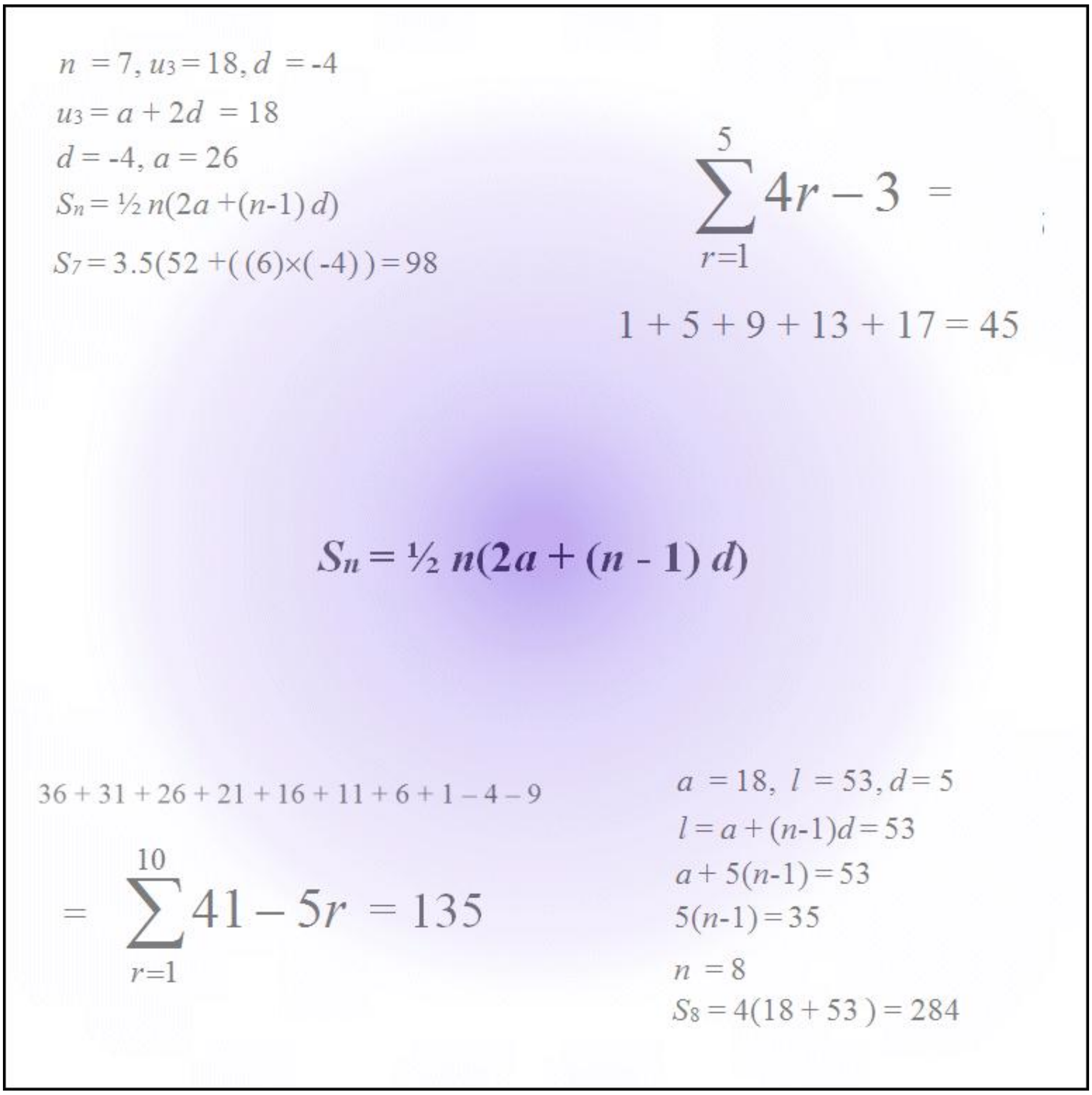
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

THE ARITHMETIC SERIES



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The Arithmetic Series. (also known as the Arithmetic Progression or A.P.)

In an A.P. successive terms have a **common difference**, such as in 1, 5, 9, 13, 17... The first term is denoted by a and the common difference is d. A recursive definition of an A.P. is therefore $u_1 = a$; $u_{k+1} = u_k + d$; a term is obtained from the previous one by **adding** the common difference. The example 1, 5, 9, 13, 17... is thus defined as $u_1 = 1$; $u_{k+1} = u_k + 4$.

The terms of an A.P. take the form a, $a + d$, $a + 2d$, $a + 3d$,... and the n^{th} term is $a + (n-1)d$.

Summing an Arithmetic Series.

An **arithmetic series** is formed by adding together the terms of an arithmetic progression.
Its sum is given by

$$S_n = a + (a + d) + (a + 2d) + ... + (a + (n-1)\,d).$$

where a is the first term, d is the common difference and n is the number of terms.
This formula can be written in reverse order starting from the last term, l.

$$S_n = l + (l - d) + (l - 2d) + ... + (l - (n-1)\,d).$$

Adding the two formulae gives

$$2S_n = (a + l) + (a + l) + (a + l) + ... + (a + l).$$

All the common differences cancel out, and the resulting sum is merely a repetition of n occurrences of $(a + l)$. As the last term can be rewritten as $a + (n-1)\,d$, we can divide the previous result by 2 to give the general formula for the sum of an A.P.

$$\boldsymbol{S_n = \tfrac{1}{2}\, n(2a + (n - 1)\,d).}$$

If the last term of the A.P. is given, we can use the formula

$$\boldsymbol{S_n = \tfrac{1}{2}\, n(a + l).}$$

Example(1): Find the sum of an A.P. with 7 terms, whose third term is 18, and whose common difference is -4.

The starting term is missing, so we need to find it before substituting into the formula.

$u_3 = a + 2d = 18$. Since $d = -4$, $a = 26$.

We can then substitute the values into the general formula $S_n = \frac{1}{2}\, n(2a + (n-1)\,d)$ to give

$S_7 = 3.5(52 + ((6)\times(-4)) = 98$.

Example(2): Find the sum of an A.P. with 8 terms and whose third and fifth terms are 12 and 18 respectively.

We know that $a + 2d = 12$, and $a + 4d = 18$. Subtracting the first expression from the second gives $2d = 6$, or $d = 3$. Another substitution gives $a + 6 = 12$, so $a = 6$.

We can now use the general formula $S_n = \frac{1}{2}\, n(2a + (n-1)\,d)$
to obtain $S_8 = 4(12 + (7\times 3)) = 132$.

Given the first term, last term and common difference, we can work out the total number of terms.

Example (3): Find the sum of an A.P. whose first and last terms are 18 and 53 respectively, and whose common difference is 5.

Here we need to find the number of terms, n.

Since $a = 18$ and $d = 5$, $l = a + (n-1)d = 53$
$\Rightarrow a + 5(n-1) = 53 \Rightarrow 5(n-1) = 35 \Rightarrow (n-1) = 7 \Rightarrow n = 8$.

We can now use the formula $S_n = ½\, n(a + l)$ to obtain $S_8 = 4(18 + 53) = 284$.

Variations on these straightforward questions sometimes come up in exams, often involving a little more algebra.

Example (4): The 10th term of an A.P. is 3 and the sum of the first 6 terms = 76.5. Find the first term a, the common difference d, and the smallest value of the number of terms n such that the sum of the A.P. becomes negative.

We have $u_{10} = 3$ and $S_6 = 3(2a + 5d) = 76.5$

Rewriting u_{10} as $a + 9d$ and dividing S_6 by 3 we have the simultaneous equations

$a + 9d = 3$	A
$2a + 5d = 25.5$	B
$-13d = 19.5$	B-2A

This gives $d = -1.5$, and substituting in A gives $a = 16.5$.

We finally need to find a value for n such that $S_n = ½\, n(33 + (n-1)(-1.5)) < 0$
or $S_n = ½\, n(33 + 1.5 - 1.5n) < 0$.

Either $½n = 0$ (not admissible, as a series cannot have zero members) or $34.5 - 1.5n < 0$
$34.5 - 1.5n = 0$ when $n = 23$, therefore the smallest value of n for which S_n is negative is 24.

Example(5): The sum of the first 10 terms of an A.P. is 120, and that of the first 20 terms is 840. What is the sum to 30 terms ?

Substitution into the general formula $S_n = ½\ n(2a + (n\text{-}1)\ d)$ gives two simultaneous equations.

$S_{10} = 5(2a + 9d) = 10a + 45d$
$S_{20} = 10(2a + 19d) = 20a + 190d$

$S_{20} - 2S_{10} = (20a + 190d) - (20a + 90d) = 100d$

Substituting $S_{20} = 840$ and $S_{10} = 120$ gives $100d = 840 – (2 \times 120) = 600$, and therefore $d = 6$.
Substituting $d = 6$ into the first equation gives $10a = 120 - (6 \times 45) = -150$, hence $a = -15$.

The sum of the series to 30 terms is therefore

$S_{30} = 15(2a + 29d) = 30a + 435d = (30 \times -15) + (435 \times 6) = (-450) + 2610 = 2160$.

Example (6):

The first term of an arithmetic series is a and the common difference is d.
The 18th term of the series is 40 and the 21st term is 52.

i) Use this equation to write down two equations for a and d.
ii) Show that $a = -28$ and find the value of d.

The sum of the first n terms of the series is 4400.

iii) Show that n satisfies the equation $n^2 – 15n = 55 \times 40$.
iv) Hence find the value of n.

i) $u_{18} = a + 17d = 40$ and $u_{21} = a + 20d = 52$.

ii) Subtracting the first from the second, $3d = 40$ and therefore $d = 4$.
Substituting in the first equation, $a + 68 = 40$, so $a = -28$.

Hence the first term of the A.P. is -28 and the common difference is 4.

iii) We need to find the value of n satisfying $S_n = ½\ n(2a + (n\text{-}1)\ d) = 4400$.

Substituting for a and d gives $S_n = ½\ n(-56 + 4(n\text{-}1)) = 4400$.

$\Rightarrow ½\ n(4n – 60) = 4400 \quad \Rightarrow n(2n – 30) = 4400$ (bring ½ inside the brackets)
$\Rightarrow n(n – 15) = 2200 \quad \Rightarrow n^2 – 15n = 2200$
$\Rightarrow n^2 – 15n = 55 \times 40$.

iv) The result in iii) gives us a hint in factorising the quadratic $n^2 – 15n – 2200 = 0$ as

$(n - 55)(n + 40) = 0$, and therefore the required value of $n = 55$. (The negative value is inadmissible in the context of the question.)

Example(7): Ryan was given £5 by his Uncle Bob on his first birthday, and on each succeeding birthday, Uncle Bob gave him £6 more than he did on the previous birthday.

On which of Ryan's birthdays did the annual gift from Uncle Bob exceed £100 ?
Give also the total sum donated up to and including that birthday.

This question can be interpreted as: "Give the first term of the A.P. 5, 11, 17... to exceed 100, and give the sum to that number of terms".

Here $d = 6$, $u_1 = 5$. Subtracting 5 from 100 gives 95, and the nearest multiple of 6 above that is 96, equal to 16 common differences, so the first term above 100 will be u_{17} or 101.

The first time Ryan receives over £100 from Uncle Bob will be on his 17th birthday, where he receives £101 on the day.

Also, the total of all the birthday gifts received by Ryan up to and including his 17th birthday (in £) is $S_{17} = 8.5(10 + 96) = 901$.

Sigma Notation.

This is a convenient shorthand method of defining a series. The Greek letter sigma (Σ) stands for 'sum'.

The series 1 + 5 + 9 + 13 + 17 can be expressed in sigma notation as $\sum_{r=1}^{5} 4r - 3$.

The "r = 1" below the symbol is the starting value for the count variable, r, and the "5" above it is the ending value for r. The "4r – 3" to the right of the symbol is the actual term itself, expressed as a function of r. When $r = 1$, $4r - 3 = 1$; when $r = 2$, $4r - 3 = 5$, and so on until the last term, 17, corresponding to $r = 5$.

Another equally valid definition for the same series is $\sum_{r=0}^{4} 4r + 1$.

The "4" above the symbol is **not** the same as number of terms in the sum here. There are still 5 terms, but this time the count variable goes from 0 to 4, not 1 to 5, and the formula for the term has been correspondingly modified.

Another example: the sum of the first n natural numbers can be expressed as $\sum_{r=1}^{n} r$, and the sum of their squares as $\sum_{r=1}^{n} r^2$.

Example (8): Rewrite the following series in sigma notation:

(a) 1 + 4 + 7 + 10 + 13 + 16 + 19
(b) 36 + 31 + 26 + 21 + 16 + 11 + 6 + 1 – 4 – 9

(In each case, we will begin with r = 1).

(a) This is an A.P. with common difference 3 and first term 1, with 7 terms.
The general term is $u_r = 3r - 2$.

Its sigma notation equivalent is $\sum_{r=1}^{7} 3r - 2$.

(b) This is an A.P. with common difference -5 and first term 36, with 10 terms.
The general term is $u_r = 41\text{-}5r$.

Its sigma notation equivalent is $\sum_{r=1}^{10} 41 - 5r$.

Example (9): Write down the series (do not sum it) corresponding to the following sigma notations:

(a) $\sum_{r=1}^{6} 2r + 6$

(a) is an A.P. with 6 terms whose first term is 8 and whose common difference is 2,
i.e. 8 + 10 + 12 + 14 + 16 + 18.