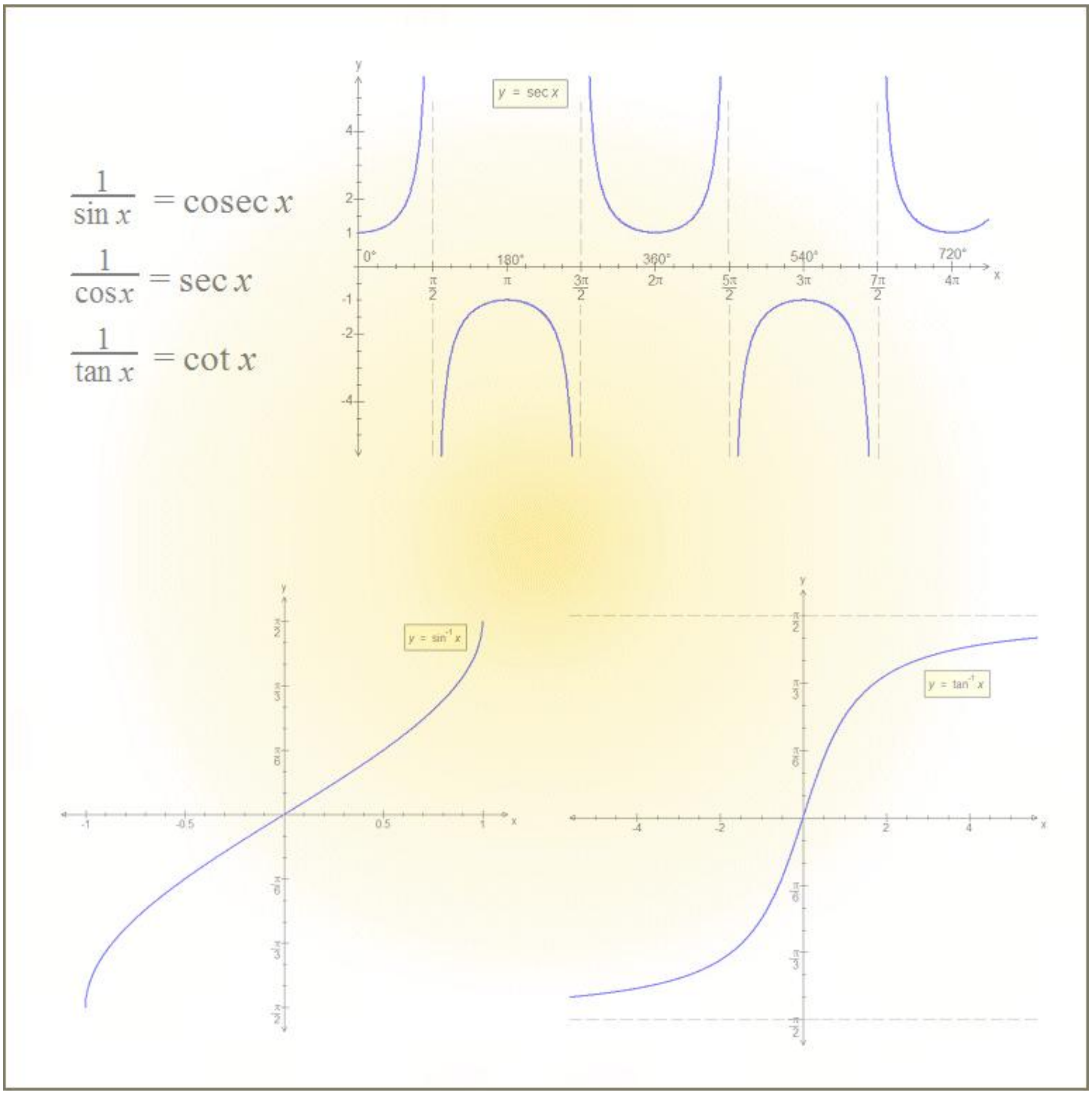
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

RECIPROCAL AND INVERSE TRIGONOMETRIC FUNCTIONS



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Reciprocal Trigonometric Functions.

The three trigonometric functions sin *x*, cos *x* and tan *x* each have a corresponding reciprocal function.

$\frac{1}{\sin x}$ = cosec *x* (the cosecant of *x*)

$\frac{1}{\cos x}$ = sec *x* (the secant of *x*)

$\frac{1}{\tan x}$ = cot *x* (the cotangent of *x*)

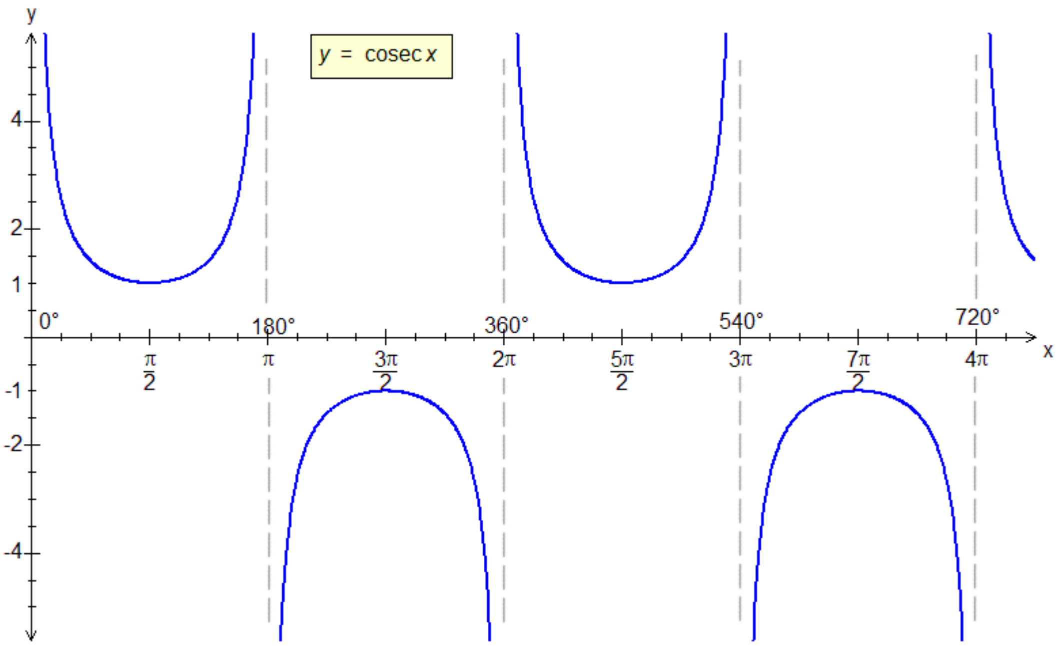
Never write these reciprocal functions as $\sin^{-1} x$, $\cos^{-1} x$ or $\tan^{-1} x$.
Those expressions are for the *inverse* functions.

The cosecant, cosec *x*.

The cosecant function has a period of 360° (2π in radians) to match that of the sine function.
It is undefined whenever sin *x* = 0, i.e. when *x* = 180*n*° or *n*π radians for integer *n*.

Like the graph of sin *x*, it has rotational symmetry of order 2 about the origin and all points on the *x*-axis at multiples of 180° (π radians) from it. It is also symmetrical through every line *x* = ½*n* × 180° (or ½*n*π in radians) where *n* is odd.

Note also that, whereas
$y = \sin x$ cannot take values *outside* the range $-1 \leq y \leq 1$,
$y = \text{cosec } x$ cannot take values *within* the range $-1 < y < 1$.



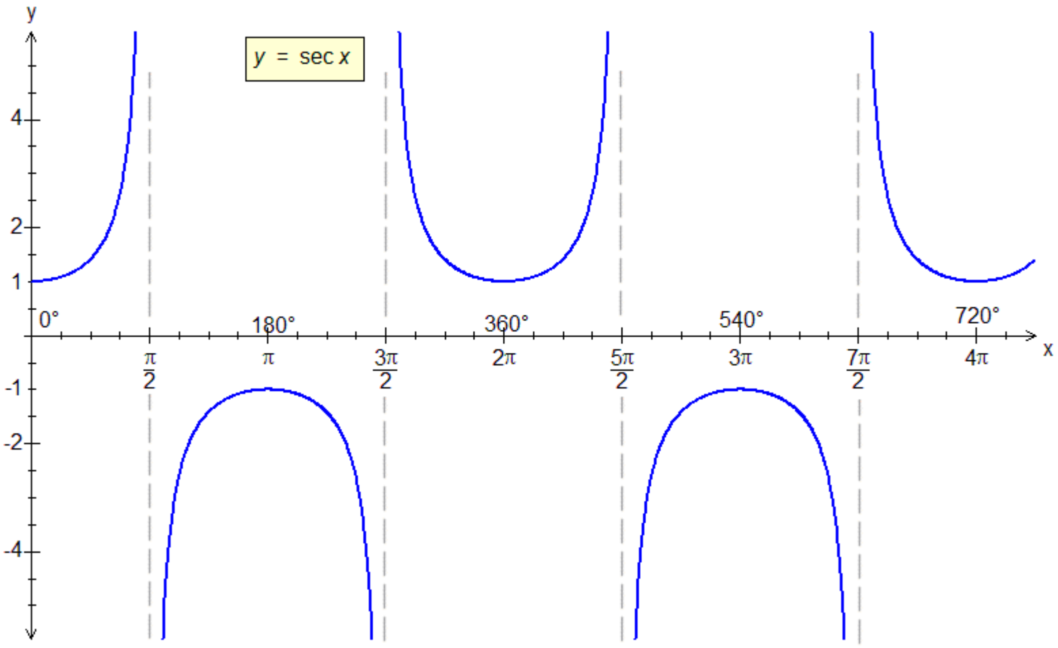
The secant, sec *x*.

The secant function has a period of 360° (2π in radians) to match that of the cosine function. Its graph is the same as that of cosec *x* translated by 90° (π/2 radians) to the left.

It is undefined whenever cos *x* = 0, i.e. when *x* = 90*n*° or ½*n*π radians for any odd integer *n*.

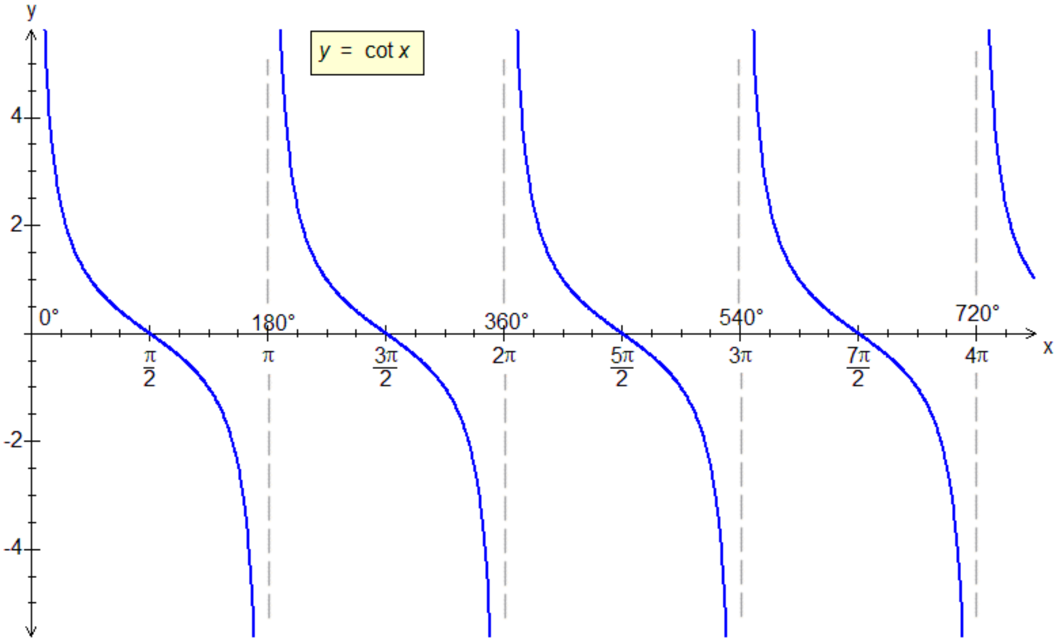
Like the graph of cos *x*, it has rotational symmetry of order 2 about the point (π/2,0) and all points on the *x*-axis at multiples of 180° (π radians) from it.
It is also symmetrical through every line *x* = 180*n*° (or *n*π radians) where *n* is an integer.

Again, just as *y* = cos *x* cannot take values *outside* the range –1 ≤ *y* ≤ 1,
y = sec *x* cannot take values *within* the range –1 < *y* < 1.



The cotangent, cot *x*.

The cotangent function has a period of 180° (π radians) to match that of the tangent function. Just as tan x is undefined when cos x = 0, cot x is undefined whenever sin x = 0. (cot x = $\frac{\cos x}{\sin x}$). Its graph is the same as that of tan x reflected in the line $x = \pi/4$ (45°) or any other line at a multiple of π (180°) from it. It also has rotational symmetry of order 2 about the origin and all points at multiples of $\pi/2$ (90°) from it.



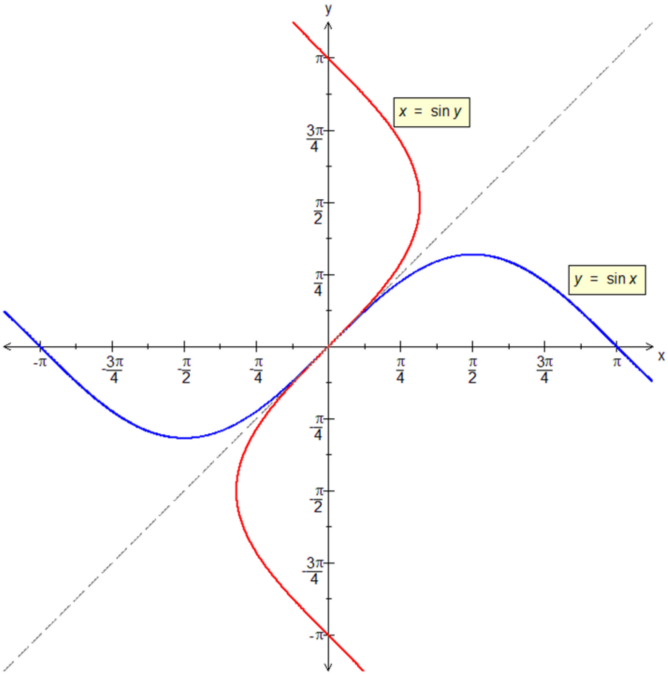
Inverse Trigonometric Functions.

If we were to take the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ and reflect them in the line $y = x$, we would obtain the graphs of $x = \sin y$, $x = \cos y$ and $x = \tan y$.

The resulting graphs are not of valid functions, because the mappings are not one-to-one.

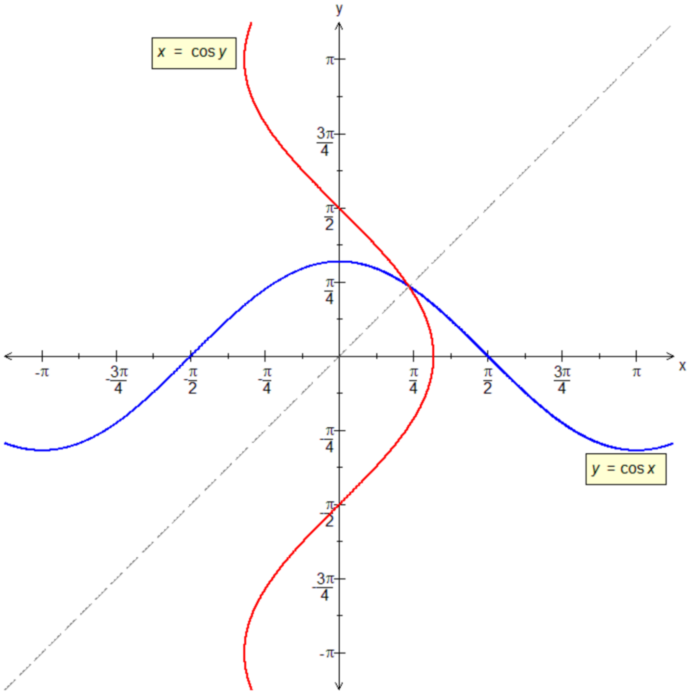
Take the graph of $x = \sin y$ on the right, where the domain of $y = \sin x$ is shown for $-\pi \le y \le \pi$.

An infinite number of angles can have a sine of 0.5; *one* such angle is ($\pi/6$), but there is also ($5\pi/6$) within the domain shown here, and an infinite number of others outside, such as ($13\pi/6$).



We can take the graph of $x = \cos y$ (right), where the domain of $y = \cos x$ is shown for $-\pi \le y \le \pi$.

Now, an angle of ($\pi/3$) has a cosine of 0.5, but then so does ($-\pi/3$) in the domain shown here. Again, there are infinitely many other angles having the same cosine outside the domain, such as ($5\pi/3$).

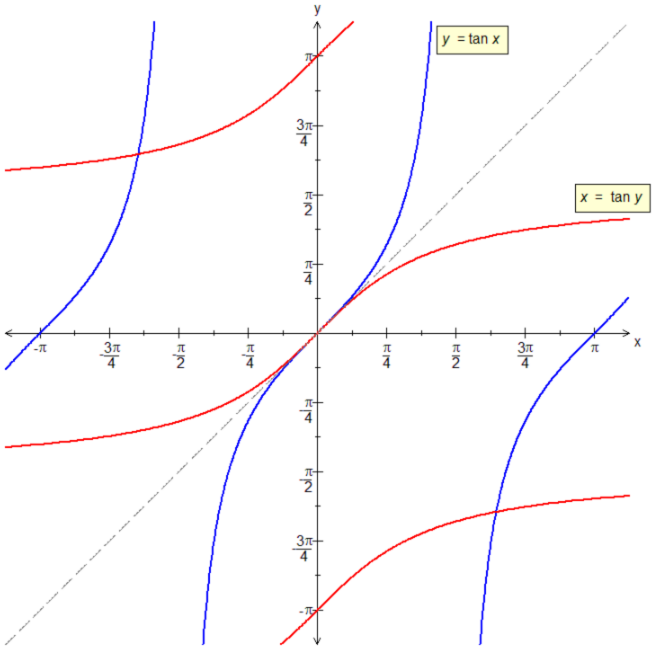


We can also take the graph of $x = \tan\ y$ (right), where the domain of $y = \tan x$ is shown for $-\pi \le y \le \pi$.

Note how vertical asymptotes on the graph of $y = \tan x$ have been transformed into horizontal asymptotes on the graph of $x = \tan\ y$.

The angle ($\pi/4$) has a tangent of 1, but then so does ($-3\pi/4$) (in domain shown here) , ($5\pi/4$) and infinitely many others.

It is therefore necessary to restrict the domains of the original trigonometric functions so that they would provide valid inverse functions.

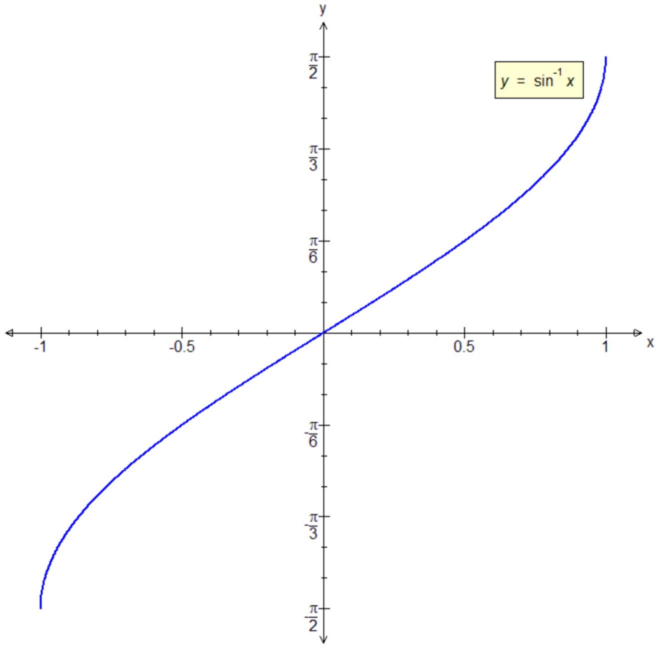


It is far more usual to employ radian measure when dealing with the inverse trigonometric functions, especially in calculus, and so the graphs are shown with angles in radians only.

Inverse sine function, $\sin^{-1} x$.

The sine function, $f(x) = \sin x$, needs to have its domain restricted to between $(-\pi/2)$ and $(\pi/2)$ inclusive so that its inverse would be a valid one-to-one function.

The inverse function ,
$y = f^{-1}(x) = \sin^{-1} x$,
has a domain $-1 \le x \le 1$
and a range of $(-\pi/2) \le y \le (\pi/2)$.

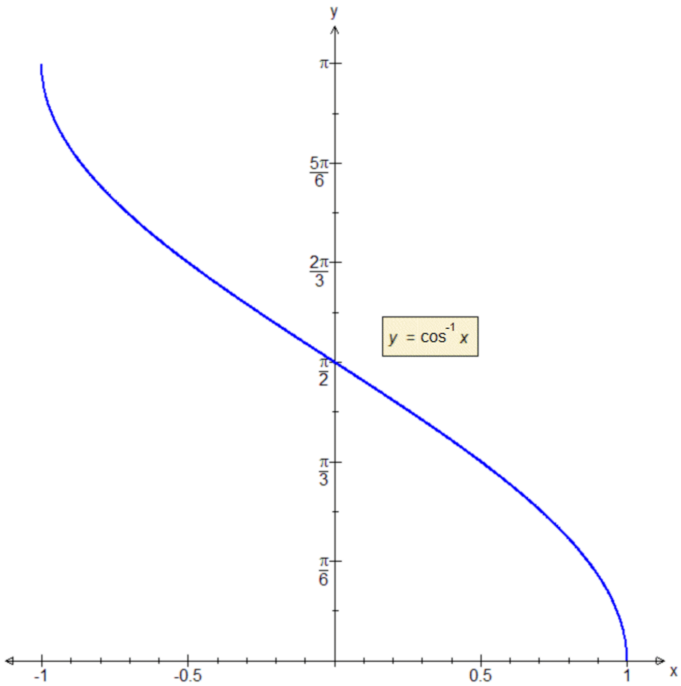


Inverse cosine function, $\cos^{-1} x$.

The cosine function $f(x) = \cos x$ would require a similar restriction to provide a valid inverse. This time the domain is restricted to between 0 and π inclusive.

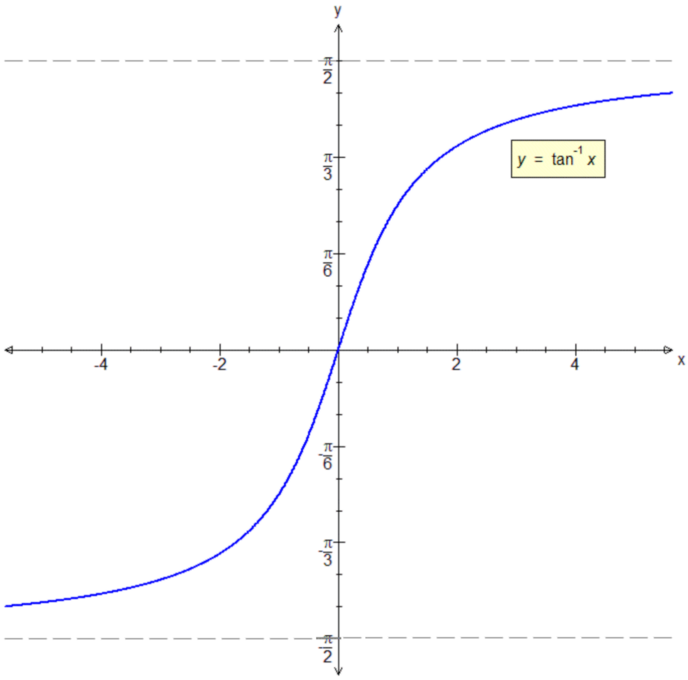
The inverse function ,

$y = f^{-1}(x) = \cos^{-1} x$,
has a domain $-1 \le x \le 1$
and a range of $0 \le y \le \pi$.



Inverse tangent function, $\tan^{-1} x$.

Just as the graph of the tangent function differs markedly from that of the other two functions, so does the graph of its inverse.



The domain of the tangent function $f(x) = \tan x$ would require x to be restricted between $(-\pi/2)$ and $(\pi/2)$ (not inclusive) so that its inverse would be a valid one-to-one function.

The inverse function, $y = f^{-1}(x) = \tan^{-1} x$, has a domain of all real x and a range of $(-\pi/2) < y < (\pi/2)$.

There are horizontal asymptotes at $y = (-\pi/2)$ and $y = (\pi/2)$.

Transformations of inverse trigonometric functions.

The graphs of the inverse trigonometric functions can also be transformed like those of other graphs, but range and domain restrictions must be also transformed in most cases.

Examples (1): Describe the following transformations, including details of any resulting changes to the range and /or domain:

i) $y = \sin^{-1} x$ to $y = \sin^{-1} (x - 1)$.

ii) $y = \cos^{-1} x$ to $y = \cos^{-1} (4x)$.

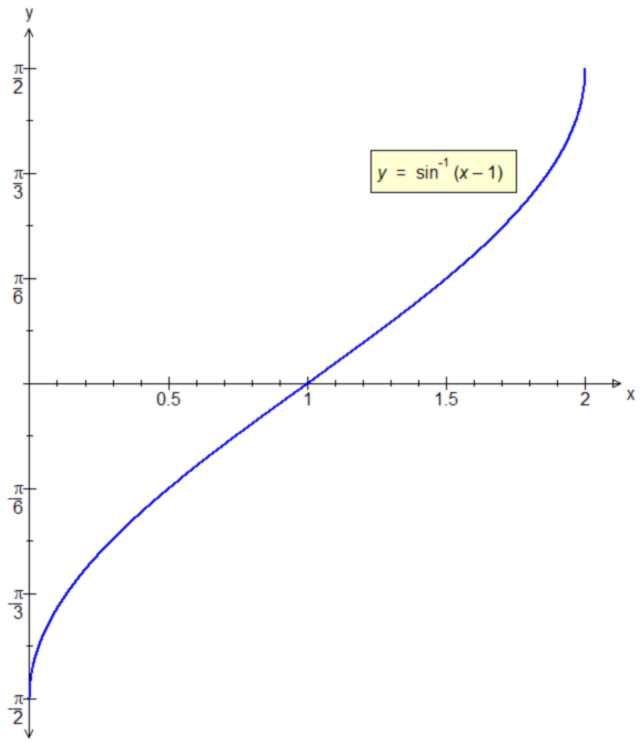
iii) $y = \sin^{-1} x$ to $y = 3 \sin^{-1} x$.

iv) $y = \tan^{-1} x$ to $y = \pi + \tan^{-1} x$.

i) The graph of $y = \sin^{-1} x$ is given an x-shift to the right by 1 unit, i.e. a translation by vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The domain of $y = \sin^{-1} (x - 1)$ is a translation of that of $y = \sin^{-1} x$ by 1 unit, namely to $0 \le x \le 2$.

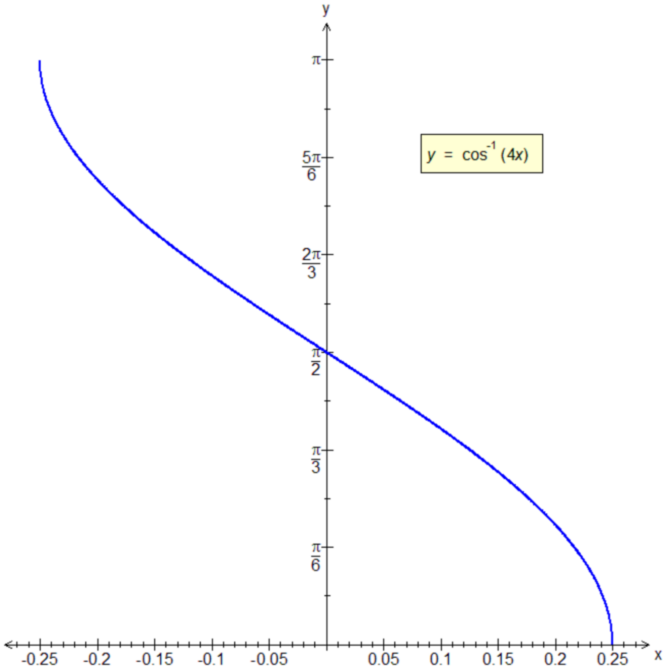
The range of $y = \sin^{-1} (x - 1)$ is the same as that of $y = \sin^{-1} x$, i.e. $(-\pi/2) \le y \le (\pi/2)$.



ii) Here, the graph of $y = \cos^{-1} x$ has an x-stretch with scale factor of $\frac{1}{4}$ applied to it.

This stretch also applies to the domain; it is transformed to $-\frac{1}{4} \leq x \leq \frac{1}{4}$.

The range of $y = \cos^{-1}(4x)$ is the same as that of $y = \cos^{-1} x$, i.e. still $0 \leq y \leq \pi$.

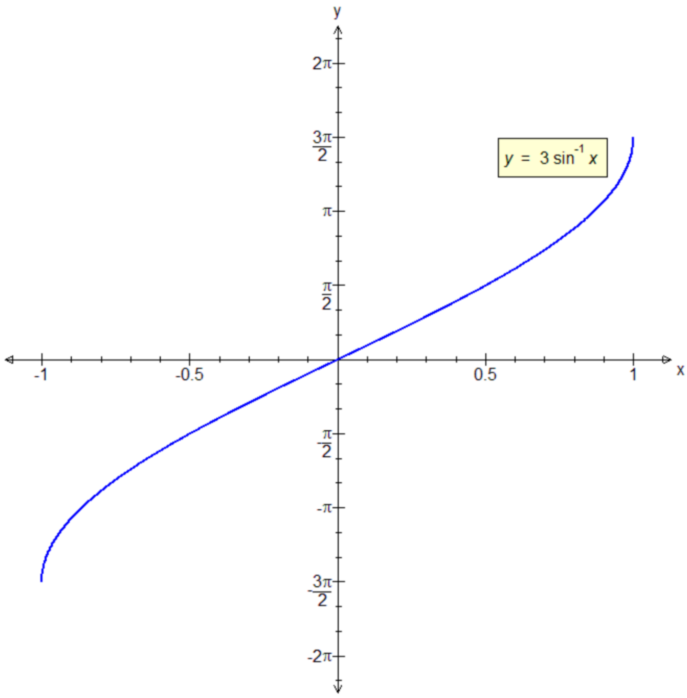


iii) In this example, the graph of $y = \sin^{-1} x$ is stretched in the y-direction with a scale factor of 3.

The domain of $y = 3 \sin^{-1} x$ is the same as that of $y = \sin^{-1} x$ – i.e. $-1 \leq x \leq 1$.

The range of $y = 3 \sin^{-1} x$ is that of $y = \sin^{-1} x$, but additionally stretched with the same scale factor.

In other words it is transformed to $(-3\pi/2) \leq y \leq (3\pi/2)$.

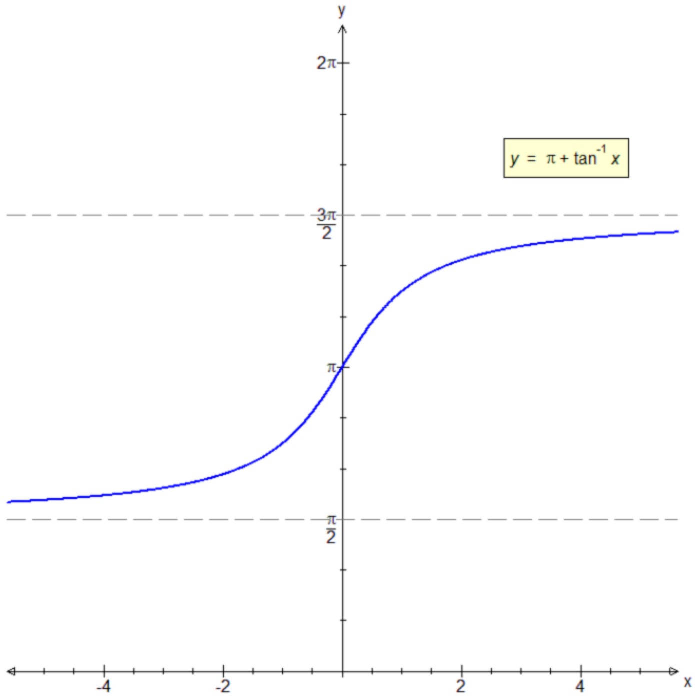


iv) This time, the graph of $y = \tan^{-1} x$ is translated by the vector $\begin{pmatrix} 0 \\ \pi \end{pmatrix}$.

The domain of $y = \pi + \tan^{-1} x$ is the same as that of $y = \tan^{-1} x$, i.e. all real x.

The range of $y = \pi + \tan^{-1} x$ is that of $y = \tan^{-1} x$, translated by π in the y-direction, namely to

$(\pi/2) \leq y \leq (3\pi/2)$.



The last four examples demonstrate the effects of the four single transformations on the domains and ranges of the original functions after transformation.

To recap the properties:

The *y*-translation:
For any function $y = f(x)$, the graph of the function $f(x) + k$ is the same as the graph of $f(x)$, but translated by k units in the y-direction. In column vector form, the transformation is $\begin{pmatrix} 0 \\ k \end{pmatrix}$.

The domain is unchanged after transformation, but the range is translated by k units.

The *x*-translation:
For any function $y = f(x)$, the graph of the function $f(x+k)$ is the same as the graph of $f(x)$, but translated by $-k$ units in the x-direction. In column vector form, the transformation is $\begin{pmatrix} -k \\ 0 \end{pmatrix}$.

The range is unchanged after transformation, but the domain is translated by $-k$ units.

The *y*-stretch:
In general, for any function $y = f(x)$, the graph of the function $kf(x)$ is the same as the graph of $f(x)$, but stretched by a factor of k in the y-direction.

The domain is unchanged after transformation, but the range is multiplied by a factor of k.

The *x*-stretch:
In general, for any function $y = f(x)$, the graph of the function $f(kx)$ is the same as the graph of $f(x)$, but stretched by a factor of $(1/k)$ in the x-direction.

The range is unchanged after transformation, but the domain is multiplied by a factor of $(1/k)$.

Note how the single transformations involving x affect the domain but not the range, but that the single transformations involving y affect the range but not the domain.