M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

THE BINOMIAL SERIES FOR RATIONAL POWERS

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n$$

$$\frac{1}{1+x} = (1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \frac{(-1)(-2)(-3)(-4)}{4!}x^4 \ldots\ldots$$

$$= 1 - x + x^2 - x^3 + x^4. \quad |x| < 1$$

$$x = 0.05 \qquad \frac{1}{1.05} = 1 - 0.05 + 0.0025 - 0.000125 + 0.00000625 = 0.952381$$

$$\frac{1}{\sqrt{4+x}} = (4+x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}\left[\left(1+\frac{x}{4}\right)^{-\frac{1}{2}}\right] = \frac{1}{2}\left[\left(1+\frac{x}{4}\right)^{-\frac{1}{2}}\right]$$

$$= \frac{1}{2}\left[1 + (-\tfrac{1}{2})\left(\frac{x}{4}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(\frac{x}{4}\right)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(\frac{x}{4}\right)^3 \ldots\ldots\right]$$

$$= \tfrac{1}{2} - \tfrac{1}{16}x + \tfrac{3}{256}x^2 - \tfrac{5}{2048}x^3 \ldots\ldots \quad |\tfrac{x}{4}| < 1 \text{, or } |x| < 4.$$

$$\frac{1}{(1-4x)^2} = (1-4x)^{-2}$$

$$= 1 + (-2)(-4x) + \frac{(-2)(-3)}{2!}(-4x)^2 + \frac{(-2)(-3)(-4)}{3!}(-4x)^3 + \ldots\ldots$$

$$= 1 + 8x + 48x^2 + 256x^3 \ldots.$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n$$

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THE BINOMIAL SERIES FOR RATIONAL *n*.

To recap, the general binomial expansion for $(a + b)^n$, where n is a positive integer, is

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \binom{n}{3} a^{n-3}b^3 + ... + b^n$$

or

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3}b^3 + ... + b^n.$$

Notice the pattern here: powers of a decrease from n to zero, powers of b increase from zero to n, and in each term there is a binomial coefficient multiplier with a factorial denominator and a product of a series of numbers in the numerator.

Recall factorials: $1! = 1$; $2! = 2 \times 1 = 2$; $3! = 3 \times 2 \times 1 = 6$; $4! = 4 \times 3 \times 2 \times 1 = 24$ and so forth.

Substituting $(1 + x)^n$ for $(a + b)^n$ we have

$$(1 + x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + ... + x^n$$

or

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + ... + x^n$$

This expression is simpler than the previous one, because all powers of 1 are equal to 1 itself.

Finally, substituting $(1 - x)^n$ for $(1 + x)^n$ in the last expression we have

$$(1 - x)^n = 1 - \binom{n}{1} x + \binom{n}{2} x^2 - \binom{n}{3} x^3 + ... + (-x)^n$$

or

$$(1 - x)^n = 1 - nx + \frac{n(n-1)}{2!} x^2 - \frac{n(n-1)(n-2)}{3!} x^3 + ... + (-x)^n$$

The terms in odd powers of x have their sign reversed – think of $(1 – x)$ as $(1 + (-x))$.

If n is a positive integer, the series will terminate at the x^n term, since the numerator of the fractional representation of the binomial coefficient will have a zero term in it, and as multiplying by 0 gives 0, this term and all subsequent ones will vanish.

However, the formulae for $(1 + x)^n$ and $(1 - x)^n$ can also be used for all other rational n, giving an infinite series. Provided x lies within certain limits, the series will converge, in other words, the terms will become smaller as we move from left to right.

A series of the form $(1 + x)^n$ converges, i.e. the expansion is valid, when $|x| < 1$.

If the second term of the binomial is kx where k is a non-zero constant, the limits of convergence are $|kx| < 1$, or $|x| < \frac{1}{k}$.

Example(1): Expand $\frac{1}{1+x}$ up to the term in x^4 and state the values for which the expansion is valid.

Use the result to find the value of $\frac{1}{1.05}$ to 6 decimal places.

The expression can be written as $(1+x)^{-1}$ and therefore its binomial expansion is

$$(1 + x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \frac{(-1)(-2)(-3)(-4)}{4!}x^4..............$$

$= 1 - x + x^2 - x^3 + x^4$.

This expansion is valid and convergent for $|x| < 1$.

When $x = 0.05$, the series sums to 1 – 0.05 + 0.0025 – 0.000125 + 0.00000625 = 0.952381 to 6 d.p.

Note that for examination purposes, questions will usually be restricted only to the term in x^2 or x^3.

Example(2): Expand $\frac{1}{(1+2x)^2}$ up to the term in x^3 , stating the values for which the expansion is valid.

Use the result to estimate $\frac{1}{(1.06)^2}$ to 4 decimal places.

The expression can be written as $(1 + 2x)^{-2}$ and therefore its binomial expansion is

$$1+(-2)(2x)+\frac{(-2)(-3)}{2!}(2x)^2+\frac{(-2)(-3)(-4)}{3!}(2x)^3+.....$$

or $1 - 4x + 12x^2 - 32x^3$ ....

This expansion is valid and convergent for $|2x| < 1$, or $|x| < \frac{1}{2}$.

When $1 + 2x = 1.06$, $2x = 0.06$, and therefore $x = 0.03$.
Substituting $x = 0.03$ in the expansion, we have $1 - 4(0.03) + 12(0.03)^2 - 32(0.03)^3$ ....
or 1 - 0.12 + 0.0108 – 0.000864 = 0.8899 to 4 d.p.

Example (3): In trigonometry, the approximation $\sin x \approx x$ is valid for small angles.
By using the identity $\cos^2 x + \sin^2 x = 1$, show by binomial expansion that a small-angle approximation to $\cos x$ is $\cos x \approx 1 - ½x^2$.

By taking $\sin x \approx x$ and using the identity $\cos^2 x + \sin^2 x = 1$, then $\cos^2 x \approx 1-x^2$ and $\cos x = \sqrt{(1-x^2)}$.

In binomial expansion, $\sqrt{1-x^2} = \left(1-x^2\right)^{\frac{1}{2}} = 1+\frac{1}{2}(-x^2)+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(-x^2)^2.....$

$= 1-\frac{1}{2}x^2$, disregarding all terms beyond the second one.

Sometimes, we may need algebraic manipulation to put the expression 'into shape', i.e as some multiple of $(1 + x)^n$ or $(1 - x)^n$.

Example(4): Expand $\sqrt{1-x}$ up to the term in x^3, stating the values for which the expansion is valid.

Use the result to find the value of $\sqrt{24}$ to 5 decimal places. (Hint : 24 = 25 × 0.96)

The expression can be written as $(1-x)^{\frac{1}{2}}$ and therefore its binomial expansion is

$$(1-x)^{\frac{1}{2}} = 1-\tfrac{1}{2}x+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^2-\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3+......$$ or treat as

$$(1-x)^{\frac{1}{2}} = 1+\tfrac{1}{2}(-x)+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(-x)^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}(-x)^3+......$$

$$= 1-\tfrac{1}{2}x-\tfrac{1}{8}x^2-\tfrac{1}{16}x^3...$$

(The binomial coefficients are more awkward here because of the fractions)

This expansion is valid and convergent for $|x| < 1$.

To find $\sqrt{24}$, we cannot substitute x = -23 and say $\sqrt{1-(-23)}$, as the series is only valid for $|x| < 1$.

We can, though, manipulate surds to give $\sqrt{24} = \sqrt{25\times 0.96} = \sqrt{25}\sqrt{0.96} = 5\sqrt{0.96}$.

Now we can calculate $\sqrt{0.96}$ by substituting $x = 0.04$ (to give $1-x = 0.96$) into the binomial series, since x now is within the valid range for expansion.

$$\sqrt{0.96} \approx 1-\tfrac{1}{2}(0.04)-\tfrac{1}{8}(0.0016)-\tfrac{1}{16}(0.000064)$$

$\approx 1 - 0.02 - 0.0002 - 0.000004 \approx 0.979796$.

Multiplying the result by 5 gives $\sqrt{24}$ = 4.89898 to 5 decimal places.

Example(5): Expand $\dfrac{1}{\sqrt{4+x}}$ up to the term in x^3 , stating the values for which the expansion is valid.

Firstly, $\dfrac{1}{\sqrt{4+x}}$ is the same as $(4+x)^{-\frac{1}{2}}$

The expression inside the square root sign is not of the correct format (1 + x) for substituting into the binomial expansion, so we have to take out a factor of 4 as follows :

Because $(4+x)=4\left(1+\dfrac{x}{4}\right)$, we can say $(4+x)^{-\frac{1}{2}}=\left(4\left(1+\dfrac{x}{4}\right)\right)^{-\frac{1}{2}}$

$= 4^{-\frac{1}{2}}\left[\left(1+\dfrac{x}{4}\right)^{-\frac{1}{2}}\right]$ (recall laws of indices !) $= \dfrac{1}{2}\left[\left(1+\dfrac{x}{4}\right)^{-\frac{1}{2}}\right]$.

We now have the all-important 1 as the first term, so we can carry out the expansion :

$$\frac{1}{2}\left[\left(1+\tfrac{x}{4}\right)^{-\frac{1}{2}}\right] = \frac{1}{2}\left[1+(-\tfrac{1}{2})\left(\frac{x}{4}\right)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(\frac{x}{4}\right)^2+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(\frac{x}{4}\right)^3\ldots\ldots\right]$$

$= \tfrac{1}{2}-\tfrac{1}{16}x+\tfrac{3}{256}x^2-\tfrac{5}{2048}x^3\ldots\ldots$ (Do not forget the external factor of $\tfrac{1}{2}$).

This expansion is valid and convergent for $|\tfrac{x}{4}|<1$, or $|x| < 4$.

Example (6):

i) Expand $\frac{1}{(1-4x)^2}$ in ascending powers of x, up to and including the term in x^3 .

ii) Find the coefficient of x^2 in the expansion of $\frac{(1+3x)^2}{(1-4x)^2}$.

i) $\frac{1}{(1-4x)^2} = (1-4x)^{-2}$

$$= 1 + (-2)(-4x) + \frac{(-2)(-3)}{2!}(-4x)^2 + \frac{(-2)(-3)(-4)}{3!}(-4x)^3 + \ldots\ldots$$

$$= 1 + 8x + 48x^2 + 256x^3 \ldots.$$

ii) Multiplying the expansion in i) by $(1+3x)^2$ gives the result

$$(1 + 8x + 48x^2 + 256x^3 \ldots.)\ (1 + 6x + 9x^2)\ .$$

The combinations of terms contributing to the quadratic term in the product are $(1 \times 9x^2)$, $(8x \times 6x)$ and $(48x^2 \times 1)$. Their sum is $(9 + 48 + 48)\ x^2$ or $105\ x^2$.

$\therefore$ the coefficient of x^2 in the expansion of $\frac{(1+3x)^2}{(1-4x)^2}$ is 105.

Example (7): In the section "Partial Fractions", we resolved the expression

$\frac{4x+5}{(x-1)(x+2)^2}$ into partial fractions as $\frac{1}{x-1} - \frac{1}{x+2} + \frac{1}{(x+2)^2}$.

An examination question could continue as follows:

i) Use the formula for the sum to infinity of a geometric series to show that the binomial expansion of $\frac{1}{x-1}$ forms the series $-1 - x - x^2 - x^3 .....$

ii) Show that the expression $\frac{1}{x+2}$ is equivalent to $\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$ and hence find the binomial expansion of $\frac{1}{x+2}$ up to and including the term in x^3.

iii) Using similar reasoning to part ii), find the binomial expansion of $\frac{1}{(x+2)^2}$ up to and including the term in x^3.

iv) Hence show that the binomial expansion (to the term in x^3) of $\frac{4x+5}{(x-1)(x+2)^2}$ can be expressed as $-\frac{1}{16}\left(20+16x+15x^2+17x^3\right)$.

v) Hence use the series from iv) to find the value of $\frac{4x+5}{(x-1)(x+2)^2}$ when $x = 0.01$.

vi) State the range of values of x for which the expansion is valid.

i) The series $-1-x-x^2-x^3.....$ **(Series A)** is recognisable as a G.P. whose first term a is -1 and whose common ratio r is x. Its sum to infinity is therefore $\frac{a}{1-r}$ or in this case $\frac{-1}{1-x}$ or $\frac{1}{x-1}$.
This expansion is valid for $|x| < 1$.

ii) The expression $\frac{1}{x+2}$ cannot be expanded in the format $(2+x)^{-1}$, so we must rewrite it as $(2^{-1})\left(1+\frac{x}{2}\right)^{-1}$ and hence as $\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$.

This expansion is valid for $|\frac{x}{2}|<1$, or $|x| < 2$.

Expanding up to the term in x^3 we have:

$$\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1} = \frac{1}{2}\left[1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3+.....\right]$$

$$= \frac{1}{2}\left[1-\frac{1}{2}x+\frac{1}{4}x^2-\frac{1}{8}x^3+.....\right] = \frac{1}{2}-\frac{1}{4}x+\frac{1}{8}x^2-\frac{1}{16}x^3..... \textbf{(Series B)}$$

iii) The binomial expansion of $\frac{1}{(x+2)^2}$ must similarly be adjusted as follows:

$$\frac{1}{(x+2)^2} = (2^{-2})\left(1+\frac{x}{2}\right)^{-2} = \frac{1}{4}\left(1+\frac{x}{2}\right)^{-2}$$

$$= \frac{1}{4}\left[1+(-2)\left(\frac{x}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{x}{2}\right)^3+.....\right]$$

$$= \frac{1}{4}\left[1-x+\frac{3}{4}x^2-\frac{1}{2}x^3+.....\right] = \frac{1}{4}-\frac{1}{4}x+\frac{3}{16}x^2-\frac{1}{8}x^3..... \textbf{(Series C)}$$

This expansion is valid for $|\frac{x}{2}|<1$, or $|x| < 2$.

iv) Combining the expressions for Series **A**, **B** and **C** gives the following results:
(Remember that series **B** must be subtracted, not added !)

Series A	-1	$-x$	$-x^2$	$-x^3$
Series B **(SUBTRACT !)**	$\frac{1}{2}$	$-\frac{1}{4}x$	$\frac{1}{8}x^2$	$-\frac{1}{16}x^3$
Series C	$\frac{1}{4}$	$-\frac{1}{4}x$	$\frac{3}{16}x^2$	$-\frac{1}{8}x^3$
Total **(A – B + C)**	$-\frac{5}{4}$	$-x$	$-\frac{15}{16}x^2$	$-\frac{17}{16}x^3$

v) The binomial expansion (to the term in x^3) of $\dfrac{4x+5}{(x-1)(x+2)^2}$ is therefore

$$-\left(\frac{5}{4}\right)-x-\left(\frac{15}{16}x^2\right)-\left(\frac{17}{16}x^3\right).....$$

By multiplying everything out by -16 and putting $-\dfrac{1}{16}$ outside the brackets, the expansion can also be written as $-\dfrac{1}{16}\left(20+16x+15x^2+17x^3\right)$.

When $x = 0.01$, $\dfrac{4x+5}{(x-1)(x+2)^2} \approx -\dfrac{1}{16}\left(20+0.16+0.0015+0.000017\right)$ or 1.260095.

Note: The actual value is $\dfrac{5.04}{(-0.99)(2.01)^2} = -1.2600948$.

vi) Two of the terms in the combined expression are valid for $\left|\frac{x}{2}\right| < 1$, or $|x| < 2$. However, the expression for $\dfrac{1}{x-1}$ is valid only for $|x| < 1$.

The series *as a whole* is therefore valid only for the range of its 'strictest' component, i.e. $|x| < 1$.