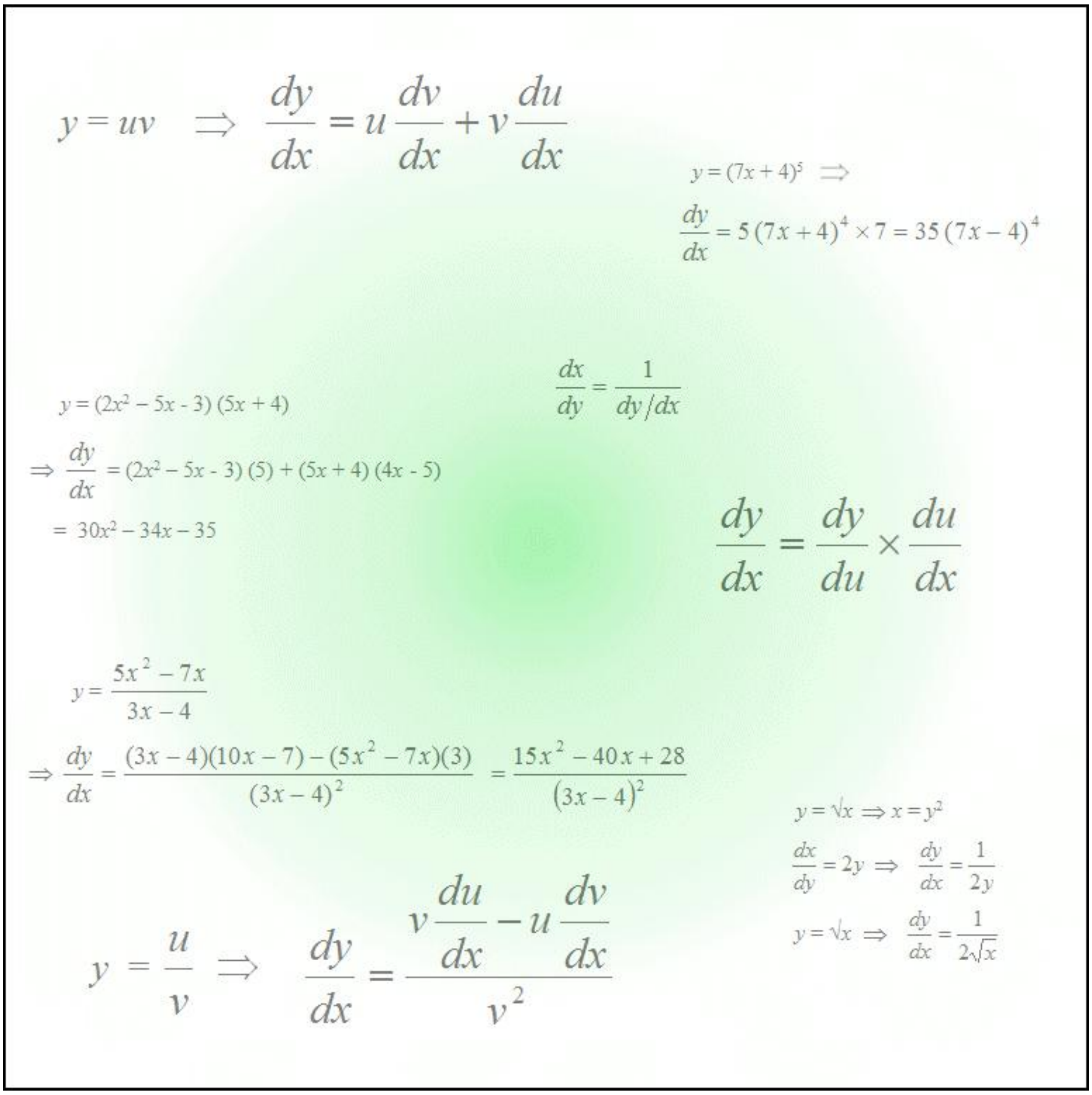
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

DIFFERENTIATION TECHNIQUES

CHAIN, PRODUCT & QUOTIENT RULES



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Differentiation Techniques.

At AS Level, we have dealt with differentiation of relatively simple functions.

Functions like $f(x) = (2x – 7)(x + 4)$ and $g(x) = \frac{x^4+1}{x^2}$ had to be manipulated or multiplied out in full to make them differentiable using AS-Level techniques.

A function like $h(x) = (3x^2 – 5)^6$ would have had to have been expanded using the binomial series and differentiated term by term - a long and rather painful process.

The Chain Rule or 'Function of a Function' rule.

Consider the function $y = (3x^2 – 5)^6$. How do we differentiate it without having to perform a messy binomial expansion ?

Starting with the variable x, we can see that there are two functions;

the inner function $3x^2 – 5$ (to apply to x) and the outer function x^6 (to apply to the result of the inner function of $3x^2 – 5$).

In function notation, it can be said that
$f(x) = 3x^2 – 5$; $g(x) = x^6$; $gf(x) = (3x^2 – 5)^6$.

Alternatively, $y = (3x^2 – 5)^6$ can be expressed as a two-stage function;

the intermediate function $u = 3x^2 – 5$ followed by $y = u^6$.

The derivative of the combined function can be evaluated by using the **chain rule**.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$, or, in function notation, $g(f(x))' = g'(f(x))\,f'(x)$.

More links are possible in the chain, such as in $\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dx}$, but such examples will not usually come up in exams !

Example (1): Use the chain rule to differentiate $y = (3x^2 – 5)^6$.

Let $u = 3x^2 – 5 \Rightarrow \frac{du}{dx} = 6x$

$y = u^6 \Rightarrow \frac{dy}{du} = 6u^5 = 6(3x^2 - 5)^5$. (Substitute $3x^2 – 5$ for u).

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 6(3x^2 - 5)^5 \times (6x) = 36x(3x^2 - 5)^5.$$

Example (2): Use the chain rule to differentiate $y = \sqrt{(8 - x^2)}$.

Here, $u = 8 - x^2 \Rightarrow \frac{du}{dx} = -2x$

$y = \sqrt{u} \Rightarrow \frac{dy}{du} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{(8-x^2)}}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{(8-x^2)}} \times (-2x) = \frac{-x}{\sqrt{(8-x^2)}}$$

With practice, the intermediate working can be done mentally.

Example (3): Use the chain rule to differentiate $y = (7x + 4)^5$, without showing the side working.

If you differentiate (‘thing’)^5 , you have 5(‘thing’)^4 × derivative of ‘thing’.

$$\frac{dy}{dx} = 5(7x+4)^4 \times 7 = 35(7x-4)^4$$

Notice how the multiplier of 35 came about; it is the original power of the expression (5) multiplied by the coefficient of the inner x-term in the brackets (7).

This holds true for all expressions of the type $(ax + b)^n$ where the inner function is linear in x:

$\frac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$. (This also includes fractional and negative n).

Examples (4): Without using side working, use the chain rule to find :

i) $\frac{d}{dx}(1-3x)^5$; ii) $\frac{d}{dx}\left(\frac{5}{(4x-3)^2}\right)$; iii) $\frac{d}{dx}\left(\sqrt{8x+5}\right)$.

i) $\frac{d}{dx}(1-3x)^5 = -15(1-3x)^4$.

ii) $\frac{5}{(4x-3)^2} = 5\,(4x-3)^{-2}$, and therefore $\frac{d}{dx}\left(5(4x-3)^{-2}\right) = -40(4x-3)^{-3}$ or $\frac{-40}{(4x-3)^3}$.

iii) $\sqrt{8x+5} = (8x+5)^{\frac{1}{2}}$, and so $\frac{d}{dx}(8x+5)^{\frac{1}{2}} = 4(8x+5)^{-\frac{1}{2}}$ or $\frac{4}{\sqrt{(8x+5)}}$.

The Product Rule.

At AS level, we differentiated $y = (2x - 7)(x + 4)$ by expanding it as $2x^2 + x - 28$, and then differentiating it to obtain $y' = 4x + 1$.

(y' is another way of writing $\frac{dy}{dx}$).

A product of two functions can be differentiated by using the following rule:

If $y = uv$, where u is a function $f(x)$ and v is another function $g(x)$, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

or in function notation, $(f(x)\,g(x))' = f(x)\,g'(x) + f'(x)\,g(x)$.

Example (5): Differentiate $y = (2x - 7)(x + 4)$ using the product rule.

Let $u = 2x - 7$ and $v = x + 4 \Rightarrow \frac{du}{dx} = 2$ and $\frac{dv}{dx} = 1$.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \Rightarrow \frac{dy}{dx} = (2x - 7)(1) + (x + 4)(2) = 2x - 7 + 2x + 8 = 4x + 1.$$

The derivative of the product is
"(first × derivative of second) + (second × derivative of first)".

Example (6): Differentiate $y = (2x^2 - 5x - 3)(5x + 4)$ using the product rule and simplify the result.

Let $u = 2x^2 - 5x - 3$ and $v = 5x + 4 \Rightarrow \frac{du}{dx} = 4x - 5$ and $\frac{dv}{dx} = 5$.

$$\Rightarrow \frac{dy}{dx} = (2x^2 - 5x - 3)(5) + (5x + 4)(4x - 5)$$

"(first × derivative of second) + (second × derivative of first)".

$$\Rightarrow \frac{dy}{dx} = 10x^2 - 25x - 15 + 20x^2 - 9x - 20 = 30x^2 - 34x - 35.$$

If the question does not ask for the result to be simplified, then the last two steps can be omitted.

Example (7): Differentiate $y = \dfrac{5x^2 - 7x}{3x - 4}$ using the product rule, simplifying the result.

Hence show that the graph of *y* has no turning points.

Although this result looks like a quotient rather than a product, we can redefine the expression as the product of $u = 5x^2 - 7x$ and $v = \dfrac{1}{3x-4}$.

From this, we work out $\dfrac{du}{dx} = 10x - 7$ and $\dfrac{dv}{dx} = \dfrac{-3}{(3x-4)^2}$ by the chain rule (working not shown).

Therefore:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \Rightarrow \frac{dy}{dx} = \frac{21x - 15x^2}{(3x-4)^2} + \frac{10x-7}{(3x-4)}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{21x - 15x^2}{(3x-4)^2} + \frac{(10x-7)(3x-4)}{(3x-4)^2}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{21x - 15x^2}{(3x-4)^2} + \frac{30x^2 - 61x + 28}{(3x-4)^2}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{15x^2 - 40x + 28}{(3x-4)^2}.$$

For the graph of *y* to have turning points, the gradient must equal zero for some value(s) of *x*.

The quadratic numerator of the derived function, however, has a discriminant of $(-40)^2 - (4 \times 15 \times 28)$, or -80, which is negative, implying no real roots, i.e. no zero gradient anywhere.

The Quotient Rule.

At AS level, we differentiated $y = \frac{x^4+1}{x^2}$ by rewriting it as $\frac{x^4+1}{x^2} = x^2 + \frac{1}{x^2}$, to obtain the result $y' = 2x - \frac{2}{x^3}$.

If $y = \frac{u}{v}$, where u is a function $f(x)$ and v is another function $g(x)$, then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}, \text{ or } \frac{\text{“bottom} \times \text{derivative of top”} - \text{“top} \times \text{derivative of bottom”}}{(\text{“bottom”})^2}$$

or in function notation, $\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$.

Example (7): Differentiate $y = \frac{x^4+1}{x^2}$ using the quotient rule, and simplify the result.

Let $u = x^4 + 1$ and $v = x^2$ $\Rightarrow$ $\frac{du}{dx} = 4x^3$ and $\frac{dv}{dx} = 2x$.

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \Rightarrow \frac{dy}{dx} = \frac{(x^2(4x^3)) - ((x^4+1)(2x))}{x^4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^5 - 2x^5 - 2x}{x^4} = \frac{2(x^5 - x)}{x^4} = 2\left(x - \frac{1}{x^3}\right)$$

Again, the last steps can be omitted if the question does not ask for the result to be simplified.

Example (8): Differentiate $y = \dfrac{3x^2+1}{x-2}$ using the quotient rule, and simplify the result.

Let $u = 3x^2 + 1$ and $v = x - 2$ $\Rightarrow \dfrac{du}{dx} = 6x$ and $\dfrac{dv}{dx} = 1$.

The order of the work is: "square the bottom line of the expression" to give the bottom line of the result, then write the "bottom × derivative of top" – "top × derivative of bottom" on the top line of the result.

$$\frac{dy}{dx} = \frac{(x-2)(6x)-(3x^2+1)(1)}{(x-2)^2} \text{ i.e. } \frac{\text{"bottom} \times \text{derivative of top"} - \text{"top} \times \text{derivative of bottom"}}{(\text{"bottom"})^2}$$

Simplifying the top line gives

$$\frac{dy}{dx} = \frac{3x^2-12x-1}{(x-2)^2}.$$

Example (9): Differentiate $y = \dfrac{5x^2-7x}{3x-4}$, using the quotient rule and simplify the result.
(This is a repeat of Example (6)).

Let $u = 5x^2 - 7x$ and $v = 3x - 4$ $\Rightarrow \dfrac{du}{dx} = 10x - 7$ and $\dfrac{dv}{dx} = 3$.

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \Rightarrow \frac{dy}{dx} = \frac{(3x-4)(10x-7)-(5x^2-7x)(3)}{(3x-4)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(30x^2-61x+28)-(15x^2-21x)}{(3x-4)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{15x^2-40x+28}{(3x-4)^2}.$$

Using the result $\frac{dx}{dy} = \frac{1}{dy/dx}$

This is another result that is useful for differentiating inverse functions, especially inverse trigonometric functions to be discussed in later study.

Example (10): Differentiate $y = \sqrt{x}$ by using $\frac{dx}{dy} = \frac{1}{dy/dx}$ and find its gradient at the point (25, 5).

How is the result related to the gradient of $y = x^2$ at the point (5, 25) ?

First, rewrite the expression as $x = y^2$

Differentiation gives $\frac{dx}{dy} = 2y$ and therefore $\frac{dy}{dx} = \frac{1}{2y}$.

Finally, we must rewrite the derivative in terms of x, so substituting $\sqrt{x}$ for y gives

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$. This is equivalent to $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$.

The gradient of $y = \sqrt{x}$ at (25, 5) is $\frac{dy}{dx} = \frac{1}{2y} = \frac{1}{10}$, whilst the gradient of $y = x^2$ at the point (5, 25) is $2x$ or 10. The product of the gradients is 1.

(Do not confuse this with the rule for perpendicular lines, whose gradients have a product of –1).

If the graph of $f(x)$ passes through the point (p, q) and its gradient at that point is equal to m, then the gradient of the graph of the inverse function $f^{-1}(x)$ is equal to $\frac{1}{m}$ at the point (q, p).

More on the chain rule – related rates of change.

The chain rule can also be used to establish results for related rates of change.

Example (11): Crude oil leaking out of a tanker produces a circular slick with radius *r* km and area *A* km^2. The radius of the slick increases with time at a rate given by $\frac{dr}{dt}$ = 0.5 km/h.

Find the related rate of change of area of the slick, $\frac{dA}{dt}$, and also the rate of increase of the area of the slick at a time when the radius is 4 km.

The area of a circle is given by $A = \pi r^2$. The change in area with respect to the change in radius is therefore $\frac{dA}{dr} = 2\pi r$.

By the chain rule, $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$. Here it is $2\pi r \times 0.5$ or πr.

$\therefore$ At the time when the radius of the slick is 4km, the rate of increase in the slick's area is 4π km^2/h or about 12.6 km^2/h.

Example (12): During a tunnel-building project, it is found that the volume V of the excavated earth is related to the height h of the resulting pile by the formula

$$V = \sqrt{\left(h^6 + h - 2\right)}$$

i) Find the value of $\frac{dV}{dh}$ at the instant when the pile of earth is 2 m high.
ii) The volume of the pile of excavated earth is increasing at a constant rate of 10 m^3 per hour.
Find the rate (per hour) at which the height of the pile is increasing at the instant the pile is 2 m high.
Give the final result to the nearest centimetre.

i) By the chain rule, $u = h^6 + h - 2$ and $V = \sqrt{u}$.

Therefore $\frac{du}{dh} = 6h^5 + 1$ and $\frac{dV}{du} = \frac{1}{2\sqrt{u}}$. (or $\frac{dV}{du} = \frac{1}{2}u^{-\frac{1}{2}}$).

Hence $\frac{dV}{dh} = \frac{dV}{du} \times \frac{du}{dh} \Rightarrow \frac{dV}{dh} = \frac{6h^5 + 1}{2\sqrt{(h^6 + h - 2)}}$.

Substituting $h = 2$ into the above derivative gives

$$\frac{dV}{dh} = \frac{6(32) + 1}{2\sqrt{(64 + 2 - 2)}} \Rightarrow \frac{dV}{dh} = \frac{193}{16}.$$

ii) From the given data, we deduce that the change (per hour) in volume of the earth, or $\frac{dV}{dt} = 10$.

We want to find the rate of change of height when the pile is 2 m high, or $\frac{dh}{dt}$, when $h = 2$.

Now $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ by the chain rule, so at $h = 2$, from (i), $\frac{dh}{dt} = \frac{16}{193} \times 10$ or 0.829.

Note that $\frac{dh}{dV} = \frac{1}{dV/dh}$.

$\therefore$ At the instant when the pile of excavated earth is 2 m high, the height is increasing at a rate of 83 cm per hour.