M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

FURTHER INTEGRATION TECHNIQUES (TRIG, LOG, EXP FUNCTIONS)

$\sin 2A = 2\sin A\cos A : A = 3x$

$$\int \sin 3x \cos 3x \, dx = \int \tfrac{1}{2}\sin 6x \, dx = -\tfrac{1}{12}\cos 6x + c$$

$$\tan x = -\frac{(-\sin x)}{\cos x}$$

$$\Rightarrow \int \tan x \, dx = -\ln|\cos x| + c$$

$$\int_0^1 \frac{2x+4}{x^2+1} dx = \int_0^1 \frac{2x}{x^2+1} + \frac{4}{x^2+1} dx$$

$$= \left[\ln(x^2+1) + 4\tan^{-1} x\right]_0^1 = (\ln 2 + \pi) - (\ln 1) = \ln 2 + \pi$$

$\tan^2 x = \sec^2 x - 1$

$$\int_0^{\pi/3} \tan^2 x \, dx = \int_0^{\pi/3} \sec^2 x - 1 \, dx = \left[\tan x - x\right]_0^{\pi/3} = \sqrt{3} - \tfrac{\pi}{3}$$

in partial fractions

$$\int_5^6 \frac{4x+5}{x^2-x-12} dx = \int_5^6 \frac{3}{x-4} + \frac{1}{x+3} dx = \left[3\ln(x-4) + \ln(x+3)\right]_5^6$$

$$= \left[\ln\left[(x-4)^3(x+3)\right]\right]_5^6 = \ln 72 - \ln 8 = \ln 9.$$

$\sin^2 x = ½(1 - \cos 2x) \quad \cos^2 x = ½(1 + \cos 2x)$

$$\int \sin^2 x \cos^2 x \, dx = \int \left(\tfrac{1}{2}(1-\cos 2x)\right)\left(\tfrac{1}{2}(1+\cos 2x)\right) dx$$

$$= \int \tfrac{1}{4}(1-\cos^2(2x)) \, dx = \int \tfrac{1}{4}(\sin^2(2x)) \, dx$$

$$= \int \tfrac{1}{4}\left(\tfrac{1}{2}(1-\cos(4x))\right) dx = \int \tfrac{1}{8}\left(1-\cos(4x)\right) dx = \tfrac{1}{8}x - \tfrac{1}{32}\sin(4x) + c$$

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Further Integration Techniques.

Trigonometric Integrals.

We have already met several trigonometric integrals in earlier sections. The following two are typical examples.

Example (1): Find $\int_{\pi/6}^{\pi/2} 2\sin^4 x \cos x\, dx$.

(Remember – radian measure must be used !)

We can use the tabled result $\int \sin^n x \cos x = \frac{1}{n+1}\sin^{n+1} x + c$

and obtain the answer $2 \times \frac{1}{5}\sin^5 x + c = \frac{2}{5}\sin^5 x + c$.

Alternatively, we can look at the integral and notice that it includes a power of sin x (the fourth power) multiplied by its derivative – a chain rule result.

We can thus guess that the integral will be something like $\sin^5 x$ (compare integrating x^4 to get $\frac{1}{5}x^5$).

Differentiating $\sin^5 x$ gives $5\sin^4 x \cos x$, but our original integral was $2\sin^4 x \cos x$. The guess is too large by a factor of $\frac{5}{2}$, so we need to multiply it by $\frac{2}{5}$ to bring it to scale .

This gives $\frac{2}{5}\sin^5 x + c$ as before.

This is a definite integral, so its value is $\left[\frac{2}{5}\sin^5 x\right]_{\pi/6}^{\pi/2} = \frac{2}{5}\left(1 - \frac{1}{32}\right) = \frac{31}{80}$.

(Remember: $\sin \pi/2 = 1$; $\sin \pi/6 = ½$).

Example (2): Find $\int \sec^4 x \tan x\, dx$.

We can use the tabled result $\int \sec^n x \tan x = \frac{1}{n}\sec^n x + c$

and obtain the answer $\frac{1}{4}\sec^4 x + c$.

Alternatively, we can rewrite the integrand as $\sec^3 x \sec x \tan x$, thus showing the product of the cube of sec x and its derivative, sec x tan x, more clearly. This suggests an answer of the form $\sec^4 x$.

Differentiating $\sec^4 x$ gives $4\sec^3 x \sec x \tan x = 4\sec^4 x \tan x$. This result is too large by a factor of 4, therefore the true integral is $\frac{1}{4}\sec^4 x + c$ as above.

There are also many other trigonometric integrals which can be evaluated by using identities and compound angle formulae to simplify complicated integrals into forms which are easier to integrate.

The double angle formulae for cos 2*A* are especially useful.

$\cos^2 A + \sin^2 A = 1$ (This is the Pythagorean identity)
$1 + \tan^2 A = \sec^2 A$.
$\cot^2 A + 1 = \mathrm{cosec}^2 A$.

$\sin(2A) = 2\sin A\cos A$
$\cos(2A) = \cos^2 A - \sin^2 A$
$= 2\cos^2 A - 1$
$= 1 - 2\sin^2 A$
$\tan(2A) = \dfrac{2\tan A}{1-\tan^2 A}$

(From the formula for cos 2*A*).

$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$
$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

The triple angle formulae for cos (3*A*) and sin (3*A*) also crop up at times:
$\cos 3A = 4\cos^3 A - 3\cos A$.
$\sin 3A = 3\sin A - 4\sin^3 A$.

$$\cos^3 A = \tfrac{1}{4}(3\cos A + \cos 3A)$$
$$\sin^3 A = \tfrac{1}{4}(3\sin A - \sin 3A)$$

Examination questions on trigonometric integrals of this type generally have an introductory hint as to the correct method to be used, and are not quite as difficult as some of the examples here.

Example (3): Find $\int \sin 3x \cos 3x \, dx$.
This integrand is based on the double-angle formula for sin 2*A*: sin 2*A* = 2 sin *A* cos *A*.
Let *A* = 3*x*, and the integrand becomes $\int \tfrac{1}{2}\sin 6x \, dx$ evaluating to $-\tfrac{1}{12}\cos 6x + c$. (The working is as in previous examples).

Interestingly, we could have guessed an integral of the form of $\sin^2(3x)$ and differentiated it to give $6\sin(3x)\cos(3x)$, and then scaled the final result to obtain $\tfrac{1}{6}\sin^2(3x) + c$. Those seemingly different results make sense because $\tfrac{1}{6}\sin^2(3x) = 1 - \tfrac{1}{12}\cos 6x$, in other words, they differ by the constant 1.

Example (4): Find $\int \cos^2\left(\tfrac{x}{2}\right) - \sin^2\left(\tfrac{x}{2}\right) dx$.
This time we have the formula $\cos 2A = \cos^2 A - \sin^2 A$ in disguise.
Let $A = \tfrac{x}{2}$, and the integrand becomes simply $\int \cos x \, dx$ or $\sin x + c$.

Example (5): Find $\int_0^{\pi/3} \tan^2 x \, dx$. (Leave the result in surds and terms of π.)

Here we use the identity $\tan^2 x = \sec^2 x - 1$, to get $\int_0^{\pi/3} \sec^2 x - 1 \, dx$.
This is a standard integral : the result is $\left[\tan x - x\right]_0^{\pi/3} = \sqrt{3} - \tfrac{\pi}{3}$.
(Remember: $\tan(\pi/3) = \sqrt{3}$).

Integration of powers of sin *x* and cos *x*.

These integrals crop up quite frequently, but they come in two types:

Even powers of sin *x* and cos *x* only

If the expression is an even power of sin *x* or cos *x* (or a product of the two), the technique is to rewrite the integral as a product of terms in $\cos^2 x$ and /or $\sin^2 x$.

From there, we can use the formulae for cos 2*x* to replace occurrences of $\cos^2 x$ with ½(1 + cos 2*x*), and occurrences of $\sin^2 x$ with ½(1 - cos 2*x*). These forms are easier to integrate.

Example (6): Find $\int \cos^2 x\, dx.$

The integrand simplifies into ½(1 + cos 2*x*),
giving a result of $\frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)$ = $\frac{1}{2}x + \frac{1}{4}\sin 2x + c.$

Example (7): Find $\int_0^{\pi/4} \sin^4 x\, dx.$

Rewrite the integrand as $\left(\sin^2 x\right)^2$ = $\left(\frac{1}{2}(1-\cos 2x)\right)^2$.

This expands to $\frac{1}{4}(1-2\cos(2x)+\cos^2(2x))$, but there is still an inner term in $\cos^2(2x)$ which needs simplifying: namely $\cos^2(2x)$ = ½(1 + cos 4*x*).

The final expansion of the integrand therefore gives $\frac{1}{4}(1-2\cos(2x)+\frac{1}{2}(1+\cos(4x))$
or $\frac{1}{8}(3-4\cos(2x)+\cos(4x))$.

This can now be integrated to give $\left[\frac{1}{8}(3x-2\sin(2x)+\frac{1}{4}\sin(4x))\right]_0^{\pi/4}$
= $\left[\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x\right]_0^{\pi/4}$ = $\left(\frac{3\pi}{32}-\frac{1}{4}+0\right)-(0)$ = $\frac{3\pi-8}{32}$.

(Remember: sin π/2 = 1; sin π = 0)

Example(8): Find $\int \sin^2 x \cos^2 x\, dx.$

Rewrite the integrand as $\left(\frac{1}{2}(1-\cos 2x)\right)\left(\frac{1}{2}(1+\cos 2x)\right)$.

This simplifies into the 'difference of squares' form of $\frac{1}{4}(1-\cos^2(2x))$ = $\frac{1}{4}(\sin^2(2x))$.

The resulting integrand can be simplified again to $\frac{1}{4}\left(\frac{1}{2}(1-\cos(4x))\right)$ = $\frac{1}{8}\left(1-\cos(4x)\right)$

Integration gives $\frac{1}{8}\left(x-\frac{1}{4}\sin(4x)\right)+c$ = $\frac{1}{8}x-\frac{1}{32}\sin(4x)+c$.

Example(9): Find $\int \cos^4(2x)\, dx.$

Rewrite the integrand as $\left(\cos^2(2x)\right)^2$ = $\left(\frac{1}{2}(1+\cos(4x))\right)^2$.

This expands to $\frac{1}{4}(1+2\cos(4x)+\cos^2(4x))$, and then we replace $\cos^2(4x)$ = ½(1 + cos 8*x*).

The final expansion of the integrand therefore gives $\frac{1}{4}(1+2\cos(4x)+\frac{1}{2}(1+\cos(8x))$
= $\frac{1}{8}(2+4\cos(4x)+1+\cos(8x))$.

This can now be integrated to give $\frac{1}{8}(2x+\sin(4x)+x+\frac{1}{8}\sin(8x))$
= $\frac{3}{8}x+\frac{1}{8}\sin(4x)+\frac{1}{64}\sin(8x)+c$.

At least one power of sin *x* or cos *x* is odd.

If the expression has at least one odd power of sin *x* and/or cos *x*, we use the identity $\cos^2 x + \sin^2 x = 1$ to convert the expression into a form which contains terms of the form $\sin^n x \cos x$ and/or $\cos^n x \sin x$.

These forms are 'reverse chain rule results' which integrate to $\frac{1}{n+1}\sin^{n+1}x + c$ and $\frac{1}{n+1}\cos^{n+1}x + c$.

Example (10): Find $\int_0^{\pi/2} \sin^5 x\,dx$.

Rewrite the integrand as $\sin^4 x \sin x$, and thus as $(1-\cos^2 x)^2 \sin x$.

Expansion by the binomial theorem gives $(1 - 2\cos^2 x + \cos^4 x)\sin x$.

The integrand then becomes a sum of 'reversed chain rule' results:

$$\int_0^{\pi/2} \sin x - 2\cos^2 x \sin x + \cos^4 x \sin x\,dx$$

This integrates to $\left[-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x\right]_0^{\pi/2}$.

$= (0) - (-1 + \frac{2}{3} - \frac{1}{5}) = \frac{8}{15}$. (Note cos (π/2) = 0, cos 0 = 1).

Some integrals can be evaluated using alternative methods:

Example (11): Find $\int_0^{\pi/2} \cos^3 x\,dx$.

Method 1: Rewrite the integrand as $\cos^2 x \cos x$, and thus as $(1-\sin^2 x)\cos x$.

The integrand thus becomes:

$$\int_0^{\pi/2} \cos x - \sin^2 x \cos x\,dx$$

Integration gives $\left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\pi/2} = 1 - \frac{1}{3} = \frac{2}{3}$. (Note sin (π/2) = 1).

Method 2: Use the triple angle formula:

$$\int_0^{\pi/2} \cos^3 x\,dx = \int_0^{\pi/2} \frac{1}{4}(3\cos x + \cos 3x)\,dx = \left[\frac{3}{4}\sin x - \frac{1}{12}\sin 3x\right]_0^{\pi/2}$$

$= \frac{3}{4} - \frac{1}{12} = \frac{2}{3}$. (Note sin (π/2) = 1, sin (3π/2) = -1).

Example (12): Find $\int \cos^3 x \sin^4 x\,dx$.

Rewrite the integrand as as $\sin^4 x(1-\sin^2 x)\cos x \Rightarrow \int \sin^4 x \cos x - \sin^6 x \cos x\,dx$,

which integrates to $\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$.

Other applications of logarithmic integrands.

Some trigonometric integrands can also lend themselves to a logarithmic function when integrated.

Examples (13) : Using the fact that the derivative of sec *x* is sec *x* tan *x* or otherwise, find i) $\int \tan x \, dx$

and use the result to ii) evaluate $\int_0^{\pi/3} \tan^3 x \, dx$.

In i) we can rewrite the identity $\tan x = \frac{\sec x \tan x}{\sec x}$, giving us an integrand where the top line is the derivative of the bottom line.

$\Rightarrow \int \tan x \, dx = \ln |\sec x| + c$.

Alternatively, we could have used the identity $\tan x = -\frac{(-\sin x)}{\cos x}$, again giving us an integrand where the top line is the derivative of the bottom line.

$\Rightarrow \int \tan x \, dx = -\ln |\cos x| + c$.

The two integrals are equivalent since cos *x* and sec *x* are reciprocals of one another.

For part ii) we rewrite the integrand as $(\sec^2 x - 1)(\tan x)$ or $\int_0^{\pi/3} \tan x \sec^2 x - \tan x \, dx$
Using the reverse chain rule (as in 'Trigonometric Integrals') and the result from part i) of the question, we have the integral $\left[\frac{1}{2}\tan^2 x - \ln(\sec x)\right]_0^{\pi/3}$ or $\frac{3}{2} - \ln 2$.

There is no need to include the modulus sign around the logarithm, since $\sec x > 0$ for *x* in the range.
Note $\tan(\pi/3) = \sqrt{3}$; $\sec(\pi/3) = 2$; $\sec(0) = 1$.

With some other rational integrands, the top line might not be exactly the derivative of the bottom line, but we can use inverse trig functions and algebraic adjustment.

Example (14): Find the value of $\int_0^1 \frac{2x+4}{x^2+1} dx$. Hint: $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.

The derivative of the denominator is $2x$, but the numerator is 'not quite right' for the integral to be a straightforward logarithmic result.

We can however rewrite the integrand as $\int_0^1 \frac{2x}{x^2+1} + \frac{4}{x^2+1} dx$.
The first term of the integrand is now a valid logarithmic function and the second term an inverse trig function, giving an integral of

$$\left[\ln(x^2+1) + 4\tan^{-1} x\right]_0^1 = (\ln 2 + \pi) - (\ln 1) = \ln 2 + \pi .$$

There is no need to include the modulus sign around the logarithm, since $x^2 + 1 > 0$ for all *x*.

(Remember $\tan^{-1}(1) = \pi/4$).

Sometimes a quadratic denominator can be factorised and the integrand rewritten in partial fractions.

Examples (15): Find the value of : i) $\int \frac{4x-9}{x^2-5x+6}dx$; ii) $\int \frac{x+7}{x^2-x-2}dx$;

iii) $\int_5^6 \frac{4x+5}{x^2-x-12}dx$, giving the result as a single logarithm.

None of the integrands have the top line equal to a multiple of the derivative of the bottom line, but each can be re-expressed in partial fractions.

In i) the denominator can be factorised to (x-2)(x-3) and the integrand rewritten as

$$\int \frac{1}{x-2}+\frac{3}{x-3}dx.$$

(Full working is in Example (1) of the document ‘Partial Fractions’).

The integral is therefore ln (|x-2|) + 3 ln (|x-3|) + c.

This same result can also be expressed as $\ln\left[\,|\,(x-2)(x-3)^3\,|\,\right]+c$ or $\ln\left[\,|\,A(x-2)(x-3)^3\,|\,\right]$ where A is non-zero.

In ii) the denominator can be factorised to (x-2)(x+1) and the integrand rewritten as

$$\int \frac{3}{x-2}-\frac{2}{x+1}dx.$$

(Full working is in the opening paragraph of the document ‘Partial Fractions’).

The integral is therefore 3 ln (|x-2|) - 2 ln (|x+1|) + c.

This same result can also be expressed as $\ln\left[\,|\,\frac{(x-2)^3}{(x+1)^2}\,|\,\right]+c$ or $\ln\left[\,|\,A\frac{(x-2)^3}{(x+1)^2}\,|\,\right]$ where $A > 0$.

In iii) the denominator can be factorised to (x - 4)(x+3) and the integrand rewritten as

$$\int_5^6 \frac{3}{x-4}+\frac{1}{x+3}dx.$$

(Full working is in Example (2) of the document ‘Partial Fractions’).

The integral is therefore $\left[3\ln(x-4)+\ln(x+3)\right]_5^6$

$= \left[\ln\left[(x-4)^3(x+3)\right]\right]_5^6$

$= \ln 72 - \ln 8$

$= \ln 9.$

APPENDIX - Parametric Integration. (Pearson/Edexcel syllabus only.)

A curve can be defined in parametric form - for instance, x could be defined as $f(t)$ and y as $g(t)$, with t as the parameter.

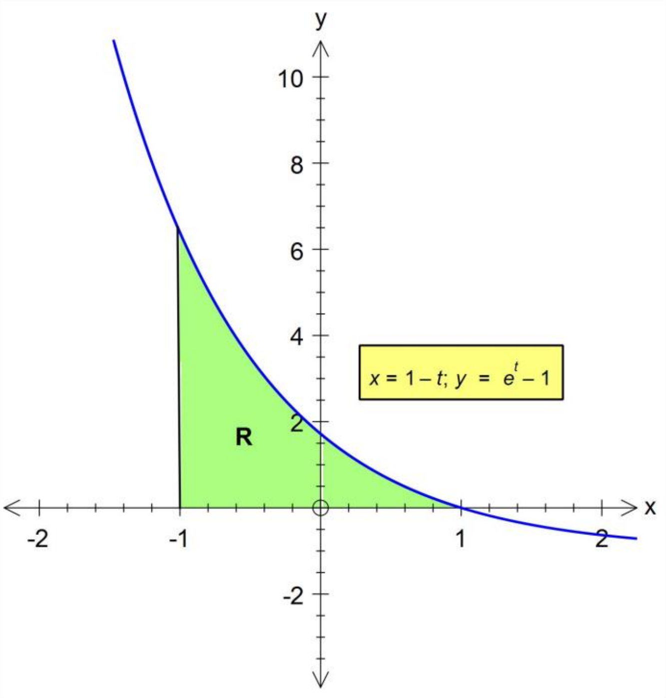
The formula for the area under a curve can be adapted for parametric integrals as follows:

$$A = \int_{x_1}^{x_2} y\,dx = \int_{t_1}^{t_2} g(t)\frac{dx}{dt}\,dt.$$

Example (16):

A curve is defined parametrically as $x = 1 - t$, $y = e^t - 1$.

Its graph is shown on the right.



Using parametric methods, find the area of region **R** enclosed by the curve, the x-axis and the line $x = -1$.

Give your answer in an exact form.

Preparatory working :

Change x – limits to t – limits; $x = 1 - t \Rightarrow t = 1 - x$, so when $x = 1$, $t = 0$; $x = -1$, $t = 2$.

Find $\frac{dx}{dt}$; here it is simply -1.

The area **R** under the curve is given by

$$\int_{x=-1}^{x=1} y\,dx = \int_{t=2}^{t=0} \left(e^t - 1\right)\frac{dx}{dt}\,dt = \int_{t=2}^{t=0} \left(e^t - 1\right)(-1)\,dt = \int_{t=0}^{t=2} \left(e^t - 1\right)dt$$

$$= \left[e^t - t\right]_0^2 = \left[\left(e^2 - 2\right) - \left(1 - 0\right)\right] = e^2 - 3.$$

Note the reversal of the limits and the multiplication by -1 in the working.

Example (17): The curve shown here has parametric equations

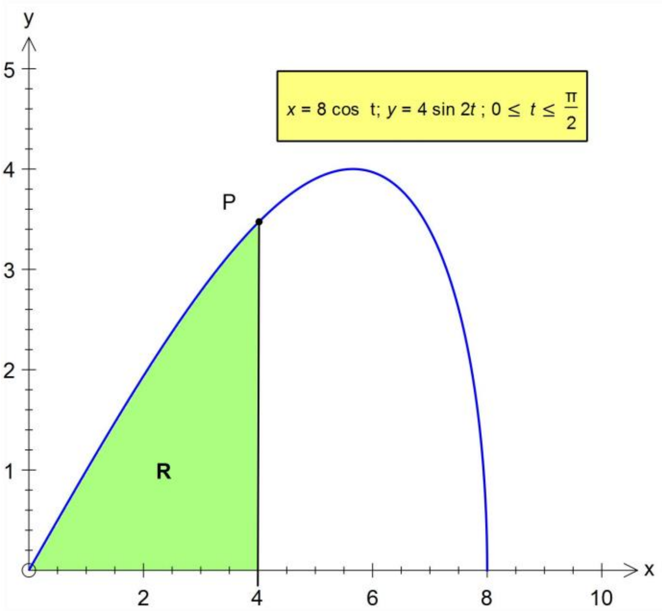
$x = 8\cos t$, $y = 4\sin 2t$, where $0 \le t \le \pi/2$.

The point P lies on the curve and its coordinates are $(4, 2\sqrt{3})$.

i) Find the value of t at the point P.

ii) Use parametric integration to find the area of region **R** enclosed by the curve, the x-axis and the line $x = 4$, giving the answer in an exact form.

Hint : $\frac{d}{dt}\left(\sin^3 t\right) = 3\sin^2 t\cos t$.



i) At point P, $8\cos t = 4 \Rightarrow \cos t = \frac{1}{2} \Rightarrow t = \frac{\pi}{3}$.

ii) $\frac{dx}{dt} = -8\sin t$; also at $x = 0$, $8\cos t = 0 \Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2}$.

To find the area of region **R** we transform the integrand from $\int_{x=0}^{x=4} y\,dx$ to

$$\int_{t=\pi/2}^{t=\pi/3} 4\sin 2t\frac{dx}{dt}dt = \int_{\pi/2}^{\pi/3} 4\sin 2t(-8\sin t)\,dt = \int_{\pi/2}^{\pi/3} 8\sin t\cos t(-8\sin t)\,dt$$

$$= \int_{\pi/2}^{\pi/3} \left(-64\sin^2 t\right)\cos t\,dt = \int_{\pi/3}^{\pi/2} 64\sin^2 t\cos t\,dt.$$

This final integrand is the inverse of the result $\frac{d}{dt}\left(\sin^3 t\right) = 3\sin^2 t\cos t$ stated earlier in the hint,

but multiplied by the scale factor of $\frac{64}{3}$.

The area of **R** is therefore $\int_{\pi/3}^{\pi/2} 64\sin^2 t\cos t\,dt = \left[\frac{64}{3}\sin^2 t\right]_{\pi/3}^{\pi/2} = \frac{64}{3}\left(1 - \frac{3\sqrt{3}}{8}\right)$

$$= \frac{64}{3} - \left(\frac{\cancel{64}^{8}}{\cancel{3}} \times \frac{\cancel{3}\sqrt{3}}{\cancel{8}}\right) = \frac{64}{3} - 8\sqrt{3}.$$

For enclosed curves, a negative integral results from tracing the curve in an anticlockwise direction; a positive one results from tracing the curve clockwise.

Example (18): Using parametric integration, find the area of the circle defined by x=6 cos θ, y= 6 sin θ, for 0 ≤ θ ≤ 2π.

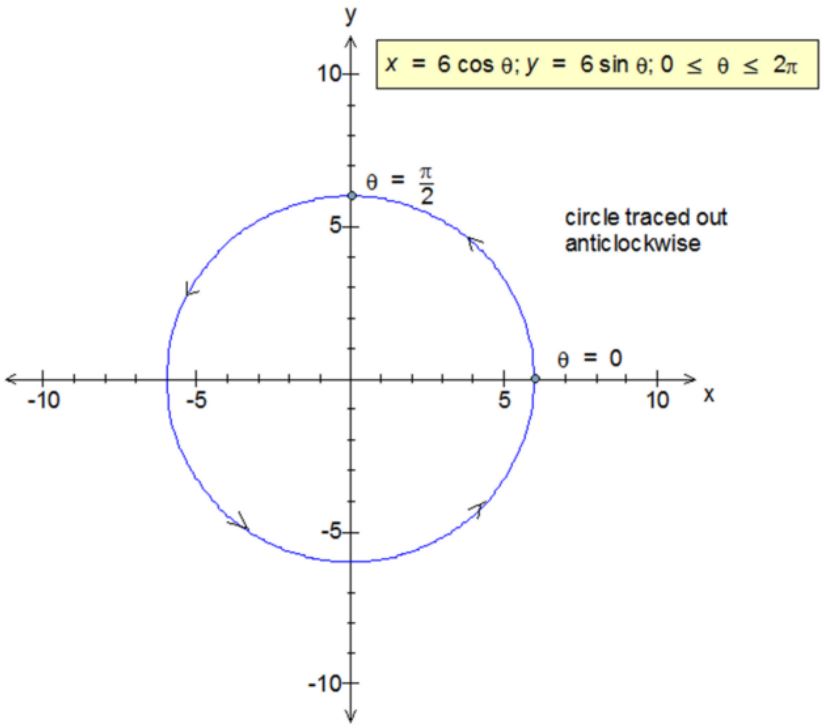
$$A = \int_0^{2\pi} y \frac{dx}{d\theta} d\theta = \int_0^{2\pi} 6\sin\theta(-6\sin\theta) d\theta = -36\int_0^{2\pi} \sin^2\theta \, d\theta$$

We use the trigonometric identity $\sin^2\theta = ½(1 - \cos 2\theta)$ to evaluate the integral.

$$-36\int_0^{2\pi} \sin^2\theta \, d\theta = -36\int_0^{2\pi} \tfrac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= -18\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} = -18\left[(2\pi - 0) - (0-0)\right] = -36\pi.$$

(The resulting integral is negative because the circle has been traced out in an anticlockwise direction.)



Example (19): Using parametric integration, find the area of the ellipse defined by x=5 cos θ, y= 2 sin θ, for 0 ≤ θ ≤ 2π.

$$A = \int_0^{2\pi} y \frac{dx}{d\theta} d\theta = \int_0^{2\pi} 2\sin\theta(-5\sin\theta) d\theta = -10\int_0^{2\pi} \sin^2\theta \, d\theta$$

Again using $\sin^2\theta = ½(1 - \cos 2\theta)$, the integrand becomes

$$-10\int_0^{2\pi} \tfrac{1}{2}(1-\cos 2\theta)\, d\theta = -5\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} = -5\left[(2\pi - 0) - (0-0)\right] = -10\pi.$$

The last examples lead us to the general result: for any ellipse defined by x=a cos θ, y= b sin θ, the area will be πab.