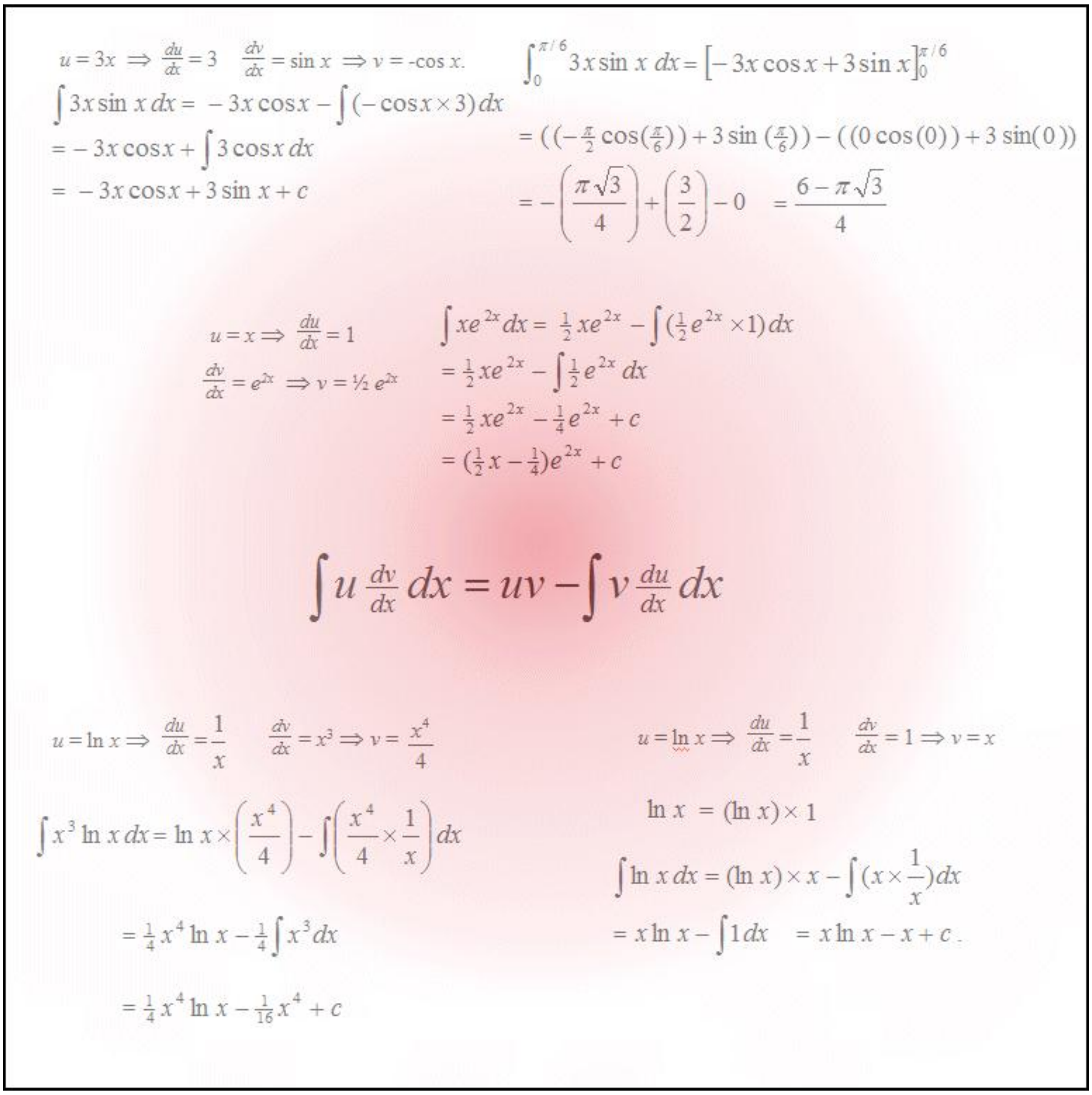
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

INTEGRATION BY PARTS



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INTEGRATION BY PARTS

This technique is a ‘reverse result’ of the product rule for differentiation.

Recall: If $y = uv$, where u and v are functions of x, then

$$\frac{dy}{dx} = \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} .$$

Integrating with respect to x , we have $uv = \int u \frac{dv}{dx}\,dx + \int v \frac{du}{dx}\,dx$.

Rearranging, we have the standard formula for integration by parts, i.e.

$\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$, or, in function notation,

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx .$$

In order to be able to use this method, the function $\frac{dv}{dx}$ must be integrable.
Additionally, the product $v\frac{du}{dx}$ should not be more difficult to integrate than the original.

Generally, if one of the factors of the product is a polynomial, then set u to be the polynomial, because its degree will be reduced by 1 by each application of the process.

Example (1): Find i) $\int 3x\sin x\,dx$ ii) $\int_0^{\pi/6} 3x\sin x\,dx$, leaving the answer as a surd in terms of π.

i) Take $u = 3x \Rightarrow \frac{du}{dx} = 3$. Take $\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$.

$\therefore \int 3x\sin x\,dx = -3x\cos x - \int(-\cos x \times 3)dx$

$= -3x\cos x + \int 3\cos x\,dx$

$= -3x\cos x + 3\sin x + c$

ii) The definite integral is worked out in the usual way:

$\int_0^{\pi/6} 3x\sin x\,dx = \left[-3x\cos x + 3\sin x\right]_0^{\pi/6}$

$= ((-\frac{\pi}{2}\cos(\frac{\pi}{6})) + 3\sin(\frac{\pi}{6})) - ((0\cos(0)) + 3\sin(0))$

$= -\left(\frac{\pi\sqrt{3}}{4}\right) + \left(\frac{3}{2}\right) - 0$

$= \frac{6 - \pi\sqrt{3}}{4}$.

Example (2): Find i) $\int xe^{2x}dx$ ii) $\int_1^2 xe^{2x}dx$, leaving the result in terms of e.

i) Let $u = x \Rightarrow \frac{du}{dx} = 1$. Let $\frac{dv}{dx} = e^{2x} \Rightarrow v = \frac{1}{2}e^{2x}$

$\therefore \int xe^{2x}dx = \frac{1}{2}xe^{2x} - \int (\frac{1}{2}e^{2x} \times 1)dx$

$= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx$

$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$

$= (\frac{1}{2}x - \frac{1}{4})e^{2x} + c$

ii) The definite integral is $\left[(\frac{1}{2}x - \frac{1}{4})e^{2x}\right]_1^2 = \frac{3}{4}e^4 - \frac{1}{4}e^2 = \frac{e^2(3e^2-1)}{4}$.

Sometimes the process might need to be carried out more than once.

Example (3): Find $\int x^2 \cos x\,dx$.

Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x$.

Let $\frac{dv}{dx} = \cos x \Rightarrow v = \sin x$.

$\therefore \int x^2 \cos x\,dx = x^2 \sin x - \int 2x \sin x\,dx$

The integrand on the right-hand side needs another application of the process, but note how the degree of the polynomial (the u function) has been reduced from 2 (quadratic) to 1 (linear).

Let $u = 2x \Rightarrow \frac{du}{dx} = 2$.

Let $\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$.

$\therefore \int 2x \sin x\,dx = -2x\cos x + \int 2\cos x\,dx$

$= -2x\cos x + 2\sin x + c$.

The two results need combining to give the final integral:
(watch out for the + and – signs !)

$\int x^2 \cos x\,dx = x^2 \sin x - \left(-2x\cos x + 2\sin x\right)$

$= x^2 \sin x + 2x\cos x - 2\sin x + c$.

Example (4):

i) Show that $\int x^2 e^{-x} dx = -e^{-x}(x^2 - 2x + 2) + c$, given that $\int xe^{-x} dx = -e^{-x}(x+1) + c$.

ii) Use the result from i) to find $\int_0^2 x^3 e^{-x} dx$, giving your answer in the form $a - be^{-2}$ where a and b are integer constants.

i) Two applications are needed here.

First application: Take $u = x^2 \Rightarrow \frac{du}{dx} = 2x$. Take $\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$.

$\therefore \int x^2 e^{-x} dx = -x^2 e^{-x} - \int -2(xe^{-x}) dx$. Label the right-hand integrand I_1.

Second application: This integrand I_1 is (-2) times the one quoted in part i),

so the integral of $I_1 = 2e^{-x}(x+1) + c$.

$\therefore \int x^2 e^{-x} dx = -x^2 e^{-x} - 2e^{-x}(x+1) + c = e^{-x}(-x^2 - 2x - 2) + c$

$= -e^{-x}(x^2 + 2x + 2) + c$.

ii) To find $\int_0^2 x^3 e^{-x} dx$, we work out the indefinite integral first :

Take $u = x^3 \Rightarrow \frac{du}{dx} = 3x^2$. Take $\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$.

$\therefore \int x^3 e^{-x} dx = -x^3 e^{-x} - \int -3(x^2 e^{-x}) dx$. Label the right-hand integrand I_2.

The integrand I_2 on the RHS is (-3) times the integrand $\int x^2 e^{-x} dx$, so its integral is $3e^{-x}(x^2 + 2x + 2) + c$.

$\therefore$ The complete integral $\int x^3 e^{-x} dx = -x^3 e^{-x} - 3e^{-x}(x^2 + 2x + 2) + c$

$= -x^3 e^{-x} - 3e^{-x}(x^2 + 2x + 2) + c = -e^{-x}(x^3 + 3x^2 + 6x + 6) + c$.

Hence $\int_0^2 x^3 e^{-x} dx = \left[-e^{-x}(x^3 + 3x^2 + 6x + 6)\right]_0^2$

$= \left[-e^{-2}(8 + 12 + 12 + 6)\right] - \left[-e^{0}(0 + 0 + 0 + 6)\right] = 6 - 38e^{-2}$.

Integrals involving products of ln x need to be handled a little differently, since ln x is not easy to integrate, but it can be differentiated. In those cases, ln x should be taken as u.

Example (5): Find $\int x^3 \ln x \, dx$.

Take $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$.

Take $\frac{dv}{dx} = x^3 \Rightarrow v = \frac{x^4}{4}$

$$\therefore \int x^3 \ln x \, dx = \ln x \times \left(\frac{x^4}{4}\right) - \int \left(\frac{x^4}{4} \times \frac{1}{x}\right) dx$$

$$= \tfrac{1}{4} x^4 \ln x - \tfrac{1}{4} \int x^3 dx$$

$$= \tfrac{1}{4} x^4 \ln x - \tfrac{1}{16} x^4 + c$$

Sometimes an expression which does not look like a product can be defined as one, by treating 1 as a term.

Example (6): Find $\int \ln x \, dx$.

Treat the expression as ln $x \times 1$.

Thus $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$.

Take $\frac{dv}{dx} = 1 \Rightarrow v = x$

$$\therefore \int \ln x \, dx = (\ln x) \times x - \int (x \times \frac{1}{x}) dx$$

$$= x \ln x - \int 1 dx \quad = x \ln x - x + c.$$

Example (7):

i) Evaluate $\int_0^{7.2} \tan^{-1} x \, dx$.

ii) The velocity v of a performance car (in metres per second) is modelled by the equation

$v = 84\tan^{-1}(0.12t)$ where radian measure is used.

Use your knowledge of transformations to evaluate $\int_0^{60} 84\tan^{-1}(0.12t) \, dt$, and hence calculate the distance travelled in those 60 seconds.

Give your result in metres to 3 significant figures.

Treat the expression as $\tan^{-1} x \times 1$.

Thus $u = \tan^{-1} x \Rightarrow \frac{du}{dx} = \frac{1}{1+x^2}$.

Take $\frac{dv}{dx} = 1 \Rightarrow v = x$

$$\therefore \int_0^{7.2} \tan^{-1} x \, dx = x\left[\tan^{-1} x\right]_0^{7.2} - \int_0^{7.2} \frac{x}{1+x^2} dx$$

$$= \left[x\tan^{-1} x - \tfrac{1}{2}\ln(1+x^2)\right]_0^{7.2} = \left[7.2\tan^{-1} 7.2 - \tfrac{1}{2}\ln(52.84)\right] - \left[0-0\right]$$

= 10.316 – 1.984 = 8.332.

ii) The integrand $\int_0^{60} 84\tan^{-1}(0.12t) \, dt$ is obtained by transforming the integrand $\int_0^{7.2} \tan^{-1} x \, dx$ by an x-stretch of scale factor $\frac{1}{0.12}$ followed by a y-stretch of scale factor 84, plus a trivial renaming of the variable from x to t.

Now $\frac{1}{0.12} \times 84 = 700$, so we multiply the result of part i) by 700 to obtain the distance travelled by the car in those 60 seconds – namely 8.332 $\times$ 700, or **5830 metres** to 3 significant figures.

Example (8): Find $\int e^x \sin x\, dx$.

Here $u = e^x \Rightarrow \frac{du}{dx} = e^x$.

Also $\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$.

$\therefore \int e^x \sin x\, dx = e^x \times (-\cos x) - \int (-\cos x) \times e^x$

$= \quad -e^x \cos x + \int e^x \cos x\, dx$

Applying integration by parts again we have

$u = e^x \Rightarrow \frac{du}{dx} = e^x$

$\frac{dv}{dx} = \cos x \Rightarrow v = \sin x$.

$\int e^x \cos x\, dx = e^x \times (\sin x) - \int (\sin x) \times e^x$

$= e^x \sin x - \int e^x \sin x\, dx$ **(1)**

Combining the two results, we finally have

$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$

The process appears to be going on for ever without simplifying the expression, but notice how the original integrand now appears on both sides of the equation. Adding this integrand to both sides will cancel it out from the RHS.

$2\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x$

$\therefore \int e^x \sin x\, dx = \tfrac{1}{2}(e^x \sin x - e^x \cos x) + c$ **(2)**

From the above results in **(1)** and **(2)** we can also deduce that

$\int e^x \cos x\, dx = e^x \sin x - \tfrac{1}{2}(e^x \sin x - e^x \cos x)$

$\therefore \int e^x \cos x\, dx = \tfrac{1}{2}(e^x \sin x + e^x \cos x) + c$