adding the last two equations gives $7A = 21$ and thus $A = 3$.
Substituting in the first equation gives $3 + B = 4$ and thus $B = 1$.

Three distinct linear factors in the denominator.

The general method of solving partial fractions of this type is the same as the case with two distinct linear factors, though there is more work involved.

Example (3): Express $\dfrac{2x^2+17x+21}{(x+2)(x+3)(x-3)}$ in partial fractions.

The partial fraction will now be of the form $\dfrac{A}{x+2}+\dfrac{B}{x+3}+\dfrac{C}{x-3}$.

(Long working)

The expanded partial fraction is now $\dfrac{A(x+3)(x-3)+B(x+2)(x-3)+C(x+2)(x+3)}{(x+2)(x+3)(x-3)}$.

We need to carry out three substitutions for *x* this time.

When $x = -2$, the terms in *B* and *C* both become zero, and thus $2x^2 + 17x + 21 = A(x+3)(x-3)$
$\Rightarrow 8 - 34 + 21 = A(1)(-5) \Rightarrow -5A = -5$ and hence $A = 1$.

When $x = -3$, the terms in *A* and *C* both become zero, and thus $2x^2 + 17x + 21 = B(x+2)(x-3)$
$\Rightarrow 18 - 51 + 21 = B(-1)(-6) \Rightarrow 6B = -12$ and hence $B = -2$.

When $x = 3$, the terms in *A* and *B* both become zero, and thus $2x^2 + 17x + 21 = C(x+2)(x+3)$
$\Rightarrow 18 + 51 + 21 = C(5)(6) \Rightarrow 30C = 90$ and hence $C = 3$.

Hence $\dfrac{2x^2+17x+21}{(x+2)(x+3)(x-3)} = \dfrac{1}{x+2}-\dfrac{2}{x+3}+\dfrac{3}{x-3}$

(Cover-up rule working)

$\dfrac{2x^2+17x+21}{(.......)(x+3)(x-3)}$ To find *A*, we cover up $(x + 2)$ and substitute $x = -2$ to give

$A = \dfrac{8-34+21}{(1)\times(-5)}$ or $\dfrac{-5}{-5}$ or 1.

$\dfrac{2x^2+17x+21}{(x+2)(.......)(x-3)}$ To find *B*, we cover up $(x + 3)$ and substitute $x = -3$ to give

$B = \dfrac{18-51+21}{(-1)\times(-6)}$ or $\dfrac{-12}{6}$ or -2.

$\dfrac{2x^2+17x+21}{(x+2)(x+3)(.......)}$ To find *C*, we cover up $(x - 3)$ and substitute $x = 3$ to give

$C = \dfrac{18+51+21}{5\times6}$ or $\dfrac{90}{30}$ or 3.

The method of equating coefficients can also be used when there are three linear factors in the denominator.

Example (3a): Use the method of equating coefficients to express $\frac{2x^2+17x+21}{(x+2)(x+3)(x-3)}$ in partial fractions (same as Example (3)). .

The partial fraction is of the form $\frac{A}{x+2}+\frac{B}{x+3}+\frac{C}{x-3}$,

and $\frac{A(x+3)(x-3)+B(x+2)(x-3)+C(x+2)(x+3)}{(x+2)(x+3)(x-3)}$ in expanded form .

We need to expand all the quadratic products in the numerator:

$A(x^2 - 9) + B(x^2 - x - 6) + C(x^2 + 5x + 6) = 2x^2 + 17x + 21.$

This gives us three simultaneous equations in three unknowns:

$A + B + C = 2$	(equating x^2 coefficients)	(1)
$5C - B = 17$	(equating x coefficients)	(2)
$6C - 6B - 9A = 21$	(equating constants)	(3)

We can substitute $B = 5C - 17$ into equations (1) and (3) to obtain

$A + 6C = 19$	(1)
$5C - B = 17$	(2)
$-24C - 9A = -81$	(3)

$4A + 24C = 76$	$4 \times$ (1)
$-24C - 9A = -81$	(3)

Adding the two equations gives $-5A = -5$ and thus $A = 1$.
Substituting back into the original equation (1) gives $C = 3$.
Finally, substituting into the original equation (2) gives $B = -2$.

Hence $\frac{2x^2+17x+21}{(x+2)(x+3)(x-3)} = \frac{1}{x+2}-\frac{2}{x+3}+\frac{3}{x-3}$

When we have three factors in the denominator, the method of equating coefficients can become increasingly difficult, but there are occasions when we do need to use it.

Example (4): Express $\frac{2x^2-5x-18}{x(x-2)(x+3)}$ in partial fractions.

The partial fraction will now be of the form $\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+3}$.

(Long working)

The expanded partial fraction is now $\frac{A(x-2)(x+3)+Bx(x+3)+Cx(x-2)}{x(x-2)(x+3)}$.

The appropriate substitutions for x are 0, 2 and –3.

When $x = 0$, only the term in A is non-zero, and thus $2x^2 - 5x - 18 = A(x - 2)(x + 3)$
$\Rightarrow -18 = A(-2)(3) \Rightarrow -6A = -18 \Rightarrow A = 3$.

When $x = 2$, only the term in B is non-zero, and thus $2x^2 - 5x - 18 = Bx(x + 3)$
$\Rightarrow 8 - 10 - 18 = B(2)(5) \Rightarrow 10B = -20 \Rightarrow B = -2$.

When $x = -3$, only the term in C is non-zero, and thus $2x^2 - 5x - 18 = Cx(x - 2)$
$\Rightarrow 18 + 15 - 18 = C(-3)(-5) \Rightarrow 15C = 15$ and hence $C = 1$.

Hence $\frac{2x^2-5x-18}{x(x-2)(x+3)} = \frac{3}{x}-\frac{2}{x-2}+\frac{1}{x+3}$.

(Cover-up rule working)

$\frac{2x^2-5x-18}{(......)(x-2)(x+3)}$ To find A, cover up x and substitute $x = 0$ to give

$A = \frac{-18}{(-2)\times 3}$ or $\frac{-18}{-6}$ or 3.

$\frac{2x^2-5x-18}{x(......)(x+3)}$ To find B, cover up $(x - 2)$ and substitute $x = 2$ to give

$B = \frac{8-10-18}{2\times 5}$ or $\frac{-20}{10}$ or -2.

$\frac{2x^2-5x-18}{x(x-2)(......)}$ To find A, cover up $(x + 3)$ and substitute $x = -3$ to give

$C = \frac{18+15-18}{(-3)\times(-5)}$ or $\frac{15}{15}$ or 1.

Example (4a): Express $\frac{2x^2 - 5x - 18}{x(x-2)(x+3)}$ in partial fractions, using the method of equating coefficients.

The partial fraction will take the form $\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}$.

Expanding gives $\frac{A(x-2)(x+3) + Bx(x+3) + Cx(x-2)}{x(x-2)(x+3)}$, giving a numerator of

$A(x^2 - x - 6) + B(x^2 + 3x) + C(x^2 - 2x) = 2x^2 - 5x - 18$.

The resulting simultaneous equations are as follows:

$A + B + C = 2$	(equating x^2 coefficients)	(1)
$-A + 3B - 2C = -5$	(equating x coefficients)	(2)
$-6A = -18$	(equating constants)	(3)

Note how equation (3) has turned out easy to solve – we can see at once that $A = 3$.

Substituting $A = 3$ into equations (1) and (2), we obtain

$B + C = -1$	(1)
$3B - 2C = -8$	(2)

$3B + 3C = -3$	$3 \times (1)$
$3B - 2C = -8$	(2)

Subtraction gives $5C = 5$, and thus $C = 1$; substitution in (2) gives $B = -2$.

Hence $\frac{2x^2 - 5x - 18}{x(x-2)(x+3)} = \frac{3}{x} - \frac{2}{x-2} + \frac{1}{x+3}$.

Example (5): Express $\dfrac{37x-81}{(x-3)(x+7)(2x-3)}$ in partial fractions.

The resulting partial fraction will be of the form $\dfrac{A}{x-3}+\dfrac{B}{x+7}+\dfrac{C}{2x-3}$.

(Long working)

The expanded partial fraction is $\dfrac{A(x+7)(2x-3)+B(x-3)(2x-3)+C(x-3)(x+7)}{(x-3)(x+7)(2x-3)}$.

The appropriate substitutions for x this time are 3, -7 and $\frac{3}{2}$. The fractional substitution is a little more messy than the others !

When $x = 3$, only the term in A is non-zero, and thus $37x - 81 = A(x+7)(2x-3)$
$\Rightarrow 111 - 81 = A(10)(3) \Rightarrow 30A = 30 \Rightarrow A = 1$.

When $x = -7$, only the term in B is non-zero, and thus $37x - 81 = B(x-3)(2x-3)$
$\Rightarrow -259 - 81 = B(-10)(-17) \Rightarrow 170B = -340 \Rightarrow B = -2$.

When $x = \frac{3}{2}$, only the term in C is non-zero, and thus $37x - 81 = C(x-3)(x+7)$

$\Rightarrow \frac{111}{2} - 81 = C(\frac{-3}{2})(\frac{17}{2})$
$\Rightarrow 222 - 324 = C(-3)(17)$ (multiplying both sides by 4 to get rid of the fractions)

$\Rightarrow -51C = -102 \Rightarrow C = 2$.

Hence $\dfrac{37x-81}{(x-3)(x+7)(2x-3)} = \dfrac{1}{x-3}-\dfrac{2}{x+7}+\dfrac{2}{2x-3}$.

(Cover-up rule working)

$\dfrac{37x-81}{(......)(x+7)(2x-3)}$ To find A, cover up $(x - 3)$ and substitute $x = 3$ to give

$A = \dfrac{111-81}{10\times 3}$ or $\dfrac{30}{30}$ or 1.

$\dfrac{37x-81}{(x-3)(......)(2x-3)}$ To find B, cover up $(x + 7)$ and substitute $x = -7$ to give

$B = \dfrac{(-259)-81}{(-10)\times(-17)}$ or $\dfrac{-340}{170}$ or -2.

$\dfrac{37x-81}{(x-3)(x+7)(......)}$ To find C, cover up $(2x - 3)$ and substitute $x = \frac{3}{2}$ (a bit messy!) to give

$A = \dfrac{\left(\frac{111}{2}\right)-81}{\left(-\frac{3}{2}\right)\times\left(\frac{17}{2}\right)}$ or $\dfrac{222-324}{(-3)\times 17}$ or $\dfrac{-102}{-51}$ or 2.

(Again we have multiplied top and bottom by 4 to get rid of the awkward fractions.)

Example (5a): Express $\frac{37x-81}{(x-3)(x+7)(2x-3)}$ in partial fractions by equating coefficients.

The partial fraction will take the form $\frac{A}{x-3}+\frac{B}{x+7}+\frac{C}{2x-3}$, expanding to

$\frac{A(x+7)(2x-3)+B(x-3)(2x-3)+C(x-3)(x+7)}{(x-3)(x+7)(2x-3)}$, and giving a numerator of

$A(2x^2 + 11x - 21) + B(2x^2 - 9x + 9) + C(x^2 + 4x - 21) = 37x - 81$.

The following simultaneous equations result:

$2A + 2B + C = 0$	(equating x^2 coefficients)	(1)
$11A - 9B + 4C = 37$	(equating x coefficients)	(2)
$9B - 21A - 21C = -81$	(equating constants)	(3)

We can rewrite equation (1) by expressing C as $-2(A + B)$, and then substituting in the other two equations.

$11A - 9B - 8(A + B) = 37 \Rightarrow$	$3A - 17B = 37$	(2)
$9B - 21A + 42(A + B) = -81 \Rightarrow$	$21A + 51B = -81$	(3)

$9A - 51B = 111$	$3 \times (2)$
$21A + 51B = -81$	(3)

Adding the above gives $30A = 30$ and hence $A = 1$

Substituting in (2) gives $3 - 17B = 37 \Rightarrow -17B = 34$ and $B = -2$.

Substituting in (1) gives $2 - 4 + C = 0 \Rightarrow C = 2$.

Hence $\frac{37x-81}{(x-3)(x+7)(2x-3)} = \frac{1}{x-3} - \frac{2}{x+7} + \frac{2}{2x-3}$.

Three linear factors in the denominator, but with one repeated pair.

The method here is again slightly different: here we have to use a combination of substitution and equating coefficients.

Example(6) : Express $\dfrac{3x^2+2}{x(x-1)^2}$ in partial fractions.

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Note that the factor (x – 1) is repeated here. The required partial fraction will be of the form

$\dfrac{A}{x}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$**.** Both ($x$ – 1) and its square are included in the denominators.

The cover-up rule should not be used here, because it may lead the student to think that the solution is of the form

$\dfrac{A}{x}+\dfrac{B}{(x-1)^2}$, thus missing out the term with (x - 1) in the denominator.

(Substitution working)

The expanded partial fraction is now $\dfrac{A(x-1)^2+B(x)(x-1)+C(x)}{x(x-1)^2}$.

We have used the L.C.M. of the denominators here, namely $x(x-1)^2$, and not $x(x-1)(x-1)^2$.

We first carry out two substitutions for x , i.e. 0 and 1.

When $x = 0$, the terms in B and C both become zero, and thus $3x^2 + 2 = A(x - 1)^2$
$\Rightarrow 0 + 2 = A(-1)^2 \Rightarrow A = 2$.

When $x = 1$, the terms in A and B both become zero, and thus $3x^2 + 2 = Cx$
$\Rightarrow 3 + 2 = C(1) \Rightarrow C = 5$.

Note that we have not been able to find B at this stage.

Final step – finding *B* by equating coefficients .

The best option is to equate the coefficients of x^2 , since the term in B has no constant, and equating x-terms looks more messy.

Looking at the expanded form $\dfrac{A(x-1)^2+B(x)(x-1)+C(x)}{x(x-1)^2}$, we can equate the coefficients of x^2 to give $A + B = 3$, and since $A = 2$, $B = 1$.

Hence $\dfrac{3x^2+2}{x(x-1)^2} = \dfrac{2}{x}+\dfrac{1}{x-1}+\dfrac{5}{(x-1)^2}$.

Example(6a) : Express $\frac{3x^2+2}{x(x-1)^2}$ in partial fractions using the method of equating coefficients throughout.

Note that the factor (x – 1) is repeated here. The required partial fraction will be of the form

$\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$. Both ($x$ – 1) and its square are included in the denominators.

The expanded partial fraction is $\frac{A(x-1)^2+B(x)(x-1)+C(x)}{x(x-1)^2}$.

We have used the L.C.M. of the denominators here, namely $x(x-1)^2$, and not $x(x-1)(x-1)^2$.

Thus, $A(x^2 - 2x + 1) + B(x^2 - x) + Cx = 3x^2 + 2$.

The following simultaneous equations result:

$A + B = 3$	(equating x^2 coefficients)	(1)
$-2A - B + C = 0$	(equating x coefficients)	(2)
$A = 2$	(equating constants)	(3)

From the above, we can deduce B = 1 and (-4 – 1 + C) = 0 and thus C = 5.

Hence $\frac{3x^2+2}{x(x-1)^2} = \frac{2}{x}+\frac{1}{x-1}+\frac{5}{(x-1)^2}$.

Example(7) : Express $\frac{4x+5}{(x-1)(x+2)^2}$ in partial fractions.

The required partial fraction will be of the form

$\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$**.** Both ($x$ + 2) and its square are included in the denominators.

(Substitution working)

Expanding the partial fraction we have $\frac{A(x+2)^2+B(x-1)(x+2)+C(x-1)}{(x-1)(x+2)^2}$.

Substitute values of 1 and -2 for x here.

When x =1 , the terms in B and C both become zero, and thus $4x + 5 = A(x + 2)^2$
$\Rightarrow 4 + 5 = A(3^2) \Rightarrow 9A = 9 \Rightarrow A = 1$.

When x = -2, the terms in A and B both become zero, and thus $4x + 5 = C(x-1)$
$\Rightarrow -8 + 5 = (-2-1)C \Rightarrow -3 = -3C \Rightarrow C = 1$.

Final step – finding *B* by equating the coefficients of x^2.

We still need to find B.

Looking at the expanded form $\frac{A(x+2)^2+B(x-1)(x+2)+C(x-1)}{(x-1)(x+2)^2}$, we can equate the

coefficients of x^2 to give $A + B = 0$, and since $A = 1$, $B = -1$. (The numerator of the top line, $4x + 5$, has no term in x^2 and thus its coefficient is 0.)

Therefore $\frac{4x+5}{(x-1)(x+2)^2} = \frac{1}{x-1}-\frac{1}{x+2}+\frac{1}{(x+2)^2}$

Example(7a) : Express $\frac{4x+5}{(x-1)(x+2)^2}$ in partial fractions, using the method of equating coefficients throughout.

The required partial fraction will be of the form

$\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$**.** Both ($x$ + 2) and its square are included in the denominators.

Expanding the partial fraction we have $\frac{A(x+2)^2+B(x-1)(x+2)+C(x-1)}{(x-1)(x+2)^2}$.

Thus, $A(x^2 + 4x + 4) + B(x^2 + x - 2) + C(x-1) = 4x + 5$.

The following simultaneous equations result:

$A + B = 0$	(equating x^2 coefficients)	(1)
$4A + B + C = 4$	(equating x coefficients)	(2)
$4A - 2B - C = 5$	(equating constants)	(3)

From Equation (1) we can substitute –A for B in Equations 2 and 3.

$3A + C = 4$	(equating x coefficients)	(2)
$6A - C = 5$	(equating constants)	(3)

Adding gives $9A = 9$ and hence $A = 1$ and $B = -1$.
Substituting in Eqn.2 gives $3 + C = 4$ and hence $C = 1$.

From the above, we can deduce $B = 1$ and $(-4 - 1 + C) = 0$ and thus $C = 5$.

Therefore $\frac{4x+5}{(x-1)(x+2)^2} = \frac{1}{x-1}-\frac{1}{x+2}+\frac{1}{(x+2)^2}$

Example(8) : Express $\dfrac{5x^2-6x-21}{(x-4)^2(2x-3)}$ in partial fractions.

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The required partial fraction will be of the form $\dfrac{A}{2x-3}+\dfrac{B}{x-4}+\dfrac{C}{(x-4)^2}$.

Both $(x-4)$ and its square are included in the denominators.

<u>(Substitution working)</u>

The expanded partial fraction is now $\dfrac{A(x-4)^2+B(2x-3)(x-4)+C(2x-3)}{(2x-3)(x-4)^2}$.

Again, there are messy fractions involved here – take care !
Substitute values of $\frac{3}{2}$ and 4 for x here.

When $x = \frac{3}{2}$, the terms in B and C both become zero, and thus $5x^2 - 6x - 21 = A(x - 4)^2$
$\Rightarrow \frac{45}{4} - 9 - 21 = A(\frac{3}{2} - 4)^2 \quad \Rightarrow (-\frac{75}{4}) = A(-\frac{5}{2})^2 \quad \Rightarrow (-\frac{75}{4}) = A(\frac{25}{4})$

$\Rightarrow A = -3$.

When $x = 4$, the terms in A and B both become zero, and thus $5x^2 - 6x - 21 = C(2x-3)$
$\Rightarrow 80 - 24 - 21 = (8-3)C \Rightarrow 35 = 5C \Rightarrow C = 7$.

Final step – finding B by equating the coefficients of x^2.

We still need to find B.
Looking at the expanded form $\dfrac{A(x-4)^2+B(2x-3)(x-4)+C(2x-3)}{(2x-3)(x-4)^2}$, we can equate the

coefficients of x^2 to give $A + 2B = 5$, and since $A = -3$, $B = 4$.

Hence $\dfrac{5x^2-6x-21}{(x-4)^2(2x-3)} = \dfrac{-3}{2x-3}+\dfrac{4}{x-4}+\dfrac{7}{(x-4)^2}$.

Example(8a) : Express $\frac{5x^2-6x-21}{(x-4)^2(2x-3)}$ in partial fractions, using the method of equating coefficients throughout.

The required partial fraction will be of the form $\frac{A}{2x-3}+\frac{B}{x-4}+\frac{C}{(x-4)^2}$.

The expanded partial fraction is now $\frac{A(x-4)^2+B(2x-3)(x-4)+C(2x-3)}{(2x-3)(x-4)^2}$.

Thus, $A(x^2 - 8x + 16) + B(2x^2 - 11x + 12) + C(2x-3) = 5x^2 - 6x - 21$.

The following simultaneous equations result:

$A + 2B = 5$ (equating x^2 coefficients) (1)
$-8A - 11B + 2C = -6$ (equating x coefficients) (2)
$16A + 12B - 3C = -21$ (equating constants) (3)

The equations are more messy than in the last one, but we can substitute $5 - 2B$ for A in Equations 2 and 3.

$-8(5 - 2B) - 11B + 2C = -6$
$\Rightarrow 0 + 16B - 11B + 2C = -6$
$\Rightarrow -40 + 5B + 2C = -6$
$\Rightarrow 5B + 2C = 34$ (2)

$16(5 - 2B) + 12B - 3C = -21$
$\Rightarrow 80 - 32B + 12B - 3C = -21$
$\Rightarrow 80 - 20B - 3C = -21$
$\Rightarrow -20B - 3C = -101$ (3)

Quadrupling Eqn. (2) gives

$20B + 8C = 136$
$-20B - 3C = -101$

Adding the two equations gives $5C = 35$ and hence $C = 7$.
Substituting in $20B + 8C = 136$ gives $20B = 80$ and hence $B = 4$.
Finally, recalling $A = 5 - 2B$ gives $A = -3$.

Hence $\frac{5x^2-6x-21}{(x-4)^2(2x-3)} = \frac{-3}{2x-3}+\frac{4}{x-4}+\frac{7}{(x-4)^2}$.

Improper Partial Fractions.

The rational expression to be reduced into partial fractions must have the numerator of lower degree than the denominator. If this is not the case, we have an **improper** expression corresponding to a top-heavy or improper fraction in arithmetic.

To convert an improper expression into the required form, we can either try factorising the numerator, or using long division.

Example (9). Express $\frac{4x^2+x-5}{x^2-x-12}$ in partial fractions. Express the result in the form

$$(Cx+D)\left(\frac{A}{x-4}+\frac{B}{x+3}\right).$$

This is an improper expression because the degree of the numerator is not less than that of the denominator.

Substituting x=1 in the numerator gives a value of 0, therefore (x - 1) is a factor of the numerator by the factor theorem.

The expression therefore factorises into $(x-1)\,\frac{(4x+5)}{(x-4)(x+3)}$, leaving the fractional part in the correct form. The full working is given here in Example (2), and the solution is

$$(x-1)\left(\frac{3}{x-4}+\frac{1}{x+3}\right).$$

Example (10). Express $\frac{3x^2-11x+9}{(x-2)(x-3)}$ in partial fractions

The numerator in the expression cannot be simplified by factorisation , so we need to use long division. We must also expand the denominator to give x^2- 5x+ 6.

			3	(Quotient)
x^2- 5x+ 6	$3x^2$	-11x	+9	
	$3x^2$	-15x	+18	
		4x	**-9**	(Remainder)

The quotient is $3+\frac{4x-9}{(x-2)(x-3)}$, again leaving the fractional part in the correct form. The working is also identical to that in Example (1).

Occasionally, the following cases turn up in exams, depending on the board and syllabus:

- One linear factor and one quadratic factor that cannot be factorised
- Two factors in the denominator, but repeated.
- Three linear factors in the denominator, all repeated.

The cover-up rule should not be used in any of those cases.

One linear factor and one quadratic factor that cannot be factorised

Example (11): Express $\dfrac{5x^2-2x-1}{(x+1)(x^2+1)}$ in partial fractions.

The quadratic cannot be factorised – in fact it does not even have real solutions.

The partial fraction form of the expression is therefore $\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+1}$.

<u>(Long working)</u>

The expanded form of the partial fraction is $\dfrac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)}$

When $x = -1$, the term in B and C becomes zero, and thus $5x^2 - 2x - 1 = A(x^2 + 1)$
$\Rightarrow 6 = 2A \Rightarrow A = 3$.

Having found A, we can substitute $x = 0$ to find C.
Thus $5x^2 - 2x - 1 = 3(0 + 1) + (0 + C)(0 + 1) \Rightarrow -1 = 3 + C \rightarrow C = -4$.

We find B by equating the terms in x^2 on each side of the expression :

$5x^2 = (3 + B)x^2 \Rightarrow 5 = 3 + B \Rightarrow B = 2$.

Hence $\dfrac{5x^2-2x-1}{(x+1)(x^2+1)} = \dfrac{3}{x+1}+\dfrac{2x-4}{x^2+1}$.

Example (11a): Express $\frac{5x^2-2x-1}{(x+1)(x^2+1)}$ in partial fractions, using the method of equating coefficients throughout.

The partial fraction form of the expression is $\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$.

The expanded form of the partial fraction is $\frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)}$

Thus, $A(x^2 + 1) + Bx(x+ 1) + C(x+1) = 5x^2 - 2x - 1$.

The following simultaneous equations result:

$A + B = 5$ (equating x^2 coefficients) (1)
$B + C = -2$ (equating x coefficients) (2)
$A + C = -1$ (equating constants) (3)

Substituting $B = 5-A$ in equation 2 gives $5-A+C = -2$ or $-A+C = -7$.

$-A + C = -7$ (equating x coefficients) (2)
$A + C = -1$ (equating constants) (3)

Hence $2C = -8$ and $C = -4$; from Eq. 3, $A = 3$ and from Eq.1, $B = 2$.

Hence $\frac{5x^2-2x-1}{(x+1)(x^2+1)} = \frac{3}{x+1}+\frac{2x-4}{x^2+1}$.

Example (12): Express $\frac{10x^2 - 11x - 15}{(x-1)(x^2-5)}$ in partial fractions.

The quadratic cannot be factorised , although it does have two real solutions of ±√5.

The partial fraction form of the expression is therefore $\frac{A}{x-1} + \frac{Bx+C}{x^2-5}$.

(Long working)

The expanded form of the partial fraction is $\frac{A(x^2-5)+(Bx+C)(x-1)}{(x-1)(x^2-5)}$

When $x = 1$, the term in $(Bx + C)(x - 1)$ becomes zero, and thus $10x^2 - 11x - 15 = A(x^2 - 5)$
$\Rightarrow -16 = -4A \Rightarrow A = 4$.

Again, the cover-up rule can only be used to find A.

$\frac{10x^2 - 11x - 15}{(......)(x^2-5)}$ Cover up $x - 1$ and substitute $x = 1$ to give $A = \frac{-16}{-4}$ or 4.

Having found A using either method, we can substitute $x = 0$ to find C.
Thus $10x^2 - 11x - 15 = 4(0 - 5) + (0 + C)(0 - 1) \Rightarrow -20 = -15 + C \Rightarrow C = -5$.

We find B by equating the terms in x^2 on each side of the expression :

$10x^2 = (4 + B)x^2 \Rightarrow 10 = 4 + B \Rightarrow B = 6$.

Hence $\frac{10x^2 - 11x - 15}{(x-1)(x^2-5)} = \frac{4}{x-1} + \frac{6x-5}{x^2-5}$.

Example (12a): Express $\frac{10x^2 - 11x - 15}{(x-1)(x^2-5)}$ in partial fractions, using the method of equating coefficients throughout.

The partial fraction form of the expression is therefore $\frac{A}{x-1} + \frac{Bx+C}{x^2-5}$.

The expanded form of the partial fraction is $\frac{A(x^2-5) + (Bx+C)(x-1)}{(x-1)(x^2-5)}$

Thus, $A(x^2 - 5) + Bx(x-1) + C(x-1) = 10x^2 - 11x - 15$.

The following simultaneous equations result:

$A + B = 10$	(equating x^2 coefficients)	(1)
$-B + C = -11$	(equating x coefficients)	(2)
$-5A - C = -15$	(equating constants)	(3)

Substituting $B = 10-A$ and hence $-B = A - 10$ in equation 2 gives $A-10+C = -11$ or $A+C = -1$.

$A + C = -1$	Eqn.(2)
$-5A - C = -15$	Eqn.(3)

$5A + 5C = -5$	5 × Eqn.(2)
$-5A - C = -15$	Eqn. (3)

Adding the equations above gives $4C = -20$ and $C = -5$. From Eqn.(2), $A-5 = -1$, so $A = 4$.
Finally, from Eqn.(1), $B = 6$.

Hence $\frac{10x^2 - 11x - 15}{(x-1)(x^2-5)} = \frac{4}{x-1} + \frac{6x-5}{x^2-5}$.

Two factors in the denominator, but repeated.

Here, the method of equating coefficients is the only feasible one.

Example (13): Express $\frac{4x+3}{(x-1)^2}$ in partial fractions.

The partial fraction form of the expression will be $\frac{A}{(x-1)}+\frac{B}{(x-1)^2}$ (compare Example (3)).

The expanded form is therefore

$$\frac{4x+3}{(x-1)^2} = \frac{A}{(x-1)}+\frac{B}{(x-1)^2}$$

Multiplying both sides by $(x - 1)^2$ gives $4x + 3 = A(x - 1) + B$.

When $x = 1$, $4 + 3 = B \Rightarrow B = 7$.

Equating the terms in x gives $A = 4$.

Hence $\frac{4x+3}{(x-1)^2} = \frac{4}{(x-1)}+\frac{7}{(x-1)^2}$.

Example (14): Express $\frac{2x-1}{(x-3)^2}$ in partial fractions.

Expanding, we have $\frac{2x-1}{(x-3)^2} = \frac{A}{(x-3)}+\frac{B}{(x-3)^2}$.

Multiplying both sides by $(x - 3)^2$ gives $2x - 1 = A(x - 3) + B$.

When $x = 3$, $6 - 1 = B \Rightarrow B = 5$.

Equating the terms in x gives $A = 2$.

Hence $\frac{2x-1}{(x-3)^2} = \frac{2}{(x-3)}+\frac{5}{(x-3)^2}$.

Three factors in the denominator, but repeated.

Here again, the method of equating coefficients is the only feasible one.

Example (15): Express $\frac{2x^2+x+3}{(x+1)^3}$ in partial fractions.

The partial fraction form of the expression will be $\frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$ (see Example (6)).

The expanded form is therefore

$$\frac{2x^2+x+3}{(x+1)^3}=\frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

Multiplying both sides by $(x + 1)^3$ gives $2x^2 + x + 3 = A(x+1)^2 + B(x+1) + C$.
$\Rightarrow A(x^2 + 2x + 1) + B(x + 1) + C$

When $x = -1$, $2 + (-1) + 3 = A(0) + B(0) + C \Rightarrow C = 4$.

A and B need to be found by equating coefficients.

$A(x+1)^2 + B(x+1) + 4 = A(x^2 + 2x + 1) + B(x+1) + 4$

$= Ax^2 + (B + 2A)x + (A + B + 4)$, (constant terms included for completeness only)

Equating the terms in x^2 gives $A = 2$.
Equating the terms in x gives $B + 2A = 1 \Rightarrow B = -3$.

Hence $\frac{2x^2+x+3}{(x+1)^3}=\frac{2}{(x+1)}-\frac{3}{(x+1)^2}+\frac{4}{(x+1)^3}$.

(Again, the cover-up rule is no help here !)

Example (16): Express $\frac{4x^2-17x+21}{(x-2)^3}$ in partial fractions.

$$\frac{4x^2-17x+21}{(x-2)^3}=\frac{A}{(x-2)}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3}$$

Multiplying both sides by $(x - 2)^2$ gives $4x^2 - 17x + 21 = A(x-2)^2 + B(x-2) + C$.

$A(x^2 - 4x + 4) + B(x - 2) + C$

When $x = 2$, $16 - 34 + 21 = A(0) + B(0) + C \Rightarrow C = 3$.

$A(x-2)^2 + B(x-2) + 3 = A(x^2 - 4x + 4) + B(x-2) + 3$.

Equating the terms in x^2 gives $A = 4$.
Equating the terms in x gives $B - 4A = -17 \Rightarrow B = -1$.

Hence $\frac{4x^2-17x+21}{(x-2)^3}=\frac{4}{(x-2)}-\frac{1}{(x-2)^2}+\frac{3}{(x-2)^3}$.