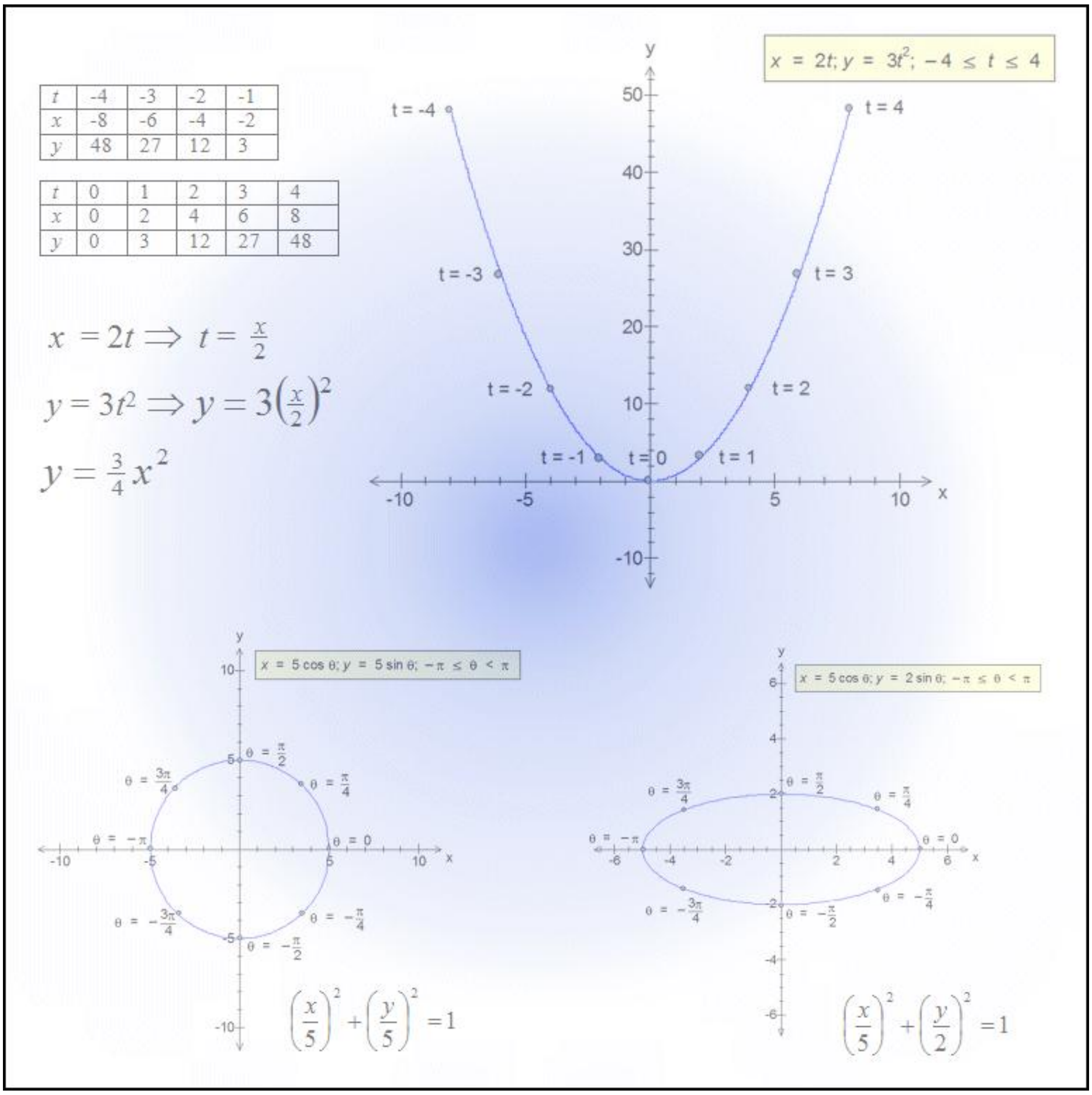
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

CARTESIAN AND PARAMETRIC FORM



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Cartesian Form.

A curve in Cartesian form is defined in terms of the two variables *x* and *y* only.

The quadratic equation $y = x^2 + 7x + 10$ is an example where *y* is given **explicitly** in terms of *x*.
By contrast, the equation $x^2 - 2xy + y^2 = 16$ is stated **implicitly**.

Parametric Form.

A curve is said to be in **parametric form** if the variables *x* and *y* are expressed in terms of a third variable. For example, *x* could be defined as *f*(*t*) and *y* as *g*(*t*). The variable *t* is the parameter in this case.

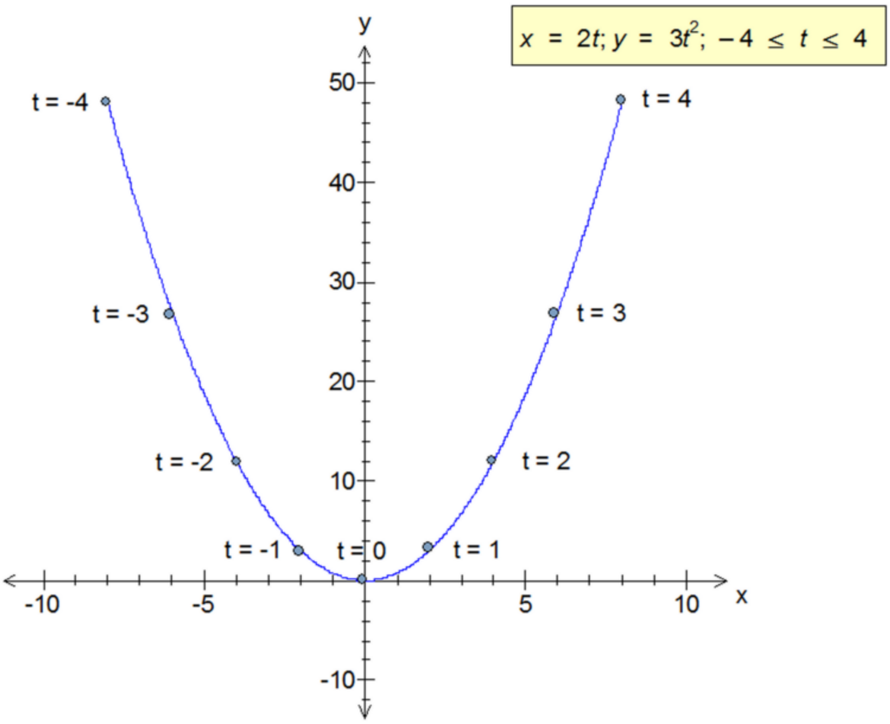
When plotting a parametric curve, it is necessary to work out *x* and *y* whilst varying the values for the parameter.

Example (1): Plot the curve $x = 2t$, $y = 3t^2$ for $-4 \le t \le 4$, and give the Cartesian equation of the curve.

t	-4	-3	-2	-1	0	1	2	3	4
x	-8	-6	-4	-2	0	2	4	6	8
y	48	27	12	3	0	3	12	27	48

The Cartesian equation for the curve can be obtained by eliminating *t*, the parameter.

If $x = 2t$, then $t = \frac{x}{2}$. Substituting into $y = 3t^2$ gives the Cartesian equation for the curve, which is a parabola with equation $y = \frac{3}{4}x^2$.



Example (2): A curve is defined by the parametric equations $x = t^2 - 8t + 6$, $y = t - 4$, for $t > 4$.

Express the equation of the curve in Cartesian form with y as the subject.

We complete the square with $x = t^2 - 8t + 6 \Rightarrow x = (t - 4)^2 - 10 \Rightarrow x = y^2 - 10$ (eliminating t)

$\Rightarrow y^2 = x + 10 \Rightarrow y = \sqrt{x+10}$.

Example (3): A curve is defined parametrically as $x = 4t + 3t^2$, $y = 4t^2 + 3t^3$.

Find its Cartesian equation in the form, $ax^3 + bxy + cy^2$ where a, b and c are integer constants.

We spot $\frac{y}{x} = \frac{4t^2 + 3t^3}{4t + 3t^2} \Rightarrow \frac{y}{x} = t$, and substituting for t in the (easier) equation for x

we have $x = \frac{4y}{x} + \frac{3y^2}{x^2}$, and after multiplying by x^2 on both sides, $x^3 = 4xy + 3y^2$

and finally $x^3 - 4xy - 3y^2 = 0$.

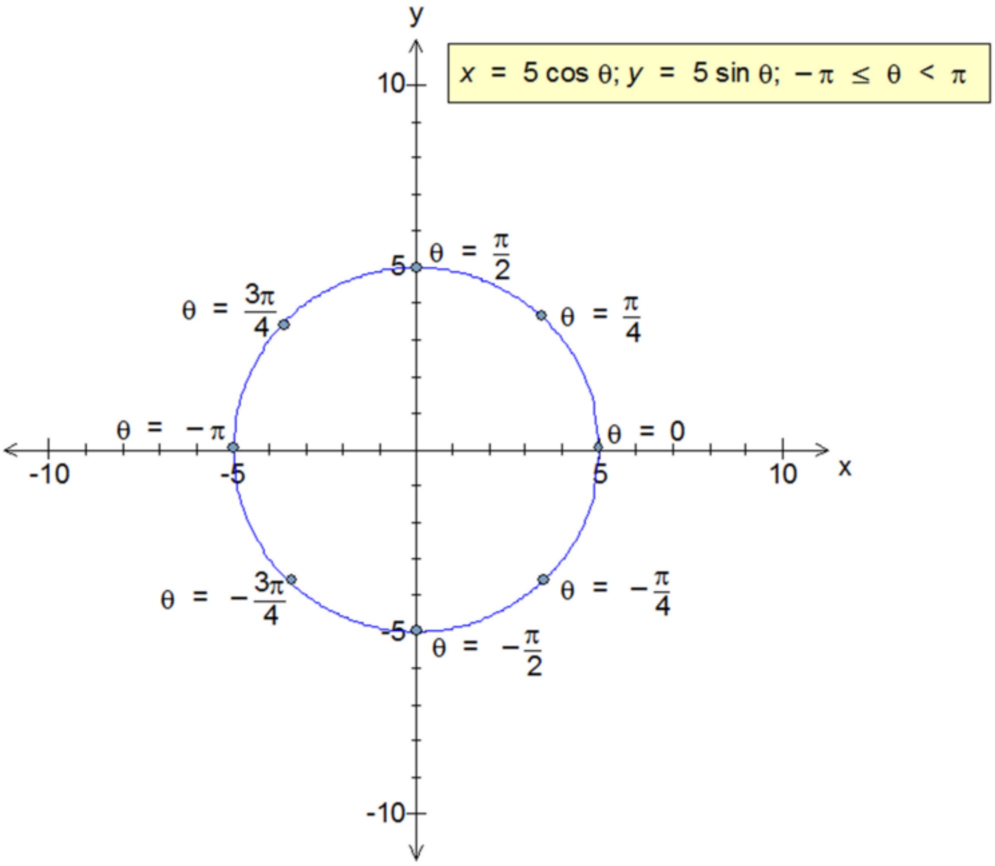
When the parameter is θ, trigonometric identities may be required.

Example (4): Plot the curve $x = 5 \cos \theta$, $y = 5 \sin \theta$ for $-\pi \le \theta < \pi$, and give the Cartesian equation of the curve.

(The approximation $5/\sqrt{2} = 3.54$ is used here.)

t	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
x	-5	-3.54	0	3.54	5	3.54	0	-3.54	-5
y	0	-3.54	-5	-3.54	0	3.54	5	3.54	0

Starting at $-\pi$, the values of θ are measured anticlockwise from the negative x –axis, here at (-5, 0).



The curve is a circle – to find the Cartesian equation we use the identity $\cos^2\theta + \sin^2\theta = 1$.

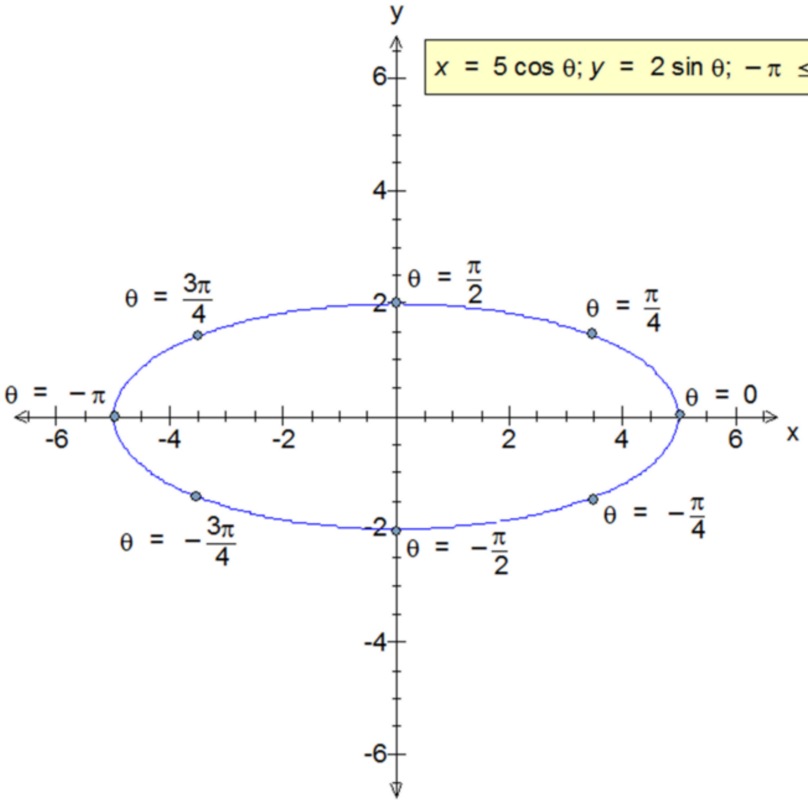
The Cartesian equation is therefore $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$ or $x^2 + y^2 = 25$.

The general Cartesian equation of the circle $x = r \cos \theta$, $y = r \sin \theta$ is thus $x^2 + y^2 = r^2$.

Example (5): Plot the curve $x = 5 \cos \theta$, $y = 2 \sin \theta$ for $-\pi \le \theta < \pi$, and give the Cartesian equation of the curve.

(The approximations $2/\sqrt{2} = 1.41$ and $5/\sqrt{2} = 3.54$ are used here.)

t	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
x	-5	-3.54	0	3.54	5	3.54	0	-3.54	-5
y	0	-1.41	-2	-1.41	0	1.41	2	1.41	0



This time, the curve is an ellipse, and again we use the identity $\cos^2\theta + \sin^2\theta = 1$ to find the Cartesian equation of the curve.

Here it is $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$.

The general Cartesian equation of the ellipse $x = a \cos \theta$, $y = b \sin \theta$ is thus $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

Example (6): The parametric equation of a curve is $x = \cos\theta$, $y = 1 + 2\cos 2\theta$, $0 \le \theta \le \pi$.

Show that it coincides with part of the curve $y = (2x + 1)(2x - 1)$, and state its domain and range.

Using the double angle identity , we have $\cos 2\theta = 2\cos^2\theta - 1$, hence $y = 1 + 2(2\cos^2\theta - 1)$.

Substituting $x = \cos\theta$ gives $y = 1 + 2(2x^2 - 1) \Rightarrow y = 4x^2 - 1$, factorising to $y = (2x + 1)(2x - 1)$..

Because $\cos\theta$ can only take values between -1 and 1 inclusive, the domain is $-1 \le x \le 1$.
The corresponding range is $-1 \le y \le 3$, as shown by the graph below.