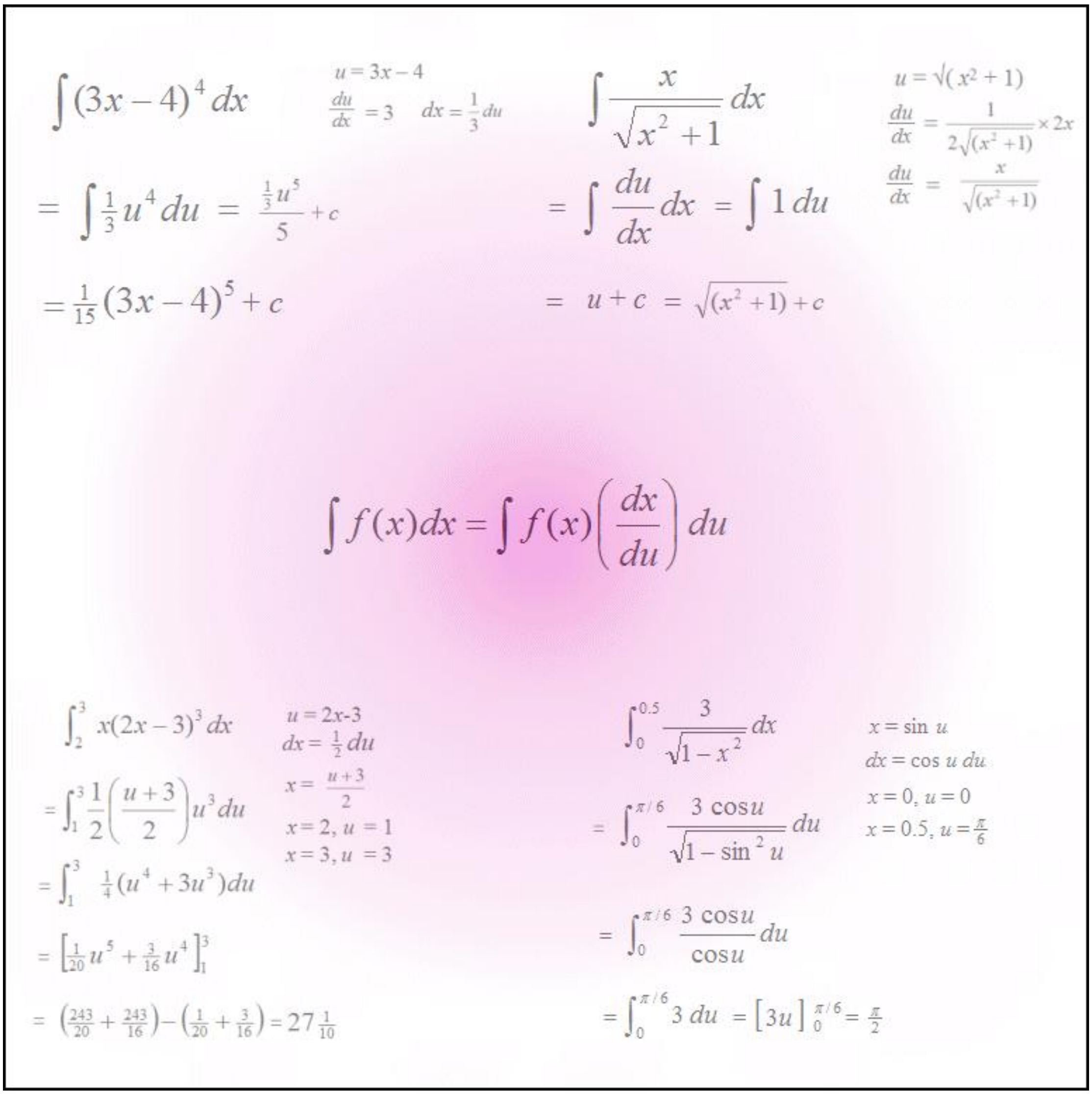
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

INTEGRATION BY SUBSTITUTION



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INTEGRATION BY SUBSTITUTION

This technique derives from the 'function of a function' rule or chain rule for differentiation.

Previous sections on integration techniques often used the method of inspecting an integral to look for a function of a function, multiplied by its derivative. From there, an educated guess was used, and then the trial integral differentiated and adjusted by a scale factor.

We also used set results such as $\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1}$ + c or

$\int f'(x)(f(x))^n\,dx = \frac{1}{n+1}(f(x))^n + c$ to evaluate such integrals.

Integration by substitution is the formal method for evaluating such integrals, as well as many others.

The method is to transform the integral with respect to one variable, *x*, into an integral with respect to another variable, *u*.

$$\int f(x)dx = \int f(x)\left(\frac{dx}{du}\right)du\ .$$

Although '*du*' and '*dx*' are meaningless when separated, the method treats them as actual algebraic expressions, to be manipulated as such.

Example (1): Find $\int (3x-4)^4\,dx$, using the substitution $u = 3x - 4$.

The informal method would have been to guess at the integral to be something like $(3x - 4)^5 + c$ and then differentiating that guess.

By using the chain rule, the derivative of $(3x - 4)^5$ is in fact $3 \times 5(3x - 4)^4$, which is 15 times the required integral. By scaling this guess, we obtain the true result, i.e. $\frac{1}{15}(3x-4)^5 + c$.

The more formal method of using the substitution $u = 3x - 4$ works as follows:

$\frac{du}{dx} = 3$ here, so we replace dx by $\frac{dx}{du}du$, namely $\frac{1}{3}du$, to redefine the integrand as $\int \frac{1}{3}u^4\,du$.

(Remember that $\frac{du}{dx}$ and $\frac{dx}{du}$ are reciprocals of each other !)

This works out as $\frac{\frac{1}{3}u^5}{5} + c$, and rewriting in terms of x gives $\frac{1}{15}(3x-4)^5 + c$.

Example (1a): Find $\int_2^3 (3x-4)^4\,dx$ from the result in Example (1).

The value of the integral is $\int_2^3 (3x-4)^4\,dx = \left[\frac{1}{15}(3x-4)^5\right]_2^3$

$= \frac{1}{15}((9-4)^5 - (6-4)^5)$ = $\frac{1}{15}(5^5 - 2^5) = \frac{3093}{15}$ = 206.2.

The last example illustrates an important point when using substitution for evaluating definite integrals. The final part of the calculation could have been simplified by transforming the limits.

Because $u = 3x - 4$, we can transform the x-limits to u-limits by substitution.
When $x = 2$, $u = 2$ and when $x = 3$, $u = 5$.

The original integrand would have been rewritten as $\int_2^5 \frac{1}{3}u^4\,du$, giving

$\left[\frac{1}{15}u^5\right]_2^5$ = $\frac{1}{15}(5^5 - 2^5)$ = $\frac{3093}{15}$ = 206.2 .

When dealing with definite integrals using substitution, it is often better to transform the *x*-limits to *u*-limits and evaluate the integral in terms of *u*.

Example (2): The velocity v of a performance car (in m/s) is modelled by the equation

$$v = 120 - \frac{(t-60)^4}{108000}.$$

Find the distance travelled by the car in 60 seconds.

The required integrand is $\int_0^{60} 120 - \frac{(t-60)^4}{108000}\,dt$.

Let $u = t - 60 \Rightarrow \frac{du}{dt} = 1$ so $dt = du$.

Also, when $t = 0$, $u = -60$, and when $t = 60$, $u = 0$.

We redefine the integrand as $\int_{-60}^{0} 120 - \frac{u^4}{108000}\,du$

$$= \left[120u - \frac{u^5}{540000}\right]_{-60}^{0} = (0) - \left(-7200 - \frac{(-60)^5}{540000}\right) = 7200 - 1440 = 5760$$

The car has therefore travelled a distance of **5760 metres.**

Example (3): Find $\int \sin^2 x \cos x \, dx.$

This integral could again have been integrated by inspection. We have a power of an expression (here the square of sin (x), multiplied by its derivative.

The formal method uses the substitution $u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow \cos x \, dx = du$.

Replacing $\cos x \, dx$ by du enables us to redefine the integrand as $\int u^2 du$ which works out as $\frac{u^3}{3}$

$= \frac{1}{3}\sin^3 x + c$.

Example (4): Find $\int 10xe^{x^2} dx.$

Using $u = x^2 \rightarrow \frac{du}{dx} = 2x \Rightarrow 10x \, dx = 5 \, du$, we redefine the integrand as

$\int 5e^u du$ which works out as $5e^u$ and finally $5e^{x^2} + c$.

Again, note the occurrence of a product of the function x^2 and a multiple of its derivative, $2x$.

Example (5): Find $\int \frac{2x+3}{x^2+3x-5} dx$.

Using $u = x^2 + 3x - 5 \Rightarrow \frac{du}{dx} = 2x + 3$. Replacing $2x + 3 \, dx$ with du redefines the integrand as $\int \frac{1}{u} du$.

This integrates to $\ln(| x^2 + 3x - 5 |) + c$ or $\ln(| A(x^2 + 3x - 5)|)$ where $c = \ln A$.

(Remember the modulus function for integrals leading to a logarithm – also note the alternative method of incorporating the constant of integration.)

Note that the numerator of the integrand is the exact derivative of the denominator.

Example (6): Find $\int_0^1 \frac{x+1}{\left(2x^2+4x+5\right)^2}\, dx$.

Here we have a definite integral, so we can change the *x*-limits to *u*-limits, and then use the latter to calculate the result.

Let $u = 2x^2 + 4x + 5 \Rightarrow \frac{du}{dx} = 4x + 4$.

Also, when $x = 0$, $u = 5$, and when $x = 1$, $u = 11$.

We replace $x + 1\ dx$ with $\frac{1}{4}du$ and use the transformed limits to redefine the integrand as

.

$$\int_5^{11} \frac{1}{4u^2}\, du = \left[\frac{-1}{4u}\right]_5^{11} = \left(\frac{-1}{44}\right) - \left(\frac{-1}{20}\right) = \frac{3}{110}.$$

Example (7): Find $\int 3x\sqrt{(1+x^2)}\, dx$.

Again, we have a product of x^2 and a multiple of its derivative, $2x$.

Using $u = 1 + x^2 \Rightarrow \frac{du}{dx} = 2x \rightarrow 2x\, dx = du$.

This simplifies the integrand to $\int \frac{3}{2}\sqrt{u}\, du = \int \frac{3}{2}u^{1/2}\, du = \frac{3}{2} \times \frac{u^{\frac{3}{2}}}{\frac{3}{2}} = u^{\frac{3}{2}} + c = \left(1+x^2\right)^{\frac{3}{2}} + c.$

The examples above were all of the form $\int f'(x)(f(x))^n\, dx = \frac{1}{n+1}(f(x))^n + c$

or $\int \frac{f'(x)}{f(x)}\, dx = \ln(|\ f(x)\ |) + c$, and were therefore integrable by inspection or by substitution.

When the method of inspection does not find an obvious product of a function and its derivative, then substitution is necessary.

Example (8): Find $\int_2^3 x(2x-3)^3\,dx.$

Here, x is not a derivative of $2x$–3, and so we cannot integrate the expression by inspection.

Incidentally, don't even think about $\int_2^3 x(2x-3)^3\,dx = x\int_2^3 (2x-3)^3\,dx$.

(You can bring a *constant* multiplier outside the integral sign, but certainly not a variable !)

A possible substitution is $u = 2x-3$, $\Rightarrow \frac{du}{dx} = 2 \Rightarrow dx = \frac{1}{2}du$.

As we are evaluating a definite integral, we can also transform the x-limits to u-limits.
When $x = 2$, $u = 1$; when $x = 3$, $u = 3$.

We also need to rewrite x in the original expression as a function of u, namely $\frac{u+3}{2}$.

The integrand then becomes $\int_1^3 \frac{1}{2}\left(\frac{u+3}{2}\right)u^3\,du$ or $\int_1^3 \frac{1}{4}(u^4+3u^3)\,du$.

This is equal to $\left[\frac{1}{20}u^5 + \frac{3}{16}u^4\right]_1^3 = \left(\frac{243}{20}+\frac{243}{16}\right)-\left(\frac{1}{20}+\frac{3}{16}\right)$
$= \frac{242}{20}+\frac{240}{16}$ or $27\frac{1}{10}$.

Sometimes different substitutions can be used, all giving rise to the same result.

Example(9): Find $\int \frac{x}{\sqrt{x^2+1}}\,dx$ using the substitutions i) $u = x^2 + 1$; ii) $u = \sqrt{(x^2 + 1)}$.

In Part i), $\frac{du}{dx} = 2x \Rightarrow 2x\,dx = du$.

This gives an integrand of $\int \frac{1}{2\sqrt{u}}\,du = \int \frac{1}{2}u^{-\frac{1}{2}}\,du$, which integrates to $u^{\frac{1}{2}} + c$, or $\sqrt{u} + c$, and finally $\sqrt{(x^2+1)} + c$.

In Part ii), we need the chain rule to work out $\frac{du}{dx}$; it is $\frac{1}{2\sqrt{(x^2+1)}} \times 2x$, or $\frac{x}{\sqrt{(x^2+1)}}$.

Although this looks more messy, we now have the case where $\frac{du}{dx}$ is identical to the original integrand.

This original integrand dissolves into $\int \frac{du}{dx}\,dx \Rightarrow \int 1\,du$,

integrating to $u + c$ and finally $\sqrt{(x^2+1)} + c$.

Example(10): Find $\int \frac{x}{\sqrt{x-2}}\,dx$ using substitutions i) $u = x - 2$; ii) $u = \sqrt{(x - 2)}$.

In Part i), $\frac{du}{dx} = 1 \rightarrow dx = du$.

This becomes $\int \frac{u+2}{\sqrt{u}}\,du \;\Rightarrow\; \int u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\,du$, integrating to

$$\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + 2\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \;=\; \frac{2}{3}u^{\frac{3}{2}} + 4u^{\frac{1}{2}} + c \;=\; \frac{2}{3}\left(u^{\frac{3}{2}} + 6u^{\frac{1}{2}}\right) + c \;=\; \frac{2}{3}(u+6)u^{\frac{1}{2}} + c$$, and finally

$$\frac{2}{3}(x+4)\sqrt{(x+2)} + c.$$

In Part ii), $u^2 = x - 2$ and thus $x = u^2 + 2$.

Also $\frac{dx}{du} = 2u$ (much easier to work out than $\frac{du}{dx}$) $\Rightarrow$ $dx = 2u\,du$.

The integrand simplifies to

$$\int \frac{x}{\sqrt{x-2}}\,dx = \int \frac{(u^2+2)(2u)}{u}\,du = \int 2u^2 + 4\,du$$

$$= \tfrac{2}{3}u^3 + 4u + c = \tfrac{2}{3}u(u^2+6) + c.$$

Replacing u by $\sqrt{(x - 2)}$ throughout gives a final integral of

$$\tfrac{2}{3}\sqrt{(x-2)}(x-2+6) + c \text{ or } \tfrac{2}{3}\sqrt{(x-2)}(x+4) + c.$$

Examination questions will usually quote a suitable substitution.

Sometimes it might be more convenient to substitute *x* as a function of *u*, as in part ii) of the previous example. This is particularly useful for inverse trigonometric functions.

Example (11): Find $\int_0^{0.5} \frac{3}{\sqrt{1-x^2}}\,dx$ by using the substitution $x = \sin u$.

$\frac{dx}{du} = \cos u \Rightarrow dx = \cos u\,du$.

This is a definite integral, so we can transform the limits without the need to convert back from *u* to *x*. When $x = 0$, $u = 0$; when $x = 0.5$, $u = \frac{\pi}{6}$.

The integrand becomes $\int_0^{\pi/6} \frac{3\cos u}{\sqrt{1-\sin^2 u}}\,du = \int_0^{\pi/6} \frac{3\cos u}{\cos u}\,du$

$= \int_0^{\pi/6} 3\,du = \left[3u\right]_0^{\pi/6} = \frac{\pi}{2}$.

Example (12): Find $\int_0^{1} \frac{8}{1+x^2}\,dx$ by using the substitution $x = \tan u$.

$\frac{dx}{du} = \sec^2 u \Rightarrow dx = \sec^2 u\,du$.

When $x = 0$, $u = 0$; when $x = 1$, $u = \frac{\pi}{4}$.

This reduces the integrand to $\int_0^{\pi/4} \frac{8\sec^2 u}{1+\tan^2 u}\,du = \int_0^{\pi/4} \frac{8\sec^2 u}{\sec^2 u}\,du$

$= \int_0^{\pi/4} 8\,du = \left[8u\right]_0^{\pi/4} = 2\pi$.

Example (13): Find $\int \frac{-2}{\sqrt{1-9x^2}}\,dx$ using the substitution $x = \frac{1}{3}\sin u$.

$\frac{dx}{du} = \frac{1}{3}\cos u \Rightarrow dx = \frac{1}{3}\cos u \; du$; $u = \sin^{-1}(3x)$

The integrand becomes $\int \frac{-2\cos u}{3(\sqrt{1-\sin^2 u})}\,du = \int \frac{-2\cos u}{3\cos u}\,du = \int \frac{-2}{3}\,du$

$= -\frac{2}{3}u \Rightarrow -\frac{2}{3}\sin^{-1}(3x) + c.$

Example (14): Find $\int \frac{4}{25+x^2}\,dx$ using the substitution $x = 5\tan u$.

$\frac{dx}{du} = 5\sec^2 u \Rightarrow dx = 5\sec^2 u \; du$; $u = \tan^{-1}(x/5)$.

The integrand becomes $\int \frac{20\sec^2 u}{25+25\tan^2 u}\,du = \int \frac{20\sec^2 u}{25\sec^2 u}\,du = \int \frac{4}{5}\,du$

$= \frac{4}{5}u = \frac{4}{5}\tan^{-1}\left(\frac{x}{5}\right) + c.$