M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

PARAMETRIC DIFFERENTIATION

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$x = 8t,\ y = 2t^3$$

$$\frac{dx}{dt} = 8 \quad \frac{dy}{dt} = 6t^2 \Rightarrow \frac{dy}{dx} = \frac{6t^2}{8} \Rightarrow \frac{dy}{dx} = \frac{3t^2}{4}$$

$$x = 3t^2,\ y = 2t^3 + 6t \Rightarrow \frac{dx}{dt} = 6t,\ \frac{dy}{dt} = 6t^2 + 6$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt},\ \frac{dy}{dx} = \frac{6t^2+6}{6t} \Rightarrow \frac{dy}{dx} = t + \frac{1}{t} \quad \therefore \text{When } t = 5,\ \frac{dy}{dx} = 5\tfrac{1}{5}$$

$$x = 2\sin\theta,\ y = \cos\theta \Rightarrow \frac{dx}{d\theta} = 2\cos\theta,\ \frac{dy}{d\theta} = -\sin\theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta},\ \frac{dy}{dx} = \frac{-\sin\theta}{2\cos\theta} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}\tan\theta \quad \therefore \text{When } \theta = \pi/4,\ \frac{dy}{dx} = -\tfrac{1}{2}$$

$$x = t^3,\ y = 5t^2 - 10$$

$$\Rightarrow \frac{dx}{dt} = 3t^2,\ \frac{dy}{dt} = 10t \Rightarrow \frac{dy}{dx} = \frac{10t}{3t^2} \Rightarrow \frac{dy}{dx} = \frac{10}{3t}$$

When $x = 8,\ t = 2,\ y = 10,\ \frac{dy}{dx} = \frac{5}{3} \Rightarrow$ gradient of normal to $(8, 10) = -\frac{3}{5}$

equation of normal to $(8, 10)$ is $y - 10 = -\frac{3}{5}(x - 8)$

$\Rightarrow 5y - 50 = -3x + 24 \Rightarrow$ $\mathbf{3x + 5y - 74 = 0}$

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Parametric Differentiation.

If x and y are expressed in terms of another variable t (a **parameter**) ,then by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \text{ which can be rearranged as } \frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \text{since } \frac{dx}{dt} = \frac{1}{dt/dx}.$$

Example (1): A curve is defined parametrically by $x = 3t^2$, $y = 2t^3 + 6t$.

Find $\frac{dy}{dx}$ in terms of t, and give its value when $t = 5$.

Differentiating with respect to t gives $\frac{dx}{dt} = 6t$ and $\frac{dy}{dt} = 6t^2 + 6$.

Using $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, $\frac{dy}{dx} = \frac{6t^2 + 6}{6t} \Rightarrow \frac{dy}{dx} = t + \frac{1}{t}$.

$\therefore$ When $t = 5$, $\frac{dy}{dx} = 5\frac{1}{5}$.

Example (2): A curve is defined parametrically by $x = 2 \sin \theta$, $y = \cos \theta$.

(Trigonometric functions use θ as a parameter rather than t).

Find $\frac{dy}{dx}$ in terms of θ, and give its value when $\theta = \pi/4$.

Differentiating with respect to θ gives $\frac{dx}{d\theta} = 2\cos\theta$ and $\frac{dy}{d\theta} = -\sin\theta$.

Using $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$, $\frac{dy}{dx} = \frac{-\sin\theta}{2\cos\theta} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}\tan\theta$.

$\therefore$ When $\theta = \pi/4$, $\frac{dy}{dx} = -\frac{1}{2}$.

Example (3): The parametric equation of a curve is $x = \cos \theta$, $y = 3 \cos 2\theta$, $0 < \theta < \pi$.

Find $\frac{dy}{dx}$ in terms of θ, and hence find the maximum value that the gradient can take at any point on the curve.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \Rightarrow \frac{dy}{dx} = \frac{-6\sin 2\theta}{-\sin\theta} \Rightarrow \frac{dy}{dx} = \frac{12\sin\theta\cos\theta}{\sin\theta} \Rightarrow \frac{dy}{dx} = 12\cos\theta$$

(We can cancel $\sin \theta$ out as a factor in the last step, since it is never zero when $0 < \theta < \pi$).

Since $\cos \theta$ takes a maximum value of 1, the gradient must take a maximum value of 12.

Examination questions usually include revision of AS-level techniques such as maxima, minima, tangents and normals.

Example (4): A curve is defined parametrically by $x = 8t$, $y = 2t^3$.
Find $\frac{dy}{dx}$ in terms of t and therefore find the equation of the tangent at the point (16, 16).

Differentiating with respect to t gives $\frac{dx}{dt} = 8$ and $\frac{dy}{dt} = 6t^2$.

$\Rightarrow \frac{dy}{dx} = \frac{6t^2}{8} \Rightarrow \frac{dy}{dx} = \frac{3t^2}{4}$ by the chain rule.

When $y = 16$, $t = 2$, therefore the gradient of the tangent to the point (16, 16) is 3 and its equation is $y - 16 = 3(x - 16)$, $\Rightarrow$ $\mathbf{y = 3x - 32}$ or $3x - 48 - y + 16 = 0$ $\Rightarrow$ $\mathbf{3x - y - 32 = 0.}$

Example (5): A curve is defined parametrically by $x = t^3$, $y = 5t^2 - 10$.
Find $\frac{dy}{dx}$ in terms of t and therefore find the equation of the normal at the point (8, 10).

Differentiating with respect to t gives $\frac{dx}{dt} = 3t^2$ and $\frac{dy}{dt} = 10t$.

$\Rightarrow \frac{dy}{dx} = \frac{10t}{3t^2} \Rightarrow \frac{dy}{dx} = \frac{10}{3t}$.

When $x = 8$, $t = 2$, and so the gradient of the tangent is $\frac{5}{3}$.
The gradient of the normal to the point (8, 10) is thus $-\frac{3}{5}$.

Its equation is therefore $y - 10 = -\frac{3}{5}(x - 8)$
$\Rightarrow 5y - 50 = -3x + 24 \Rightarrow$ $\mathbf{3x + 5y - 74 = 0}$,

or $\mathbf{y = \frac{74}{5} - \frac{3}{5}x.}$

Parametric Second Derivatives (Not all syllabuses)

Example (6): Extension to Example (4)

A curve is defined parametrically by $x = 8t$, $y = 2t^3$.

(We need the following results from Example (3): $\frac{dy}{dx} = \frac{3t^2}{4}$, $\frac{dx}{dt} = 6t^2$).

Find the value of $\frac{d^2y}{dx^2}$ when $t = 4$.

To find the second derivative **with respect to *x***, we must use the chain rule again.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{3t^2}{4}\right)$$

$$= \frac{d}{dt}\left(\frac{3t^2}{4}\right) \times \frac{dt}{dx}$$

This is important – we cannot just differentiate with respect to *t* and leave it as $\frac{3t}{2}$.

Given $\frac{dx}{dt} = 6t^2$ from Example (5) , it follows that $\frac{dt}{dx} = \frac{1}{6t^2}$.

Therefore $\frac{d^2y}{dx^2} = \frac{3t}{2} \times \frac{1}{6t^2} = \frac{3t}{12t^2} = \frac{1}{4t}$.

When $t = 4$, $\frac{d^2y}{dx^2} = \frac{1}{16}$.