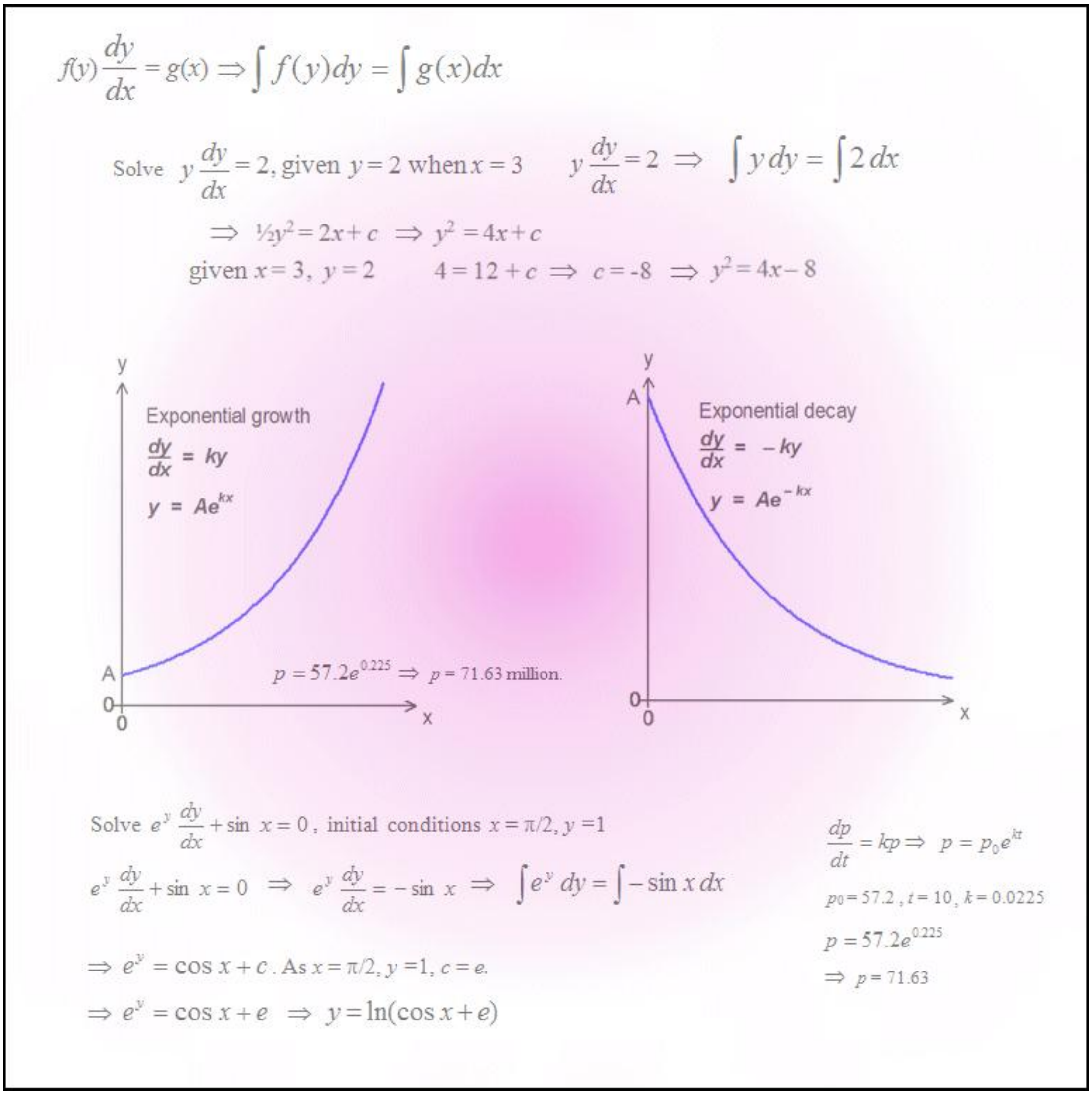
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

DIFFERENTIAL EQUATIONS



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DIFFERENTIAL EQUATIONS

A **differential equation** in terms of the variables *x* and *y* is one that contains a differential coefficient, namely a term like $\frac{dy}{dx}$ (or y').
If only the first derivative is present, as in the example above, we have a **first-order** differential equation.

Second-order differential equations (beyond the scope of A-level) would include the second derivative, i.e. $\frac{d^2y}{dx^2}$ (or y'').

Examples of first-order differential equations are $\frac{dy}{dx} = 2x$, $\frac{dy}{dx} = \frac{x}{y}$ and $y\frac{dy}{dx} = 2$.

If $\frac{dy}{dx} = 2x$, then integrating with respect to *x* gives

$y = x^2 + c$.

This is the **general solution** of the differential equation , and it can be illustrated by a family of curves .

Each curve has the equation
$y = x^2 + c$, with the case where $c = 0$ shown in bold.
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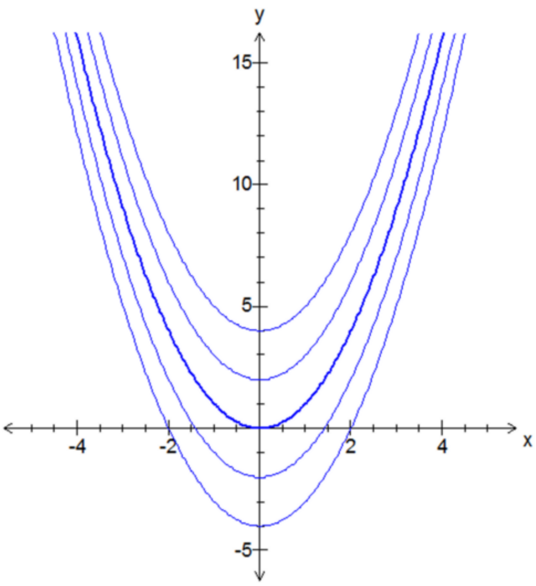


A **particular solution** is one specific curve of the family of general solutions, and it can be found from additional information.

Example (1): Find the particular solution of the differential equation $\frac{dy}{dx} = 4x + 3$, given that $y = 8$ when $x = 2$.

Integrating using standard methods gives a general solution of $y = 2x^2 + 3x + c$.
Substituting $x = 2$ and $y = 8$ gives $8 + 6 + c = 8$ and hence $c = -6$.

The particular solution is therefore $y = 2x^2 + 3x - 6$.

Example (2): Find the particular solution of the differential equation $\frac{dy}{dx} = 2\cos x$, given that $y = 3$ when $x = \pi$.

Integration gives a general solution of $y = 2\sin x + c$. Substituting $x = \pi$ and $y = 3$ gives $-2 + c = 3$ and hence $c = 5$.

The particular solution is therefore $y = 2\sin x + 5$.

The above examples were both solved using standard techniques learnt earlier in the course, but to tackle an equation like $\frac{dy}{dx} = \frac{x}{y}$, we need a new method.

Separating the variables.

If a differential equation can be written in the form $f(y)\frac{dy}{dx} = g(x)$, then it can be solved by separating the variables. By treating *dy* and *dx* as if they were algebraic quantities and integrating both sides, this expression can be rewritten as

$$\int f(y)dy = \int g(x)dx.$$

Example (3): Solve $\frac{dy}{dx} = \frac{x}{y}$.

This expression must first be manipulated to separable form

$$y\frac{dy}{dx} = x \Rightarrow \quad y\,dy = x\,dx \quad \Rightarrow \int y\,dy = \int x\,dx.$$

Note that the stage " *y dy = x dx* " is mathematically nonsensical and is only shown as a step in the working – we must prefix integral signs to give the expression meaning !

Integrating, we have $\frac{1}{2}y^2 = \frac{1}{2}x^2 + c$. This can be rewritten as $y^2 = x^2 + c$.

(Since c is an arbitrary constant, we still use c and not $2c$ after the doubling.)

The constant could have been attached to either side of the expression.

Example (4): Solve $y\frac{dy}{dx} = 2$, given that $y = 2$ when $x = 3$.

Manipulating to separable form we have

$$y\frac{dy}{dx} = 2 \Rightarrow \int y\,dy = \int 2\,dx$$

Integrating, we have the general solution $\frac{1}{2}y^2 = 2x + c$. This can be rewritten as $y^2 = 4x + c$.

Therefore, $y^2 = 4x + c$.

Using the result $y^2 = 4x + c$ and substituting $x = 3$, $y = 2$, we have
$4 = 12 + c \Rightarrow c = -8$.

The particular solution of the differential equation is $y^2 = 4x - 8$.

In the last examples, we had rewritten general solutions such as $\frac{1}{2}y^2 = 2x + c$ in the form $y^2 = 4x + c$. There was no need to write $y^2 = 4x + 2c$, as both c and $2c$ were simply numbers.

Simplifying the arbitrary constant in this way is allowable only when basic arithmetic is applied to both sides of the differential equation.

Care must be taken with the arbitrary constant if any mathematical operations other than basic arithmetic are used, such as raising to powers or taking exponents.

Example (5):

i) Find the general solution of $\frac{dy}{dx} = y$.

ii) Find the particular solution of the same equation, given initial conditions of $x = 1$ and $y = 5e$.

Separating the variables, $\frac{1}{y}\frac{dy}{dx} = 1 \Rightarrow \frac{1}{y}dy = 1dx$.

This is a special case of the form $f(y)\frac{dy}{dx} = g(x)$ where $g(x) = 1$.

Integrating we have $\int \frac{1}{y}dy = \int 1dx$ or $\ln y = x + c$.

Taking exponentials of both sides we have $y = e^{(x+c)}$.

This last expression can be rewritten as $y = Ae^x$ where $A = e^c$.

Notice how the arbitrary constant started off as a quantity to be *added*, but when we took exponentials, the constant was transformed into a *multiplier* by the log laws.

The constant must be incorporated immediately after performing the original integral. It can be added to either side – practice will tell you where it is more suitable to place it !

Thus we could have said $\int \frac{1}{y}dy = \int 1dx$ or $\ln y + c = x$.

Taking exponentials of both sides we have $ye^c = e^x$, and then $Ay = e^x$ where $A = e^c$.
The former arrangement is more logical and is the one usually quoted in textbooks.

ii) Substituting $x = 1$ and $y = 5e$ into the general solution gives $A = 5$, and hence the particular solution for the differential equation is $y = 5e^x$.

N.B. The method below is totally incorrect !

It is very bad practice to disregard the arbitrary constant throughout the working and just 'tag it on' at the end, as shown by the wrong method below.

Begin with $\int \frac{1}{y}dy = \int 1dx$, then $\ln y = x$, then take exponentials to get $y = e^x$, and finally, as an afterthought, tag the constant on at the end to obtain $y = e^x + c$.

Substituting $x = 1$ and $y = 5e$ into the incorrect 'general solution' gives $C = 4e$, and the incorrect 'particular solution' for the differential equation of $y = e^x + 4e$.

Example (6):

i) Solve the differential equation $\frac{dx}{dt} = \sqrt{x}\cos\left(\frac{t}{2}\right)$, to find *x* in terms of *t*.

ii) Given the initial conditions of *t* = 0 and *x* = 4, show that the solution can be written in the form $x = (\sin at + b)^2$ where *a* and *b* are positive constants to be determined.

i) Manipulating to separable form we have $\frac{dx}{dt} = \sqrt{x}\cos\left(\frac{t}{2}\right) \Rightarrow \int \frac{1}{\sqrt{x}}\,dx = \int \cos\left(\frac{t}{2}\right)dt$

$\Rightarrow \quad 2\sqrt{x} = 2\sin\left(\frac{t}{2}\right) + c \Rightarrow$ (halving both sides) $\sqrt{x} = \sin\left(\frac{t}{2}\right) + c$

$\Rightarrow$ (squaring both sides) $x = \left(\sin\left(\frac{t}{2}\right) + c\right)^2$.

Note how we did not need to say $\sqrt{x} = \sin\left(\frac{t}{2}\right) + \frac{1}{2}c$ when we halved both sides, but when we squared both sides, we had to put the constant inside the brackets and not say $x = \left(\sin\left(\frac{t}{2}\right)\right)^2 + c$.

ii) Given that *x* = 4 when *t* = 0, we substitute into the general solution $(\sin(0) + c)^2 = 4$ to give *c* = 2.
Hence the particular solution of the differential equation is $x = (\sin(0.5t) + 2)^2$.

Example (7): Solve the differential equation $\frac{dy}{dx} = 4xy$, given initial conditions of *x* = 0, *y* = 5.

$\frac{dy}{dx} = 4xy \Rightarrow \int \frac{1}{y}\,dy = \int 4x\,dx \Rightarrow \ln y = 2x^2 + c$

(taking exponentials) $\Rightarrow y = e^{2x^2+c} \Rightarrow y = Ae^{2x^2}$ where $A = e^c$.

Substituting the initial conditions of *x* = 0, *y* = 5 $\Rightarrow Ae^0 = 5 \Rightarrow A = 5$.

The particular solution is therefore $y = 5e^{2x^2}$.

Example (8): Solve the differential equation
$\frac{dy}{dx} = (y-3)(4x+3)$, given initial conditions of *x* = -1, *y* = $3(e^{-1}+1)$.

$\frac{dy}{dx} = (y-3)(4x+3) \Rightarrow \int \frac{1}{y-3}\,dy = \int 4x + 3\,dx \Rightarrow \ln(y+3) = 2x^2 + 3x + c$

(taking exponentials) $\Rightarrow y - 3 = e^{2x^2+3x+c}$.

Initial conditions: *x* = -1, *y* = $3(e^{-1}+1)$ $\Rightarrow 3e^{-1} = e^{2-3+c} \Rightarrow 3e^{-1} = e^{c-1}$

(taking natural logs) $\Rightarrow (\ln 3) - 1 = c - 1 \Rightarrow c = \ln 3 \Rightarrow y - 3 = 3e^{2x^2+3x}$

$\Rightarrow y = 3(e^{2x^2+3x} + 1)$.

Example (9): Solve the differential equation $e^y \frac{dy}{dx} + \sin x = 0$,
given initial conditions of $x = \pi/2$, $y = 1$.

$$e^y \frac{dy}{dx} + \sin x = 0 \;\Rightarrow\; e^y \frac{dy}{dx} = -\sin x \;\Rightarrow\; \int e^y \, dy = \int -\sin x \, dx$$

$\Rightarrow e^y = \cos x + c$. As $x = \pi/2$, $y = 1$, $c = e$.

Hence $e^y = \cos x + e$ and $y = \ln(\cos x + e)$.

Example (10): Solve $\frac{dy}{dx} = \frac{x^2 y + y}{x^2 - 1}$, given initial conditions of $x = 0$, $y = 2$.

$$\frac{dy}{dx} = \frac{x^2 y + y}{x^2 - 1} \Rightarrow \frac{dy}{dx} = y\left(\frac{x^2+1}{x^2-1}\right).$$

Before continuing with the process, we need to simplify the bracketed term into an integrable form, using partial fractions.

$$\frac{x^2+1}{x^2-1} = \frac{x^2-1}{x^2-1} + \frac{2}{x^2-1} = 1 + \frac{A}{(x+1)} + \frac{B}{(x-1)}$$

Partial fraction decomposition: $A(x-1) + B(x+1) = 2$.

$A + B = 0$ (equating x terms) ; $B - A = 2$ (equating constants) $\Rightarrow A = -1$; $B = 1$.

We now continue with $\frac{dy}{dx} = y\left(\frac{x^2+1}{x^2-1}\right) \Rightarrow \frac{dy}{dx} = y\left(1 - \frac{1}{x+1} + \frac{1}{x-1}\right)$.

Separating variables,

$$\Rightarrow \int \frac{1}{y}\, dy = \int \left(1 - \frac{1}{x+1} + \frac{1}{x-1}\right) dx$$

$$\Rightarrow \ln y = x - \ln(x+1) + \ln(x-1) + c$$

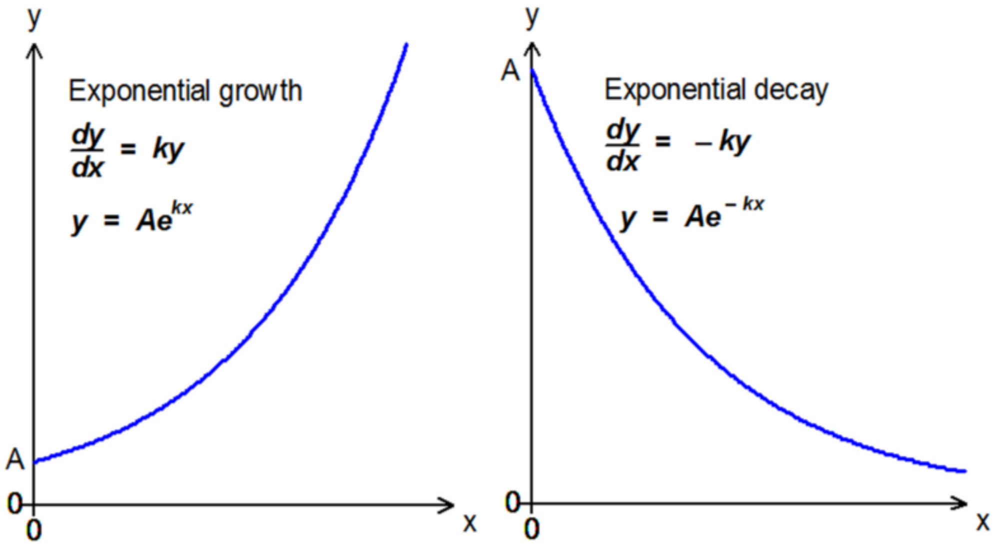
$\Rightarrow y = Ae^x \left(\frac{x-1}{x+1}\right)$ (exponentiating both sides) ($A = e^c$).

Substituting $x = 0$, $y = 2$ gives $A\left(\frac{-1}{1}\right) = 2$ and hence $A = -2$.

The particular solution to the differential equation is therefore $y = -2e^x\left(\frac{x-1}{x+1}\right)$ or $y = 2e^x\left(\frac{1-x}{x+1}\right)$.

Exponential Growth and Decay.

If we were to repeat example (5) using $\frac{dy}{dx} = ky$ and $\frac{dy}{dx} = -ky$ where *k* is a *positive* constant, we have exponential growth and decay functions.



Because exponential growth and decay take place over an interval of *time*, it is more common to use *t* for *x* as the variable with respect to which we are differentiating.

One everyday example of exponential growth is that of a population where $\frac{dp}{dt} = kp$

$\Rightarrow \quad p = p_0 e^{kt}$, where t is the elapsed time and p_0 is the population at the start of the time interval. This formula can also be applied to compound interest problems where the interest is compounded continuously (to give another example).

Examples of exponential decay include:

- Disintegration of radioactive materials where $\frac{dm}{dt} = -km \Rightarrow \quad m = m_0 e^{-kt}$, where t is the elapsed time and m_0 is the mass of the material at the start of the time interval.

- Newton's Law of Cooling, where $\frac{d\Theta}{dt} = -k(\Theta - \Theta_s) \Rightarrow \Theta - \Theta_s = (\Theta_0 - \Theta_s)e^{-kt}$ where t is the elapsed time, Θ is temperature of the object, Θ_s is the temperature of the surroundings, and Θ_0 is the temperature of the object at the start of the time interval.

Example (11): The population of the United Kingdom was estimated at 57.2 million in 2000. Assuming an annual rate of growth of 2.25%, what will be the population estimate for 2010 ?

This problem can be expressed as a differential equation, i.e.

$\frac{dp}{dt} = kp \Rightarrow \quad p = p_0 e^{kt}$ where $p_0 = 57.2$, $t = 10$ and $k = 0.0225$.

We want to find p such that $p = p_0 e^{kt}$, or $p = 57.2e^{0.225} \Rightarrow$ $p = 71.63$ million.

(Note that this is not the same as $57.2 \times (1.0225)^{10}$ or 71.45 million , since here the growth is continuous, and not just at the end of each year !).

Example (12): The radioactivity of a sample of sulfur-35 was taken in terms of a Geiger counter reading, and the adjusted count was recorded as 1852 'hits' per minute.

The same sample was then locked away and another reading taken 28 days later, and then the adjusted count was recorded as 1485 'hits' per minute.

Assuming that the number of 'hits' is proportional to the mass of the sulfur-35 remaining, what is the daily decay constant k and hence the half-life of sulfur-35 ? (The half-life of a radioactive substance is the time taken for the mass, and hence the activity, to decline to one-half of its original value).

$\frac{dm}{dt} = -km \Rightarrow \quad m = m_0 e^{-kt}$, where t is the elapsed time (in days here) and m_0 is the mass of the material at the start of the experiment.

Here k is the decay constant which is unknown, $m_0 = 1852$, $m = 1485$, and $t = 28$. (The number of 'hits' in each case represents the mass).

The decay constant k is found by $m = m_0 e^{-kt} \Rightarrow \frac{m}{m_0} = e^{-kt}$.

Taking logs of both sides, $\Rightarrow -k = \dfrac{\ln\left(\dfrac{m}{m_0}\right)}{t} \Rightarrow -k = \dfrac{\ln 1485 - \ln 1852}{28}$.

$\therefore$ -k = -0.00789.

To find the half-life, we use $\dfrac{m}{m_0} = e^{-kt}$, and find the value t which makes $\dfrac{m}{m_0} = \dfrac{1}{2}$.

We must solve $e^{-kt} = \dfrac{1}{2} \Rightarrow -kt = \ln\left(\dfrac{1}{2}\right)$ (taking logs) $\Rightarrow kt = \ln 2 \Rightarrow t = \dfrac{\ln 2}{k}$

$\therefore$ the half-life of sulfur-35 is $\dfrac{\ln 2}{0.00789}$ days or 87.8 days.

Example (13): Tea was poured into a vacuum flask and the temperature was recorded as 95°C immediately before the flask was stoppered and sealed. After one hour the temperature of the tea was measured as 76°C. Given an external temperature of 12°C, for how long would the flask have been expected to keep the tea hot ? (‘Hot’ means a temperature of 50°C or above.).

We use Newton’s Law of Cooling here:

$\frac{d\Theta}{dt} = -k(\Theta - \Theta_s) \Rightarrow \Theta - \Theta_s = (\Theta_0 - \Theta_s)e^{-kt}$ where t is the elapsed time (in hours here), Θ is temperature of the tea after one hour, Θ_s is the temperature of the surroundings, and Θ_0 is the starting temperature of the tea.

We want to make k the subject of $\Theta - \Theta_s = (\Theta_0 - \Theta_s)e^{-kt}$:

$$e^{-kt} = \frac{\Theta - \Theta_s}{\Theta_0 - \Theta_s} \Rightarrow -kt = \ln\left(\frac{\Theta - \Theta_s}{\Theta_0 - \Theta_s}\right) \text{ (take logs)}$$

$$\Rightarrow kt = \ln\left(\frac{\Theta_0 - \Theta_s}{\Theta - \Theta_s}\right) \text{ (logarithm of a reciprocal is –1 times the logarithm of the number)}$$

$$\Rightarrow k = \frac{\ln(\Theta_0 - \Theta_s) - \ln(\Theta - \Theta_s)}{t} \text{ (log laws)}$$

Substituting $t = 1$ (elapsed time in hours), $\Theta = 76$ (temperature of tea after 1 hour) , $\Theta_s = 12$ (surrounding temperature), and $\Theta_0 = 95$ (temperature at start), we have

$$k = \frac{\ln(83) - \ln(64)}{1} = 0.2600.$$

Having found k, we can therefore find the value of t corresponding to a tea temperature of 50°C. To do this we rearrange the last formula to make t the subject and substituting $\Theta = 50$.

$$t = \frac{\ln(\Theta_0 - \Theta_s) - \ln(\Theta - \Theta_s)}{k} \Rightarrow t = \frac{\ln(83) - \ln(38)}{0.2600} \text{ or 4.67 hours.}$$

$\therefore$ the tea can be expected to remain hot (above 50°C) for about 4 hours and 40 minutes.