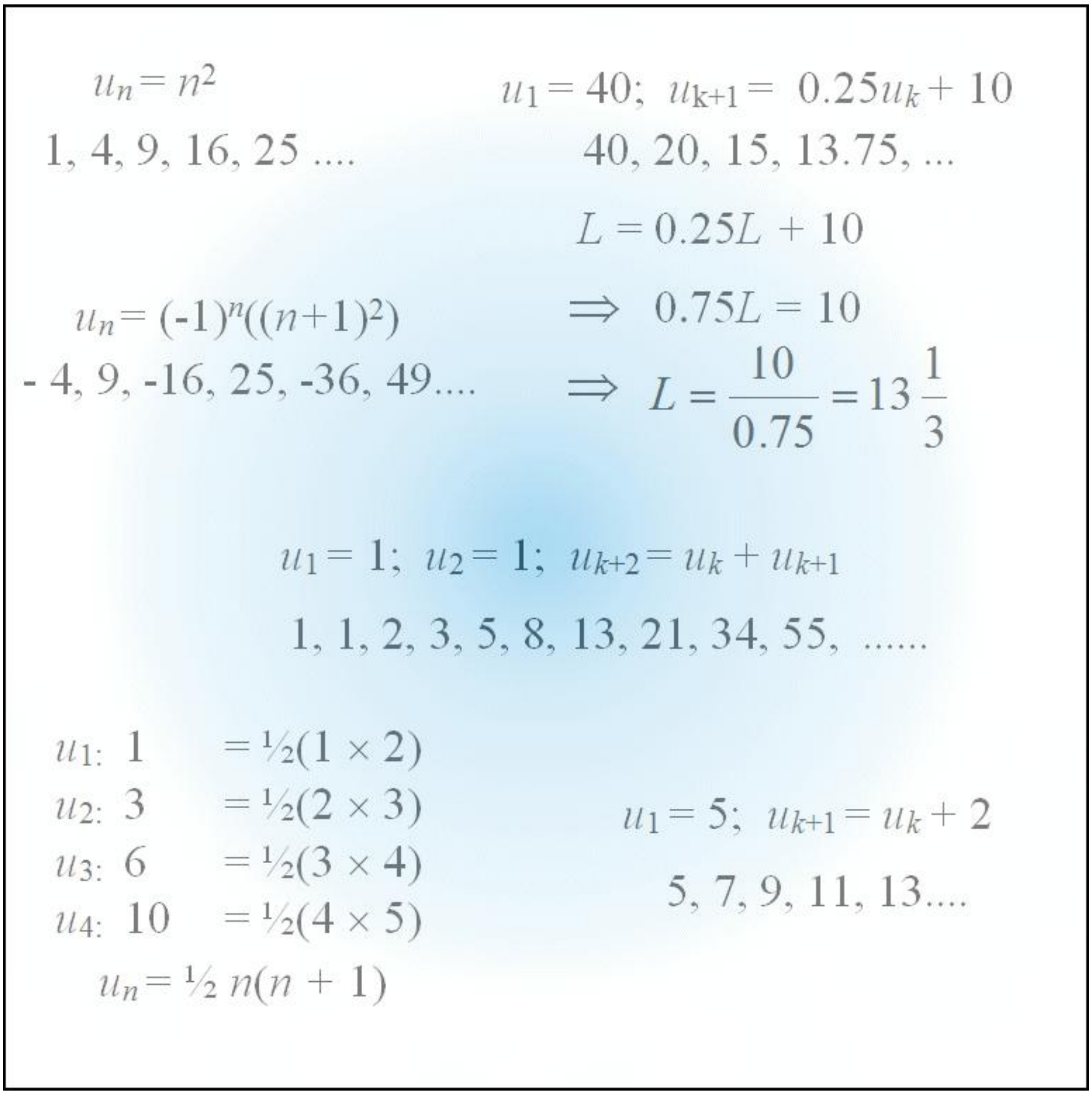
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

SEQUENCES



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Sequences.

A **sequence** is a list of numbers following some rule for finding succeeding values. Each number in a sequence is called a **term**. Terms are usually denoted by a letter u with a subscript. The first term of a sequence is u_1, the second u_2, and the n^{th} term is u_n.

A sequence can be defined by a formula for the n^{th} term.
Thus the definition $u_n = n^2$ generates the sequence of square numbers 1, 4, 9, 16, 25 .... by substituting n = 1, 2, 3, 4, 5 .... into the definition. The 100 th term, u_{100}, would be 100^2 or 10000.

Another way of defining a sequence is to use a **recursive** or **inductive** definition. This uses a starting value (or values) and a rule showing how successive terms are derived. This definition is also known as a **recurrence relation**.

Thus the rule $u_1 = 5$; $u_{k+1} = u_k + 2$ generates the sequence 5, 7, 9, 11, 13.... (You start with 5, and then each term is obtained by adding 2 to the previous one).

Example (1): Define the following sequences, using either a direct or an inductive definition.

(a) 6, 11, 16, 21, 26, 31..........

The first term is 6, and each later term can be obtained by adding 5 to the preceding one.
An inductive definition is $u_1 = 6$; $u_{k+1} = u_k + 5$
Inspection would also reveal the pattern 6 = 1+ (1 × 5); 11 = 1 + (2 × 5); 16 = 1+ (3 × 5), suggesting $u_n = 5n + 1$.

(b) 1, 3, 6, 10, 15, 21......

This is the pattern of the triangular numbers, obtained as follows:
u_1: 1 = 1
u_2: 3 = 1+2
u_3: 6 = 1+2+3
u_4: 10 = 1+2+3+4

An inductive definition is $u_1 = 1$; $u_{k+1} = u_k + (k + 1)$.

Another pattern is:

u_1: 1 = ½(1 × 2)
u_2: 3 = ½(2 × 3)
u_3: 6 = ½(3 × 4)
u_4: 10 = ½(4 × 5)

A direct definition is thus $u_n = \frac{1}{2} n(n + 1)$.

(c) 2, -3, 4, -5, 6, -7.....

This sequence oscillates between positive and negative numbers, so we use the fact that odd powers of –1 are equal to -1 and even powers are equal to 1.

u_1: 2 = $(1+1) \times (-1)^2$
u_2: -3 = $(1+2) \times (-1)^3$
u_3: 4 = $(1+3) \times (-1)^4$
u_4: -5 = $(1+4) \times (-1)^5$

A direct definition is thus $u_n = (-1)^{n+1}(n + 1)$.

Example (2): Write out the next eight terms in the sequence $u_1 = 1$; $u_2 = 1$; $u_{k+2} = u_k + u_{k+1}$

The first terms are 1 and 1.
The third term, and all subsequent ones, are obtained by adding together the previous two.

The sequence therefore goes 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

Aside: This is the 'Fibonacci' sequence, named after a 13^{th} – Century mathematician, Leonardo of Pisa, 'Fibonacci – 'son of Bonaccio' who first investigated it.

Example (3): Write out the first six terms of the sequence $u_n = (-1)^n((n+1)^2)$.

This is basically a sequence of the squares of the natural numbers, but each term is the square of the number 1 greater than its position, and it also oscillates between positive and negative values.
Since $(-1)^n$ is –1 for odd n, the first term is –4 and other odd terms are negative.

The sequence therefore goes - 4, 9, -16, 25, -36, 49.........

A sequence is said to be **convergent** if a general term u_n approaches a limiting value as n increases.

An example is the sequence 40, 20, 15, 13.75, 13.4375, where each term in the sequence is obtained from the previous one by dividing by 4 and then adding 10.

Example (4): Write down the inductive definition of the sequence 40, 20, 15, 13.75, ... shown above.

The first term, u_1, is 40, and any subsequent term is obtained by dividing the previous one by 4 and adding 10 to the result.

The inductive definition is therefore $u_1 = 40$; $u_{k+1} = 0.25u_k + 10$.

If we were to find the limit of convergence using a calculator, we would find this limiting value to be about 13.33. How can we find it analytically ?

When we reach the limit (call it L), we can say that $u_{k+1} = u_k = L$ as the terms become ever closer in value. From the details above, we can form an equation in L, where

$$L = 0.25L + 10 \Rightarrow 0.75L = 10 \Rightarrow L = \frac{10}{0.75} = 13\frac{1}{3} .$$

The sequence therefore converges to a limiting value of $13\frac{1}{3}$.

Example (5): The inductive definition of a sequence is as follows:
$u_1 = 30$; $u_{k+1} = pu_k + q$ where p and q are constants.

We are also given that $u_2 = 20$ and $u_3 = 14$.

i) Find the values of p and q.

ii) The value of the term u_k converges to a limit L as k increases. Find the limiting value L.

i) Substituting $u_1 = 30$ and $u_2 = 20$ into the inductive definition gives $30p + q = 20$.
Similarly, substituting $u_2 = 20$ and $u_3 = 14$ gives $20p + q = 14$.

Solving simultaneously gives

$30p + q = 20$ A
$20p + q = 14$ B

$10p = 6$ B-A

Therefore $p = 0.6$, and substituting into A gives $18 + q = 20$, so $q = 2$.

ii) The inductive definition of the sequence is $u_{k+1} = 0.6u_k + 2$

At the limit L, $u_{k+1} = u_k = L$. , so we can form the equation $L = 0.6L + 2 \Rightarrow 0.4L = 2 \Rightarrow L = 5$.

The terms in the sequence converge to a limiting value of 5.

Two special sequences are the **arithmetic progression (A.P.)** and the **geometric progression (G.P.)**
Those will be studied in greater detail in the relevant sections.