

M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

IMPLICIT DIFFERENTIATION

$$\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \times \frac{dy}{dx} \qquad \frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \times \frac{dy}{dx} = 2y \frac{dy}{dx}$$

$$\frac{d}{dx}(2x^2 + y^2) = \frac{d}{dx}(2x^2) + \frac{d}{dx}(y^2) \Rightarrow \frac{d}{dx}(2x^2 + y^2) = 4x + 2y \frac{dy}{dx}$$

$$y^2 + 6x = x^2$$

$$\Rightarrow \frac{d}{dx}(y^2) + \frac{d}{dx}(6x) = \frac{d}{dx}(x^2) \Rightarrow 2y \frac{dy}{dx} + 6 = 2x \Rightarrow 2y \frac{dy}{dx} = 2x - 6 \Rightarrow \frac{dy}{dx} = \frac{x-3}{y}$$

$$xy^2 + 5x + 20 = 8xy$$

$$\Rightarrow \frac{d}{dx}(xy^2) + \frac{d}{dx}(5x + 20) = \frac{d}{dx}(8xy) \Rightarrow y^2 + 2xy \frac{dy}{dx} + 5 = 8y + 8x \frac{dy}{dx}$$

$$\Rightarrow 2xy \frac{dy}{dx} = 8y + 8x \frac{dy}{dx} - y^2 - 5 \Rightarrow (2xy - 8x) \frac{dy}{dx} = 8y - y^2 - 5 \Rightarrow \frac{dy}{dx} = \frac{y(8-y) - 5}{2x(y-4)}$$

$$\frac{d}{dx}(xy^2) = \left(\frac{d}{dx}(x) \times (y^2) \right) + \left((x) \times \frac{d}{dx}(y^2) \right) \Rightarrow \frac{d}{dx}(xy^2) = y^2 + 2xy \frac{dy}{dx}$$

$$\frac{d}{dx}(2xy) = \left(\frac{d}{dx}(2x) \times (y) \right) + \left((2x) \times \frac{d}{dx}(y) \right) \Rightarrow \frac{d}{dx}(2xy) = 2y + 2x \frac{dy}{dx}$$

IMPLICIT DIFFERENTIATION

A function involving x and y is said to be **explicit** if y is clearly defined as a function of x alone.

$y = 2x^2 + 3x - 5$ is an example of an explicit function; y is on the LHS, and the RHS is expressed fully in terms of x .

$y^2 + 3xy = 3x^2$ and $x^2 + y^2 = 16$, on the other hand, are **implicit** functions involving x and y .

Sometimes an implicit equation in x and y can be rearranged into an explicit one by separating the variables.

Example (1): Differentiate $xy = 3$ with respect to x .

The equation can be rearranged as $y = \frac{3}{x}$, and from there we can directly work out $\frac{dy}{dx} = \frac{-3}{x^2}$.

Example (2): Differentiate $x^2 - 2y^2 = 7$ with respect to x .

Again we can separate the variables: $x^2 - 2y^2 = 7 \Rightarrow x^2 - 7 = 2y^2$

$$\Rightarrow y^2 = \frac{x^2 - 7}{2} \Rightarrow y = \sqrt{\frac{x^2 - 7}{2}}$$

Having obtained y explicitly in terms of x , we can differentiate using the chain rule.

$$\text{Let } u = \frac{x^2 - 7}{2} \Rightarrow \frac{dy}{du} = \frac{1}{2\sqrt{u}} \text{ and } \frac{du}{dx} = x, \Rightarrow \frac{dy}{dx} = \frac{x}{2\sqrt{\left(\frac{x^2 - 7}{2}\right)}}$$

When an equation in x and y is given implicitly and the variables cannot be separated, we find $\frac{dy}{dx}$ by differentiating each term with respect to x . We must also remember that

$$\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \times \frac{dy}{dx} \text{ by the chain rule.}$$

Example (3): Differentiate $2x^2 + y^2$ with respect to x .

Here there is no right-hand side to the expression, and so we cannot separate the variables as in the first two examples, and so the result will be given in terms of x , y and $\frac{dy}{dx}$.

$$\frac{d}{dx}(2x^2 + y^2) = \frac{d}{dx}(2x^2) + \frac{d}{dx}(y^2) \Rightarrow \frac{d}{dx}(2x^2 + y^2) = 4x + 2y \frac{dy}{dx}.$$

Note the use of the chain rule in when differentiating y^2 with respect to x .

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \times \frac{dy}{dx} = 2y \frac{dy}{dx}.$$

Example (4): Differentiate $2xy$ with respect to x .

We need to use the chain rule and the product rule here.

$$\frac{d}{dx}(2xy) = \left(\frac{d}{dx}(2x) \times (y) \right) + \left((2x) \times \frac{d}{dx}(y) \right) \Rightarrow \frac{d}{dx}(2xy) = 2y + 2x \frac{dy}{dx}.$$

Example (5): Differentiate xy^2 with respect to x .

Again, we need to use the chain rule and the product rule.

$$\frac{d}{dx}(xy^2) = \left(\frac{d}{dx}(x) \times (y^2) \right) + \left((x) \times \frac{d}{dx}(y^2) \right) \Rightarrow \frac{d}{dx}(xy^2) = y^2 + 2xy \frac{dy}{dx}.$$

Example (6): Find $\frac{dy}{dx}$ in terms of x and y if $y^2 + 6x = x^2$.

This time, there is a little more algebra involved, where we need to bring the $\frac{dy}{dx}$ term over to the LHS.

Differentiating each term with respect to x we have

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(6x) = \frac{d}{dx}(x^2).$$

$$\Rightarrow 2y \frac{dy}{dx} + 6 = 2x \Rightarrow 2y \frac{dy}{dx} = 2x - 6 \Rightarrow \frac{dy}{dx} = \frac{x-3}{y}.$$

Example (7): Find the gradient of the tangent to the curve $x^2 - 2y^2 = 7$ (from Example 2) at the point $(5, 3)$ by implicit methods.

We could work out the gradient at $x = 5$ by substituting into the explicit derivative

$$\frac{dy}{dx} = \frac{x}{2\sqrt{\left(\frac{x^2-7}{2}\right)}} \quad (\text{from Ex. 2}) \Rightarrow \frac{dy}{dx} = \frac{5}{2\sqrt{9}} = \frac{5}{6}.$$

The question asks for implicit methods, so rewriting as $x^2 - 2y^2 - 7 = 0$ and differentiating term by term will produce an expression for $\frac{dy}{dx}$ in terms of x and y .

$$\frac{d}{dx}(x^2 - 2y^2 - 7) = \frac{d}{dx}(x^2) - \frac{d}{dx}(2y^2) = 2x - 4y \frac{dy}{dx}.$$

Rearranging $= 2x - 4y \frac{dy}{dx} = 0$ to make $\frac{dy}{dx}$ the subject, we have $4y \frac{dy}{dx} = 2x$

and finally $\frac{dy}{dx} = \frac{x}{2y}$.

Substituting $x = 5$ and $y = 3$ gives a gradient of $\frac{5}{6}$ at that point.

Example (8): Find $\frac{dy}{dx}$ in terms of x and y if $xy^2 + 5x + 20 = 8xy$.

This example is a little trickier, since here we need to use the product rule to differentiate xy^2 and $8xy$.

$$\frac{d}{dx}(xy^2) = \left(\frac{d}{dx}(x) \times (y^2) \right) + \left((x) \times \frac{d}{dx}(y^2) \right) \Rightarrow \frac{d}{dx}(xy^2) = y^2 + 2xy \frac{dy}{dx}.$$

$$\frac{d}{dx}(8xy) = \left(\frac{d}{dx}(8x) \times (y) \right) + \left((8x) \times \frac{d}{dx}(y) \right) \Rightarrow \frac{d}{dx}(8xy) = 8y + 8x \frac{dy}{dx}.$$

Having worked those two products out, we evaluate the whole derivative term by term and rearrange to make $\frac{dy}{dx}$ the subject.

$$\frac{d}{dx}(xy^2) + \frac{d}{dx}(5x + 20) = \frac{d}{dx}(8xy)$$

$$\Rightarrow y^2 + 2xy \frac{dy}{dx} + 5 = 8y + 8x \frac{dy}{dx}$$

$$\Rightarrow 2xy \frac{dy}{dx} = 8y + 8x \frac{dy}{dx} - y^2 - 5$$

$$\Rightarrow (2xy - 8x) \frac{dy}{dx} = 8y - y^2 - 5$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(8 - y) - 5}{2x(y - 4)}$$

Examination questions usually include revision of AS-level techniques such as maxima, minima, tangents and normals.

Example (9): (Extension to Example (8)).

Find the equation of i) the tangent to the curve $xy^2 + 5x + 20 = 8xy$ at the point $(-4, 0)$; ii) the normal to the same curve at the point $(2, 5)$.

Part i) Substitute $(x, y) = (-4, 0)$ into the expression

$\frac{dy}{dx} = \frac{y(8-y)-5}{2x(y-4)}$ obtained in the previous question to obtain the gradient of the tangent at that

point. The gradient at $(-4, 0)$ is $\frac{dy}{dx} = \frac{-5}{(-8)(-4)} = -\frac{5}{32}$.

The equation of the tangent to the curve, in the form $y = mx + c$, is

$$y - 0 = -\frac{5}{32}(x + 4) \Rightarrow y = -\frac{5}{32}x - \frac{5}{8}.$$

(In the form $ax + by + c = 0$, the equation is $32y = -5x - 20 \Rightarrow 5x + 32y + 20 = 0$.)

Part ii) Substitute $(x, y) = (2, 5)$ into the expression for $\frac{dy}{dx}$ in i) to obtain the gradient of the tangent at that point.

This gradient is $\frac{5(8-5)-5}{4(5-4)} = \frac{10}{4} = \frac{5}{2}$, so the gradient of the normal at $(2, 5)$ is $-\frac{2}{5}$.

The equation of the normal to the curve, in the form $y = mx + c$, is

$$y - 5 = -\frac{2}{5}(x - 2) \Rightarrow y = 5 - \frac{2}{5}x + \frac{4}{5} \Rightarrow y = 5\frac{4}{5} - \frac{2}{5}x.$$

(In the form $ax + by + c = 0$, the equation is $5y - 25 = -2x + 4 \Rightarrow 2x + 5y - 29 = 0$.)