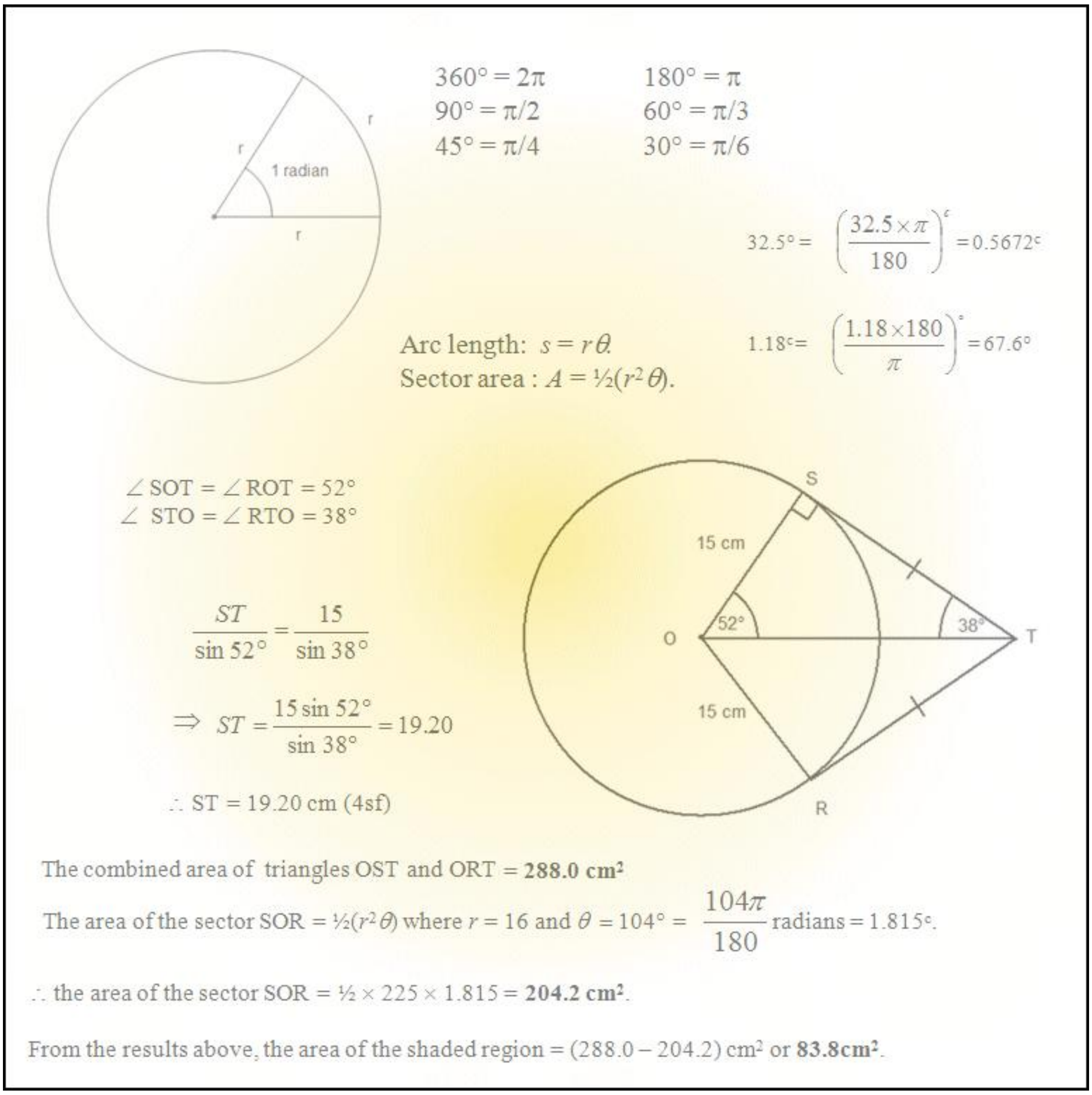
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

RADIAN MEASURE OF ANGLES



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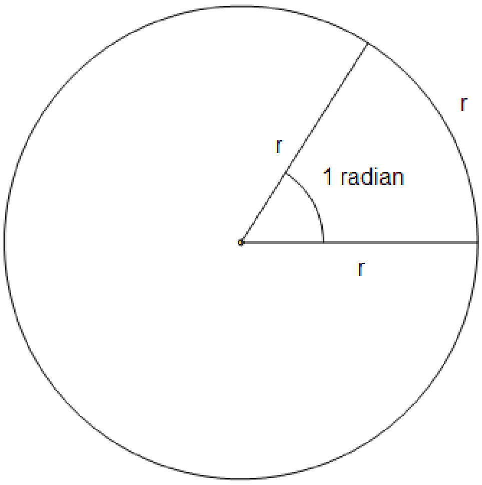
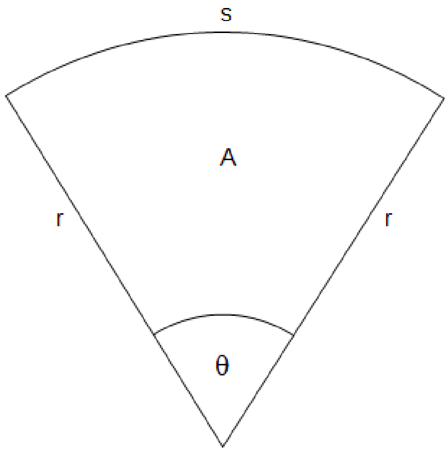
Radian measure.

Although angles are usually measured in degrees, sometimes it is more convenient to use what is known as **radian** measure, especially in calculus.

A radian is the angle subtended by the arc of a circle, where the arc's length is equal to the circle's radius.





Because the formula for the circumference is C = 2πr, it follows that 360° is equal to 2π radians.
1 radian, abbreviated to 1 rad or 1^c , is therefore (180/π)°, or about 57.3°.
The radian symbol is not normally written when the angle is given in terms of π.

Familiar angles in radian measure:

360° = 2π   180° = π
90° = π/2   60° = π/3
45° = π/4   30° = π/6

To convert degrees into radians, multiply by π and divide by 180.

To convert radians into degrees, multiply by 180 and divide by π.

Arc length and sector area.

When an angle θ is given in radians (see figure on upper right), the sector area and arc length are given by the following formulae:

The length of the arc is $s = r\theta$.
The area of the sector is $A = ½(r^2\theta)$.

Example (1): Convert 32.5° to radians, and 1.18 radians to degrees.

32.5° = $\left(\frac{32.5 \times \pi}{180}\right)^c$ = 0.5672^c to 4 decimal places.

1.18^c = $\left(\frac{1.18 \times 180}{\pi}\right)^\circ$ = 67.6° to 1 decimal place.

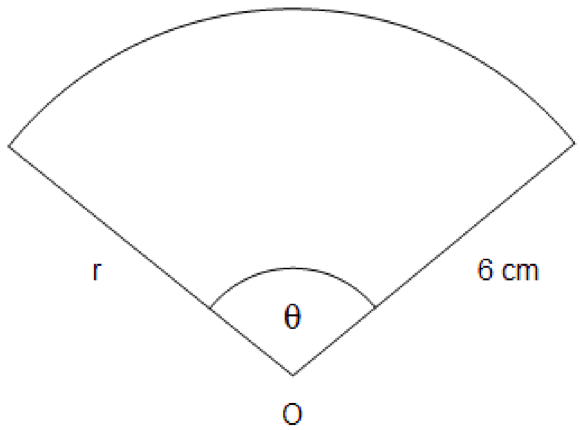
Example (2): Find the arc length and area of a) a 28° sector of a circle whose radius is 12cm; a 45° sector of a circle whose radius is 16cm . Leave the answers in part (b) in terms of π.

In a) we must first convert 28° to radians, giving $\theta = 0.4887^c$. Given that r = 12 cm, the arc length of the sector is therefore 12 × 0.4887 cm, or 5.86 cm to 2 decimal places.
The area, A is ½($r^2\theta$) or 72 × 0.4887 cm^2 , or 35.19 cm^2 to 2 decimal places.

In (b) we use the fact that $\theta = 45° = \frac{\pi}{4}$. With the radius r equal to 16 cm, the arc length of the sector is 4π cm. The area is ½($r^2\theta$) or 128 × $\frac{\pi}{4}$ cm^2, or 32π cm^2.

Example (3):

The figure on the right shows a sector of a circle of radius r = 6 cm and an angle of θ radians. The sector has an area of 44 cm^2.



Find the value of θ and hence the perimeter of the sector.

We are given the area as A = ½($r^2\theta$), but here it is θ that is missing, so we rearrange the formula as $r^2\theta = 2A$ and finally as $\theta = \frac{2A}{r^2}$.

Hence $\theta = \frac{88}{36} = 2.444^c$.

The perimeter of the sector can therefore be obtained by finding the arc length, here rθ or 14.67 cm, and adding twice the radius, or 12 cm, to the result.

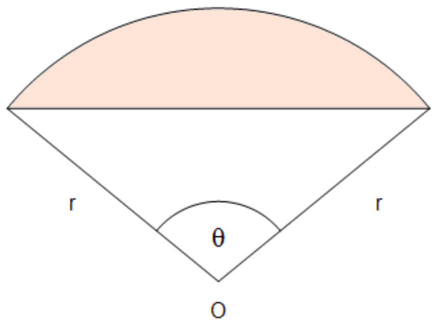
∴ the perimeter of the sector is 26.67 cm.

Other questions of this type might require knowledge of geometry and trigonometric identities.

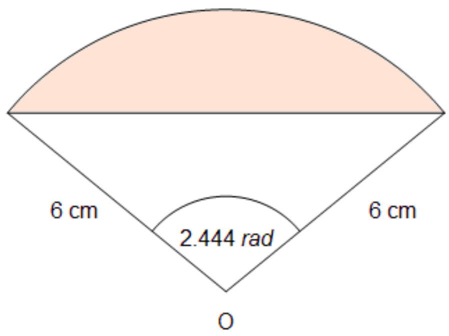
Example (4):

i) Show that the area of a segment subtended by an angle θ can be given as $\frac{1}{2}r^2(\theta - \sin\theta)$.

Use the triangle area formula $= \frac{1}{2}ab \sin C$, where a and b are two sides and C is the included angle.



ii) Using the results from Example (3), find the area and perimeter of the shaded segment in the diagram on the right.



i) The area of the general segment is most easily obtained by subtracting the area of the unshaded triangle from that of the sector.

The formula for the area of a triangle can be adapted here as $A = \frac{1}{2}r^2 \sin\theta$.

Therefore, area of segment = area of sector – area of subtended triangle $= \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta$.
Factorising, we have the area of the segment = $\mathbf{\frac{1}{2}r^2(\theta - \sin\theta)}$ **.**

ii) Substituting $r = 6$ cm and $\theta = 2.444^c$, the area of the segment is $18(2.444 - \sin 2.444^c)$ or 32.44cm^2.

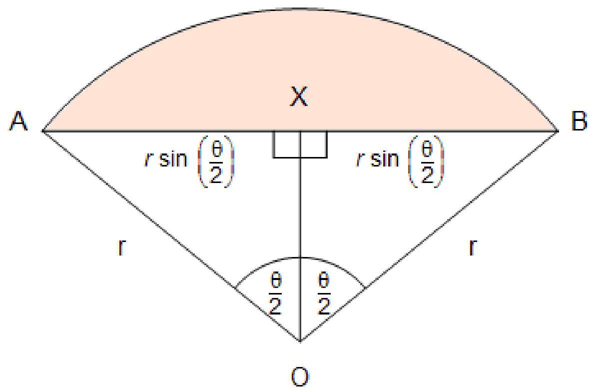
Alternately, we could have obtained the area of the triangle as $\frac{1}{2}(36 \sin 2.444^c) = 11.56$ cm^2 , and then subtracted that from the area of the sector in the last example, i.e. 44 cm^2, to obtain the same result .

Finding the perimeter of the segment is a little more tricky – we need to find the chord length AB.

The centre of the circle is at O and the radius is *r*.

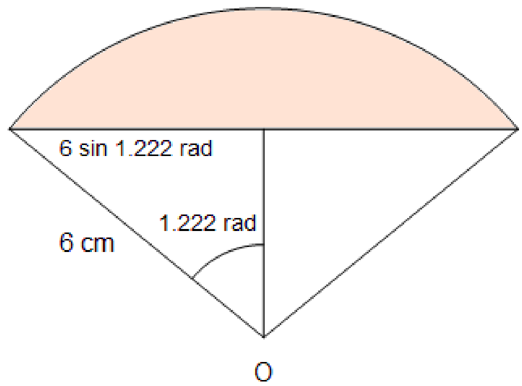
The perpendicular bisector OX of the chord AB is also the bisector of the subtended angle, and so we can apply Pythagoras to the triangle AOX to find the half-chord AX.

AX = *r* sin ½(θ), and so the length of the chord AB is twice that, or **2*r* sin ½(θ).**



We have already worked out θ = 2.444^c and the arc length , *r*θ , = 14.67 cm.
The particular chord therefore has a length of 12 sin 1.222^c or 11.27 cm.

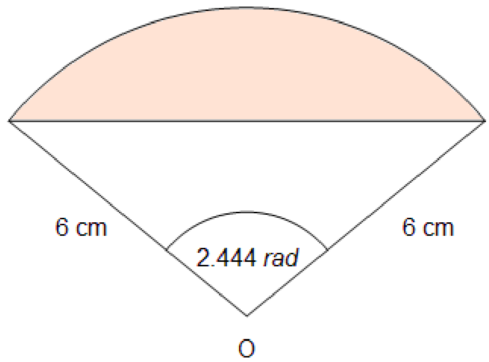
The perimeter of the segment is therefore (14.67 + 11.27) cm or 25.9 cm to 3 s.f.



Another way of finding the length of the chord AB would be to use the cosine formula: $c^2 = a^2 + b^2 - 2ab\cos C$:

Remember *a* and *b* are both radii and so can be replaced by *r*,

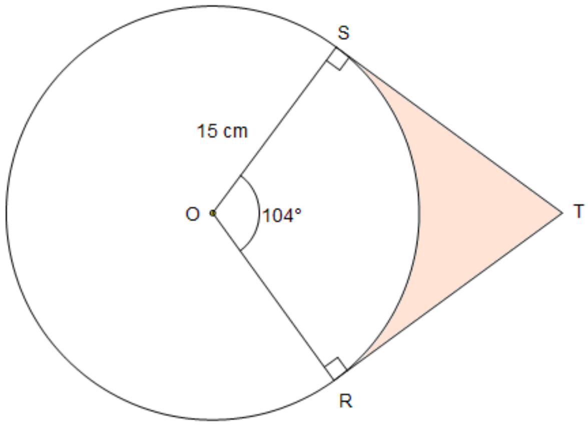
so *r* = 6 and *C* = 2.444^c :



$$c^2 = 2r^2 - 2r^2\cos C$$
$$\Rightarrow c^2 = 2r^2(1-\cos C).$$
$$\Rightarrow c^2 = 72(1-\cos 2.444^c) = 127.2 \Rightarrow c = 11.27 \text{ cm.}$$

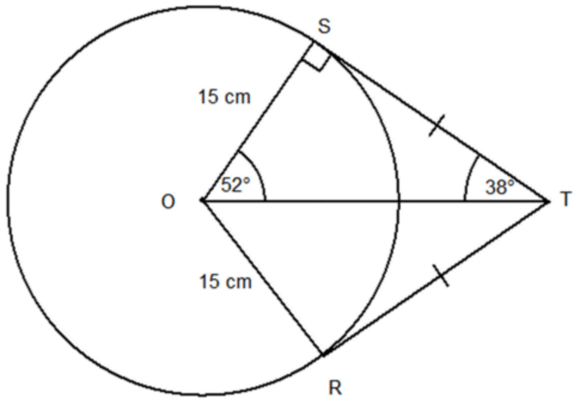
Example (5): The lines ST and RT are tangents to the circle of radius 15 cm centred on O. Angle SOR is 104°.

Find the area of the shaded region .



Recalling the properties of tangents of a circle,
we see that the lengths ST and RT are equal, and that the angles ORT and OST are right angles.

Because OR and OS are both radii, they are also equal, so triangles OST and ORT are congruent.



$\therefore$ $\angle$ SOT = $\angle$ ROT = 52°
and $\angle$ STO = $\angle$ RTO = 38°

The length ST can be found by the sine rule:

$$\frac{ST}{\sin 52°} = \frac{15}{\sin 38°}$$

$$\Rightarrow \; ST = \frac{15 \sin 52°}{\sin 38°} = 19.20$$

$\therefore$ ST = 19.20 cm (4sf)

To find the area of the shaded region in the first diagram, we must first take the combined area of the two triangles OST and ORT, and then subtract the area of the sector SOR from the result.

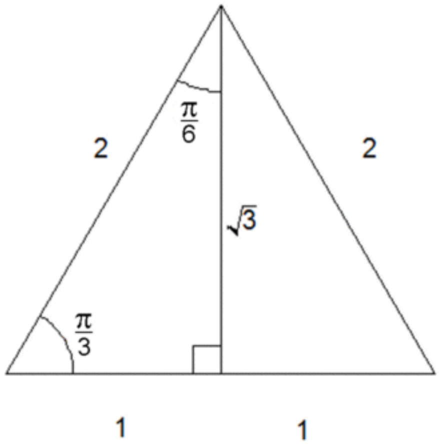
The combined area of the two triangles OST and ORT = 2 × ½(base × height) = **288.0 cm^2**.

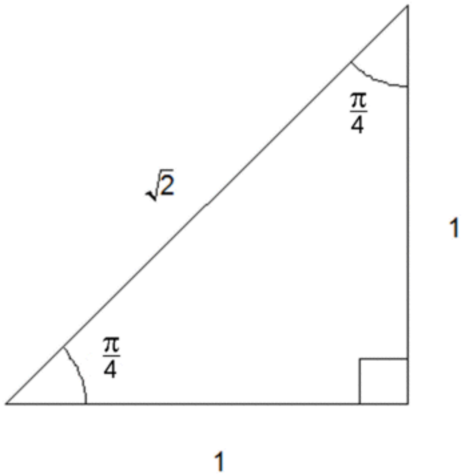
The area of the sector SOR = ½($r^2\theta$) where r = 16 and θ = 104° = $\frac{104\pi}{180}$ radians = 1.815^c.

$\therefore$ the area of the sector SOR = ½ × 225 × 1.815 = **204.2 cm^2**.

From the results above, the area of the shaded region = (288.0 – 204.2) cm^2 or **83.8cm^2**.

Ratios of special angles in radian measure.





sin (π/6) = cos (π/3) = $\frac{1}{2}$

sin (π/3) = cos (π/6) = $\frac{\sqrt{3}}{2}$

tan (π/6) = $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

tan (π/3) = $\sqrt{3}$

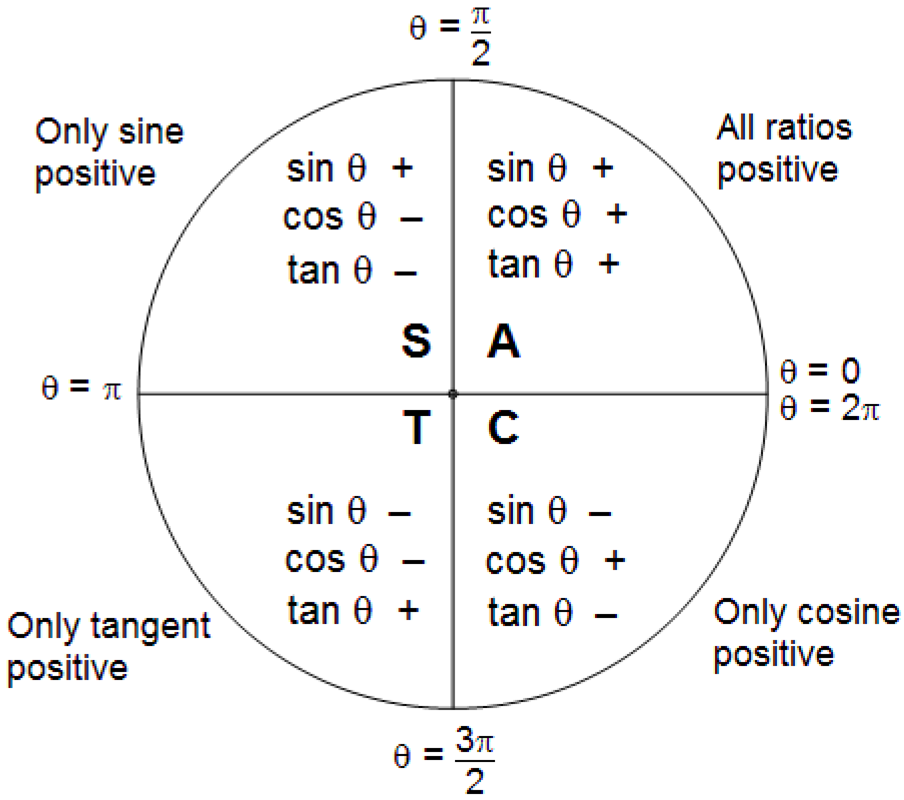
sin (π/4) = cos (π/4) = $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

tan (π/4) = 1

sin (π/2) = cos 0 = 1

sin 0 = cos (π/2) = 0

Quadrant rules in radian measure.



The working range here is $0 \le \theta < 2\pi$.

First quadrant: angle between 0 and π/2.
For any angle θ in the first quadrant, its trig ratios are all positive.

Second quadrant: angle between π/2 and π.
For any angle θ in the second quadrant, we have the trig ratios of the angle $(\pi - \theta)$ related as follows:
$\sin(\pi - \theta) = \sin\theta$; $\cos(\pi - \theta) = -\cos\theta$; $\tan(\pi - \theta) = -\tan\theta$.

Third quadrant: angle between π and 3π/2.
For any angle θ in the third quadrant, we have the trig ratios of the angle $(\pi + \theta)$ related as follows:
$\sin(\pi + \theta) = -\sin\theta$; $\cos(\pi + \theta) = -\cos\theta$; $\tan(\pi + \theta) = \tan\theta$.

Fourth quadrant: angle between 3π/2 and 2π.
For any angle θ in the third quadrant, we have the trig ratios of the angle $(2\pi - \theta)$ related as follows:
$\sin(2\pi - \theta) = -\sin\theta°$; $\cos(2\pi - \theta) = \cos\theta$; $\tan(2\pi - \theta) = -\tan\theta$.

The following ratios hold at the boundaries of the quadrants:

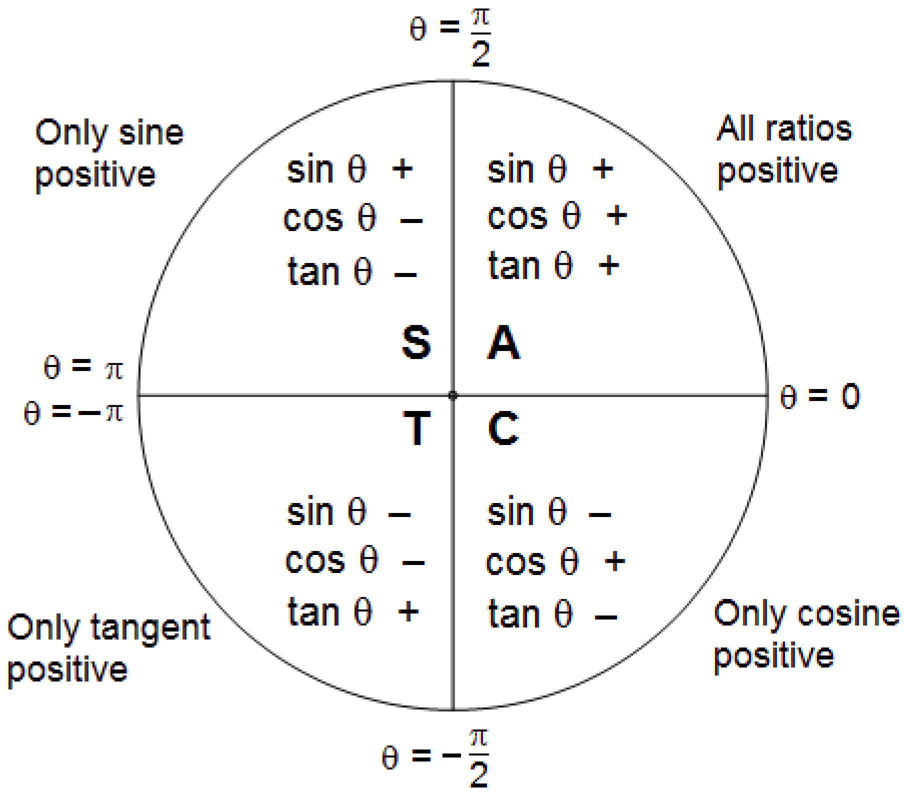
$\sin 0 = 0$; $\cos 0 = 1$; $\tan 0 = 0$

$\sin(\pi/2) = 1$; $\cos(\pi/2) = 0$; $\tan(\pi/2)$ is undefined

$\sin(\pi) = 0$; $\cos(\pi) = -1$; $\tan(\pi) = 0$

$\sin(3\pi/2) = -1$; $\cos(3\pi/2) = 0$; $\tan(3\pi/2)$ is undefined

Here is the alternative CAST diagram for the angle range $-\pi \le \theta < \pi$.



First quadrant: angle between 0 and $\pi/2$.
For any angle θ in the first quadrant, its trig ratios are all positive.

Second quadrant: angle between $\pi/2$ and π.
For any angle θ in the second quadrant, we have the trig ratios of the angle $(\pi - \theta)$ related as follows:
$\sin(\pi - \theta) = \sin\theta$; $\cos(\pi - \theta) = -\cos\theta$; $\tan(\pi - \theta) = -\tan\theta$.

Third quadrant: angle between $-\pi$ and $-\pi/2$.
For any angle θ in the third quadrant, we have the trig ratios of the angle $(-\pi - \theta)$ related as follows:
$\sin(-\pi - \theta) = -\sin\theta$; $\cos(-\pi - \theta) = -\cos\theta$; $\tan(-\pi - \theta) = \tan\theta$.

Fourth quadrant: angle between $-\pi/2$ and 0.
For any angle θ in the fourth quadrant, we have the trig ratios of the angle $(-\theta)$ related as follows:
$\sin(-\theta) = -\sin\theta°$; $\cos(-\theta) = \cos\theta$; $\tan(-\theta) = -\tan\theta$.

The following ratios hold at the boundaries of the quadrants:

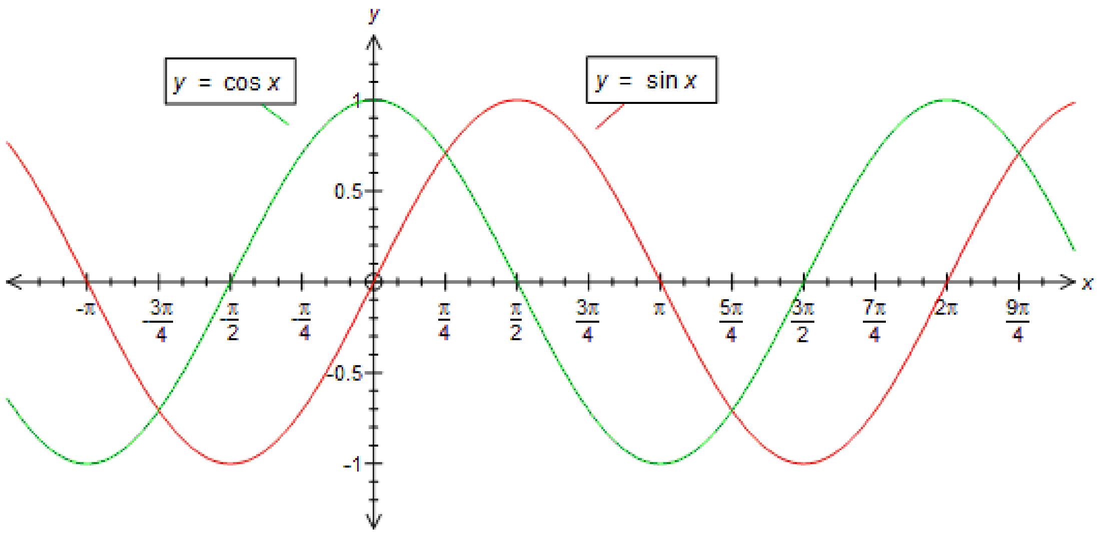
$\sin 0 = 0$; $\cos 0 = 1$; $\tan 0 = 0$

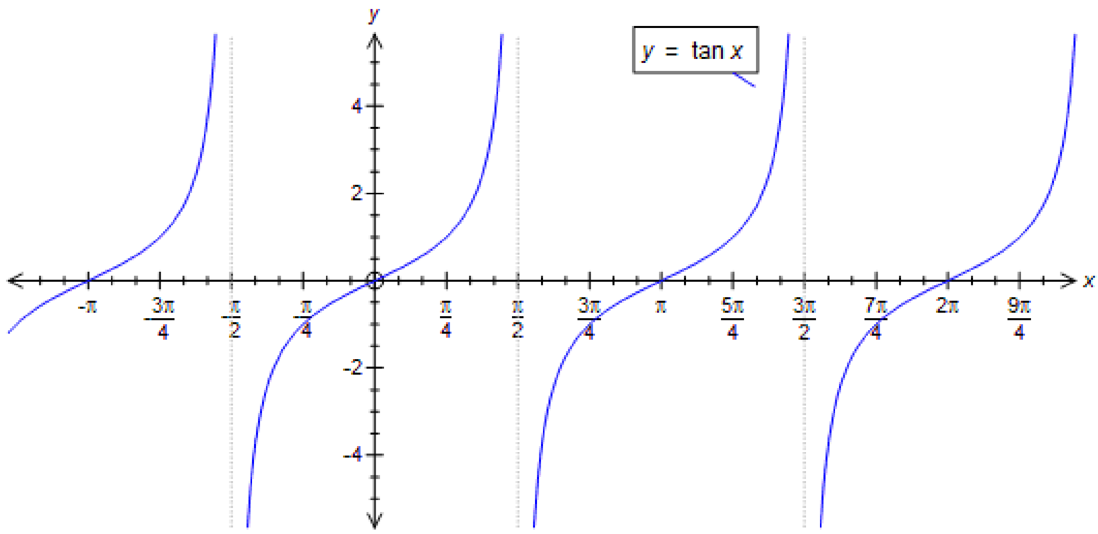
$\sin(\pi/2) = 1$; $\cos(\pi/2) = 0$; $\tan(\pi/2)$ is undefined

$\sin(-\pi) = 0$; $\cos(-\pi) = -1$; $\tan(-\pi) = 0$

$\sin(-\pi/2) = -1$; $\cos(-\pi/2) = 0$; $\tan(-\pi/2)$ is undefined

Graphs of trig functions in radian measure.





(Full details are in the document "Trigonometric Ratios and Graphs", with angles measured in degrees).

The graphs of sin *x* and cos *x* each have a repeating period of 2π radians. Indeed, the graph of cos *x* is the same as that of sin *x* translated (π/2) radians to the left , i.e. by the vector $\begin{pmatrix} -\frac{\pi}{2} \\ 0 \end{pmatrix}$.

The graph of tan *x* has a repeating period of π radians, and the function itself is undefined for certain values of *x*, such as (π/2), (3π/2), and all angles consisting of an odd number of right angles.

In other words, the tangent graph has asymptotes at (π/2), (3π/2), and all angles (*n*π/2) where *n* is an odd integer.

Solving trigonometric equations in radian measure.

At AS-Level, we had been solving a variety of trigonometric equations, although degrees had been used throughout. The following examples highlight the use of radian measure.

Example (6): Find the solutions of the following, in the interval $0 \le \theta < 2\pi$.
i) $\sin x = 0.5$; ii) $\cos x = 0.5$; iii) $\tan x = 1$.

i) In radians, the solutions of $\sin x = 0.5$ are $\boldsymbol{\pi/6}$ and $(\pi - \pi/6)$ or $\boldsymbol{5\pi/6}$ in the given interval.

As an aside, the graph of the sine function repeats every 2π radians, so other solutions are $13\pi/6$, $17\pi/6$ (positive x-direction) and $-7\pi/6$, $-11\pi/6$ (negative x-direction).

ii) The solutions of $\cos x = 0.5$ are $\boldsymbol{\pi/3}$ and $(2\pi - \pi/3)$ or $\boldsymbol{5\pi/3}$ in the given interval.
The cosine graph repeats every 2π radians, so other solutions are $7\pi/3$, $11\pi/3$ (positive x-direction) and $-\pi/3$, $-5\pi/3$ (negative x-direction).

iii) The solutions of $\tan x = 1$ are $\boldsymbol{\pi/4}$ and $\boldsymbol{5\pi/4}$ in the given interval.
Further solutions outside that range can be obtained by adding or subtracting the period of the tangent function, namely π, to those solutions, so $9\pi/4$ and $-3\pi/4$ are two others.

In general, if $\sin x = a$, then :
$\mathbf{\sin(\pi - x) = a}$
$\sin(n\pi + x) = a$ for even integer n ; $\sin(n\pi - x) = a$ for odd integer n.
Therefore, for instance, $\sin(2\pi+x)$ $\sin(3\pi-x)$ and $\sin(-\pi-x)$ are also equal to $\sin x$.

In general, if $\cos x = a$, then :
$\mathbf{\cos(-x) = a}$ or $\mathbf{\cos(2\pi-x)° = a}$
$\cos(2n\pi + x) = a$ for any integer n
$\cos(2n\pi - x) = a$ for any integer n.

Therefore, for instance, $\cos(2\pi+x)$, $\cos(4\pi-x)$ and $\cos(-2\pi-x)$ are also equal to $\cos x$.

In general, if $\tan x = a$, then $\mathbf{\tan(n\pi + x) = a}$ for any integer n.

The following examples will be shown without graphs or CAST diagrams – the details are shown in the "Solving Trig Equations " at AS-Level.

Example (7): Solve $\sin x = -0.8$ for $0 \le x \le 2\pi$, giving answer in radians to three decimal places.

The principal value, and the one given on a calculator, is $\mathbf{-0.927^c}$, from which we can derive the other solutions. Note that this solution is not in the quoted range, and so we must add an appropriate multiple of 2π (the period of $\sin x$) to it.

Here, adding 2π gives one solution, i.e. $\mathbf{5.356^c}$.

To obtain the other solution, we use $\sin(\pi-x) = \sin x$ with the negative value of -0.927^c for the answer of $\mathbf{4.069^c}$.

Example (8): Solve tan x = 1.5 for $-\pi \leq x \leq \pi$, giving the answer in radians to 3 decimal places .

The principal value is $\mathbf{0.983^c}$, but because the tangent function repeats itself every π radians, there is another solution at $(0.983 - \pi)^c$ or $\mathbf{-2.159^c}$.

Example (9): Solve cos x = 0.75 for $0 \leq x \leq 2\pi$, giving answers in radians to 3 decimal places.

The principal value is $\mathbf{0.723^c}$ **;** to find the other solution, we can use the identity $\cos(2\pi - x) = \cos x$. Now, $2\pi - 0.723 = 5.560$, so the other solution of cos x = 0.75 for $0 \leq x \leq 2\pi$ is $\mathbf{5.560^c}$.

Example (10): Use the results from Example (9) to solve cos 2x = 0.75 for $0 \leq x \leq \pi$, giving your answers in radians to three decimal places.

Here the limits for x are $0 \leq x \leq \pi$, but we are asked to solve for 2x. To ensure that no values are omitted, we must substitute A for 2x and multiply the limiting values by 2 to get the transformed limit of $0 \leq A \leq 2\pi$.

From Example (9), we see that the solutions of cos A = 0.75 for $0 \leq A \leq 2\pi$ are 0.723^c and 5.560^c. To convert the A-values back to x-values, we must divide by 2.

$\therefore$ the solutions of cos 2x = 0.75 for $0 \leq x \leq \pi$ are $\mathbf{0.361^c}$ and $\mathbf{2.780^c}$.

Example (11): Solve $\tan(2x + (\pi/3)) = 1$ for $0 \leq x \leq 2\pi$, giving exact results in terms of π.

By letting A stand for $2x + (\pi/3)$, we first transform the x-limits to A-limits:

$0 \leq x \leq 2\pi \Rightarrow (\pi/3) \leq A \leq (13\pi/3)$ (x-values doubled, $\pi/3$ added).

The substitution back to x-values must be done afterwards.

The principal value of A where tan A = 1 is $\pi/4$, but that value is not within the A-limits.
We therefore keep adding multiples of π to obtain $A = 5\pi/4$, $9\pi/4$, $13\pi/4$ and $17\pi/4$.

Since we doubled the x-values and then added $\pi/3$ to get the A-values, we must perform the inverse operations to change the A-values back to x-values – i.e. subtract $\pi/3$ and then halve the result.

Therefore $A = 5\pi/4 \Rightarrow x = \frac{1}{2}((5\pi/4) - (\pi/3)) = 11\pi/24$.
Similarly $A = 9\pi/4 \Rightarrow x = \frac{1}{2}((9\pi/4) - (\pi/3)) = 23\pi/24$.
The other A-values of $13\pi/4$ and $17\pi/4$ transform to x-values of $35\pi/24$ and $47\pi/24$.

The solutions in the range are therefore x = $\mathbf{11\pi/24}$, $\mathbf{23\pi/24}$, $\mathbf{35\pi/24}$ and $\mathbf{47\pi/24}$.

Example (12): Solve $\sin(3x - (\pi/4)) = -0.3$ for $0 \le x \le \pi$, giving answers in radians to 3 decimal places.

Again, we must substitute the x-limits of $0 \le x \le \pi$ with A – limits. By using $A = 3x - (\pi/4)$), the A-limits transform to $-\pi/4 \le A \le 11\pi/4$.

The principal value of A is here -0.3047^c, which is inside the A-limits, as $-\pi/4 = -0.7854^c$.
The second value is $\pi - (-0.3047^c)$, or 3.4463^c.

To complete the full set of solutions for A in the required range, we try adding multiples of 2π to the two above values. In fact the only additional one is $A = (-0.3047 + 2\pi) = 5.9785^c$.

These three solutions will then need converting back from A-values to x-values by the inverse operation of

$x = \dfrac{A + (\pi/4)}{3}$, so when $A = -0.3047^c$, $x = 0.1602^c$

Similarly, when $A = 3.4463^c$, $x = 1.4106^c$ and when $A = 5.9785^c$, $x = 2.2546^c$.

Hence the solutions of $\sin(3x - (\pi/4)) = -0.3$ for $0 \le x \le \pi$ are $\mathbf{0.160^c}$, $\mathbf{1.411^c}$ and $\mathbf{2.255^c}$ to 3 d.p.

Quadratic equations in the trigonometric functions are solved in the same way, but care must be taken when factorising them.

Example (9) : Solve the equation $2\cos^2 x - \cos x = 0$ for $-\pi \le x \le \pi$.

This factorises at once into $(\cos x)(2\cos x - 1) = 0$.

$\cos x = 0$ when $x = \pi/2$ or $x = -\pi/2$.

$2\cos x = 1$, or $\cos x = 0.5$, when $x = \pi/3$ or $x = -\pi/3$.

The solutions to the equation are therefore $x = \pm\pi/2$ and $\pm\pi/3$ (illustrated below).

Important: we cannot simply cancel out cos x from the equation as follows:

$2\cos^2 x - \cos x = 0 \Rightarrow 2\cos^2 x = \cos x \Rightarrow 2\cos x = 1 \Rightarrow x = \pm\pi/3$.

The final division of both sides by cos x has led to a loss of the solutions satisfying $\cos x = 0$,
i.e. $x = \pm\pi/2$.

Example (10) : Solve the equation $\cos^2 x - \cos x - 1 = 0$ for $0 \le x \le 2\pi$. Give the answer in radians to three decimal places.

This quadratic does not factorise and so the general formula must be used.

Substitute $x = \cos x$, a = 1, $b = -1$ and $c = -1$ into the formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\cos x = \frac{1 \pm \sqrt{5}}{2} \ .$$

These are the possible solutions, and are 1.618 and -0.618 to 3 decimal places.

The first value can be rejected, since the cosine function cannot take values outside the range
$-1 \le \cos x \le 1$.

The only solutions are those where $\cos x = \dfrac{1-\sqrt{5}}{2}$. The principal value of x is 2.237^c .

Since $\cos(2\pi - x) = \cos x$, another solution would be $(6.283 - 2.237)^c$ or 4.046^c.

$\therefore$ the solutions of $\cos^2 x - \cos x - 1 = 0$ where $0 \le x \le 2\pi$, are $\mathbf{2.237^c}$ and $\mathbf{4.046^c}$ to 3 decimal places.

Example (11): Find the angle(s) between 0 and 2π satisfying the equation

$1 - 2\cos^2 x + \sin x = 0$

This equation can be manipulated into a quadratic in sin x by replacing $2\cos^2 x$ with $2(1 - \sin^2 x)$ using the Pythagorean identity.

$\Rightarrow 1 - 2(1 - \sin^2 x) + \sin x = 0$
$\Rightarrow 1 - 2 + 2\sin^2 x + \sin x = 0$
$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$

This is now a quadratic in sin x which factorises into $(2 \sin x - 1)(\sin x + 1) = 0$

Hence $\sin x = 0.5$ or -1.

This gives $x = \pi/6$ or $5\pi/6$ (where $\sin x = 0.5$); also $x = 3\pi/2$ (where $\sin x = -1$).

Summary. (The principal solution is the one displayed on the calculator.)

The rules are designed to find solutions in the range $-\pi \leq x < \pi$.
This range has the virtue of having the principal solution coincide with the calculator display.

If a different range is stipulated, it is only a matter of adding multiples of 2π to the solutions so obtained (when solving equations of the form $\sin x = k$ or $\cos x = k$), or multiples of π (when solving equations of the form $\tan x = k$). In each case, n is an integer.

Solving $\sin x = k$.

Value of k	**Principal soln.**	**Companion solution**	**Additional solutions**
0	$x = 0^c$	(none)	Add/subtract $n\pi$
1	$x = \pi/2$	(none)	Add/subtract $2n\pi$
-1	$x = -\pi/2$	(none)	Add/subtract $2n\pi$
positive	$0 < x < \pi/2$	$\pi - x$	Add/subtract $2n\pi$
negative	$-\pi/2 < x < 0$	$-\pi - x$	Add/subtract $2n\pi$

Solving $\cos x = k$.

Value of k	**Principal soln.**	**Companion solution**	**Additional solutions**
0	$x = \pi/2$	(none)	Add/subtract $n\pi$
1	$x = 0^c$	(none)	Add/subtract $2n\pi$
-1	$x = -\pi$	(none)	Add/subtract $2n\pi$
positive	$0 < x < \pi/2$	$-x$	Add/subtract $2n\pi$
negative	$\pi/2 < x < \pi$	$-x$	Add/subtract $2n\pi$

Solving $\tan x = k$.

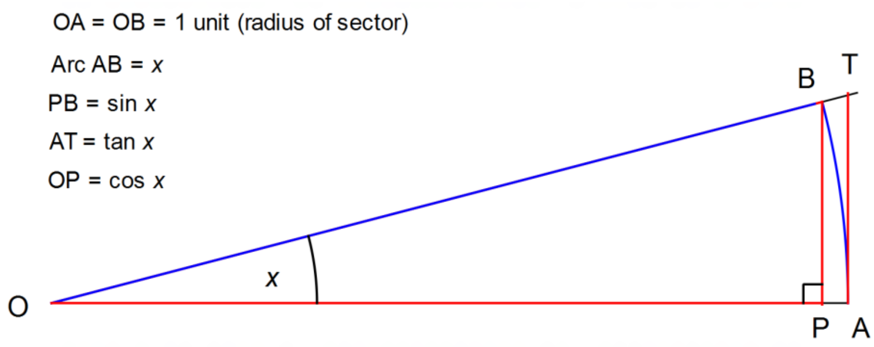
Value of k	**Principal soln.**	**Companion solution**	**Additional solutions**
0	$x = 0^c$	(none)	Add/subtract $n\pi$
positive	$0 < x < \pi/2$	(none)	Add/subtract $n\pi$
negative	$-\pi/2 < x < 0$	(none)	Add/subtract $n\pi$

Small angle approximations.

The diagram below shows a sector of a circle, of angle x radians, where *OA* and *OB* are unit radii.
By drawing a perpendicular from B, we describe a right-angled triangle *POB.*

Now $OP = \cos x$ and $BP = \sin x$.
The arc length $AB = x$.

The line *OT* is a tangent to the circle sector at point *A*, so $AT = \tan x$.



As can be seen from the diagram, the lengths *BP* and *AT* are not very different from that of the arc *AB*, even though the angle *AOB* is not particularly small at about 0.3 of a radian.

This angle was chosen for ease of illustration, but if x were to become smaller, the arc *AB* and the lines *BP* and *AT* would coincide even more closely.

If an angle x is small (say, about 0.1 of a radian), we can use approximations.

As the distance *PB* approximates to arc length *AB*, we can say that for small angles, $\mathbf{\sin x \approx x}$.
The distance *AT* also approximates to arc length *AB*, we can say that for small angles, $\mathbf{\tan x \approx x}$.

Additionally, the distance *OP* is only slightly less than the unit radius *OA*, so we can say that for small angles, $\cos x \approx 1$.

A better approximation for cos x, however, is $\mathbf{\cos x \approx 1 - \frac{1}{2}x^2}$.

By taking $\sin x \approx x$ and using the identity $\cos^2 x + \sin^2 x = 1$, then $\cos^2 x \approx 1 - x^2$ and $\cos x = \sqrt{(1-x^2)}$.
The transition from $\sqrt{(1-x^2)}$ to $1 - \frac{1}{2}x^2$ is a result of binomial expansion (see document "Binomial Series for Rational Powers").