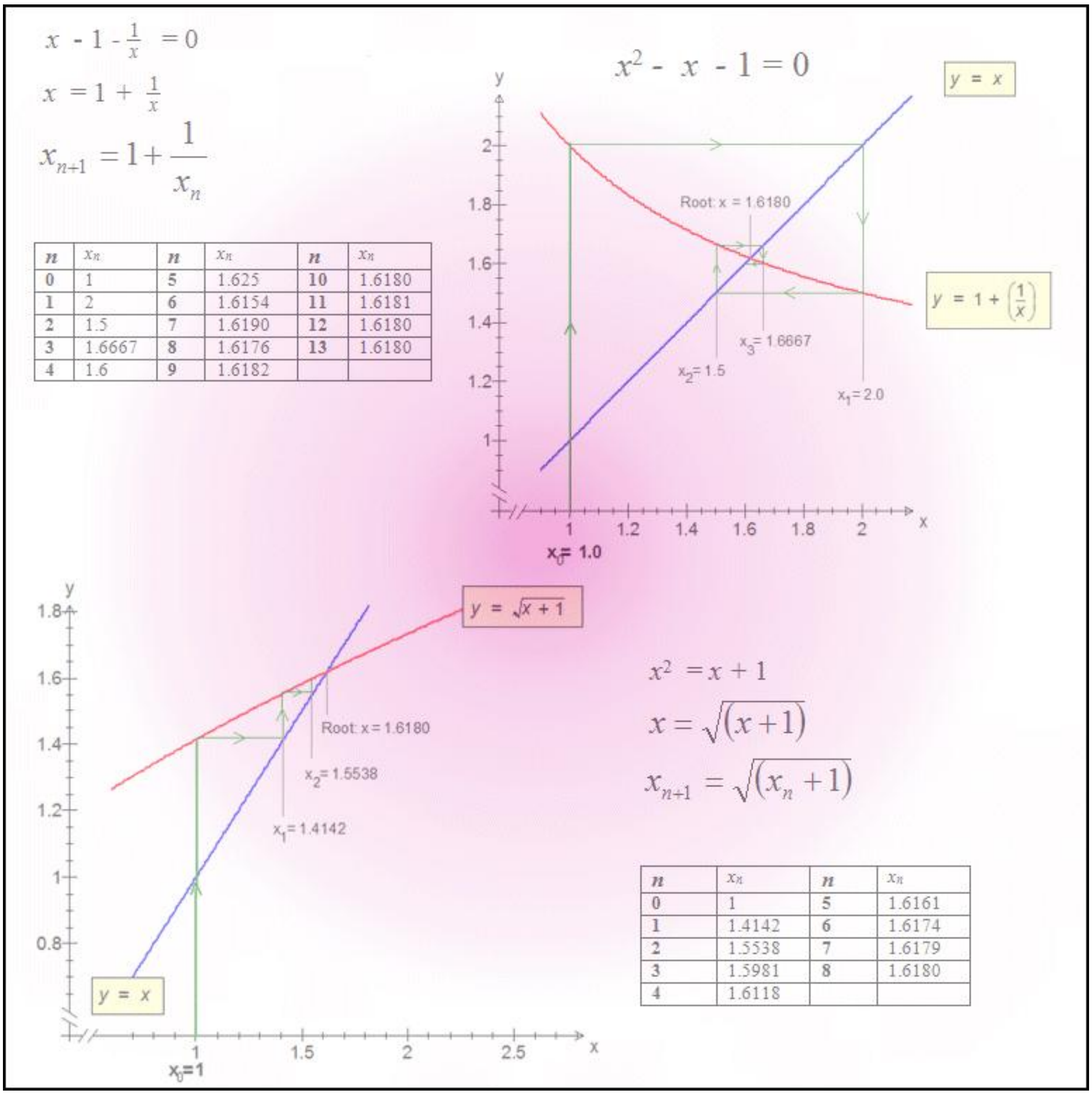
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

NUMERICAL ITERATIVE METHODS FOR SOLVING EQUATIONS



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NUMERICAL METHODS.

Overview .

Most equations cannot be solved algebraically to give an exact answer, and therefore we have to resort to numerical approximations.

There are two main methods discussed at A-level :

- Formula iteration
- The Newton-Raphson method (in its own section)

Both methods rely on choosing a single 'starter' approximation to the root, with the intention of improving the accuracy with each successive step.

Continuous Functions – change of sign.

If the graph of a function has no breaks within an interval $x = a$ to $x = b$, then the function is said to be **continuous** on the interval. For example, all polynomial functions are continuous everywhere, but the reciprocal function $1/x$ is not continuous on any interval which contains the value $x = 0$.

- If a function $f(x)$ is continuous between $x = a$ and $x = b$, and if $f(a)$ and $f(b)$ have different signs, then the interval from a to b will contain a root of $f(x) = 0$.

Example (1): Show that $f(x) = x^3 - 4x^2 + 6 = 0$ has a root between 1 and 2.

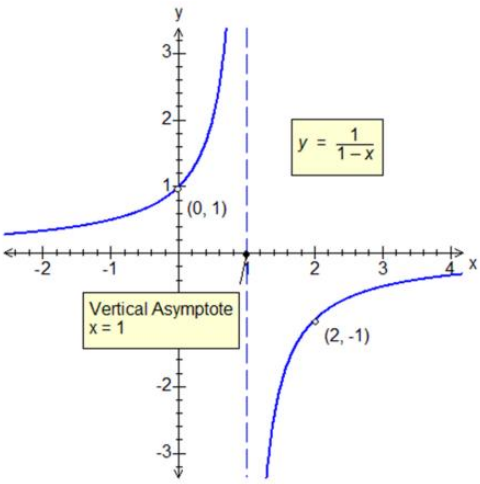
Substituting for x gives $f(1) = 3$ and $f(2) = -2$; there is a change of sign between $x = 1$ and $x = 2$. .
The function is continuous, therefore there is a root between 1 and 2.

Example (2): The function $f(x) = \dfrac{1}{1-x}$ takes a value of 1 when $x = 0$, and -1 when $x = 2$.

There is a sign change in the interval $0 < x < 2$, but it does not signify a root here !

This is because the interval in question includes a discontinuity at $x = 1$.

In fact the equation $f(x) = 0$ has no solution.



Formula iteration.

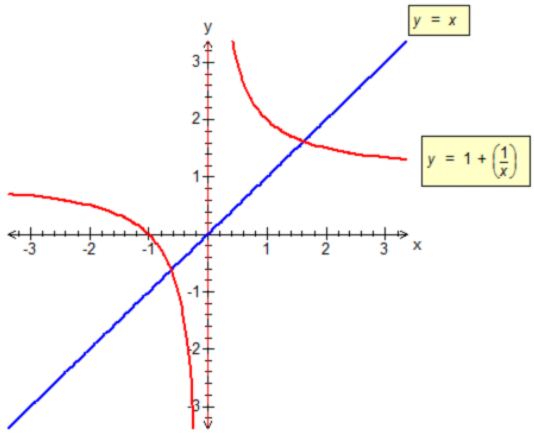
With this method, we start with an approximation to a root and improve its accuracy by substituting into a formula. We can then repeat this process until we have the desired level of accuracy.

To create a simple iterative formula, we rearrange the equation so that *x* is expressed as a function of itself. From there, we can turn it into an iterative formula. The example below will be given in considerable illustrated detail, but there is no need to memorise it for examination questions.

Example (3): Solve the equation $x^2 - x - 1 = 0$ by creating an iteration formula, and use 1 and –0.5 as the starting values for *x*. Stop the iteration when the result has converged to 4 decimal places.

(Although this is a quadratic, which can be solved using the general formula, we are choosing this example to illustrate the method. Its solutions are 1.618 and –0.618 to 3 decimal places.)

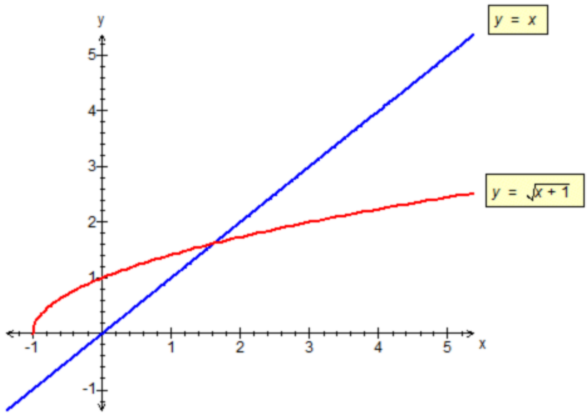
The equation can be rearranged in several ways.



One way is to re-express it (dividing by *x*) as
$x - 1 - \frac{1}{x} = 0 \Rightarrow x = 1 + \frac{1}{x}$.
This can be turned into an iterative formula by subscripting *x* on each side as follows:

$$x_{n+1} = 1 + \frac{1}{x_n}$$ **(Formula 3a)**

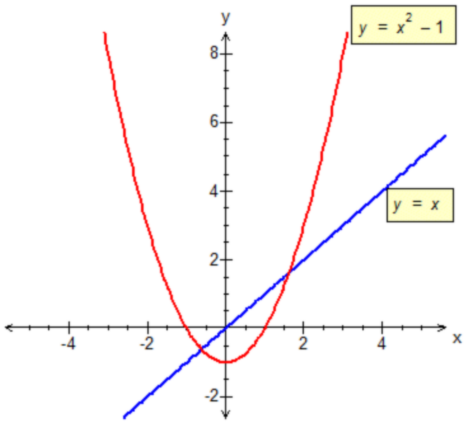
The roots of the original equation correspond to the *x*-coordinates of the points of intersection between the line $y = x$ and the curve $y = 1 + \frac{1}{x}$.



Another rearrangement is $x^2 = x + 1 \Rightarrow$
$x = \sqrt{(x+1)}$.
The corresponding iterative formula is

$$x_{n+1} = \sqrt{(x_n + 1)}$$ **(Formula 3b)**

The positive root of the original equation corresponds to the *x*- coordinate of the point of intersection between the line $y = x$ and the curve $y = \sqrt{(x+1)}$.



A third one is to re-express it as $x = x^2 - 1$.
The derived iterative formula is

$$x_{n+1} = x_n^{\ 2} - 1$$ **(Formula 3c)**

The roots of the original equation correspond to the *x*-coordinates of the points of intersection between the line $y = x$ and the curve $y = x^2 - 1$.

Using formula (3a): $x_{n+1} = 1 + \frac{1}{x_n}$

On the calculator, it is a matter of pressing the 1 and = keys, followed by entering $1 + \frac{1}{Ans}$, and then pressing the = key throughout until sufficient accuracy has been reached .

n	x_n	**n**	x_n	**n**	x_n
0	1	**5**	1.625	**10**	1.6180
1	2	**6**	1.6154	**11**	1.6181
2	1.5	**7**	1.6190	**12**	1.6180
3	1.6667	**8**	1.6176	**13**	1.6180
4	1.6	**9**	1.6182		

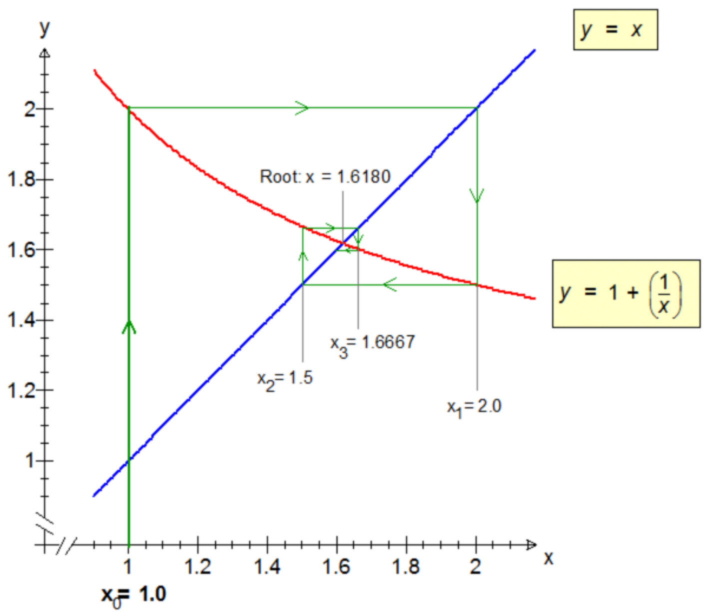
This converges to the positive root when choosing a starting value of 1.

This can be shown diagrammatically:
Start from point (1, 0) on the *x*-axis (corresponding to the starting *x*).

Move up to the curve, and then across to the line $y = x$.
This gives the next value of *x*, i.e. 2.

Move down to the curve, and across to the line $y = x$ again.
This gives the next *x*-value , i.e. 1.5.

Repeating this process creates a 'cobweb' diagram as each *x*-value oscillates around the root (1.6180 to 4 dp) in ever-smaller intervals.



When trying to find the negative root, choosing a starting value of 0 or -1 causes the iterative process to 'fall over' – one of several things that can go wrong with a formula iteration.

n	x_n
0	-1
1	0
2	Falls over - division by zero !

Different starting values also lead to wildly varying results:

n	x_n
0	-0.6
1	-0.6667
2	-0.5
3	-1
4	0
5	Falls over !

n	x_n	n	x_n
0	-0.7	6	1.8333
1	-0.4286	7	1.5455
2	-1.3333	8	1.6471
3	0.25	9	1.6071
4	5	10	1.6222
5	1.2	11	1.6164

A starting value of -0.6 causes the iteration to fall over, and a starting value of -0.7 ends up converging to the positive root of the equation.

Using formula (3b):

$$x_{n+1} = \sqrt{(x_n + 1)}$$

Calculator: key 1 and = keys, followed by entering $\sqrt{1+Ans}$ and then repeatedly pressing the = key.

n	x_n	***n***	x_n
0	1	**5**	1.6161
1	1.4142	**6**	1.6174
2	1.5538	**7**	1.6179
3	1.5981	**8**	1.6180
4	1.6118		

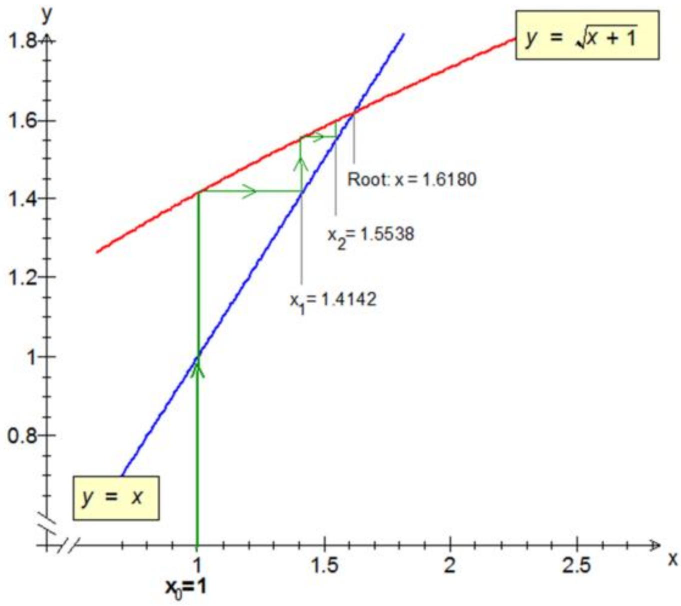
This also converges to the positive root when choosing a starting value of 1.

Start from point (1, 0) on the *x*-axis (corresponding to the starting *x*).

Move up to the curve, and then across to the line *y* = *x* to give the next value of *x*, or 1.4142 to 4 d.p.

Move up to the curve, and across to the line *y* = *x* again. This gives the next *x*-value, 1.5538 to 4 d.p.

Repeating this process creates a 'staircase' diagram as each *x*-value approaches the root (1.6180 to 4 dp) more closely. This is a different pattern from the 'cobweb' from formula (3a).



If we tried to find the other root (between 0 and –1) , there would be a problem:

n	x_n
0	-0.5
1	0.7071
2	1.3066
3	1.5187
4	1.5871
5	1.6084

This looks to be converging to the positive root as well.

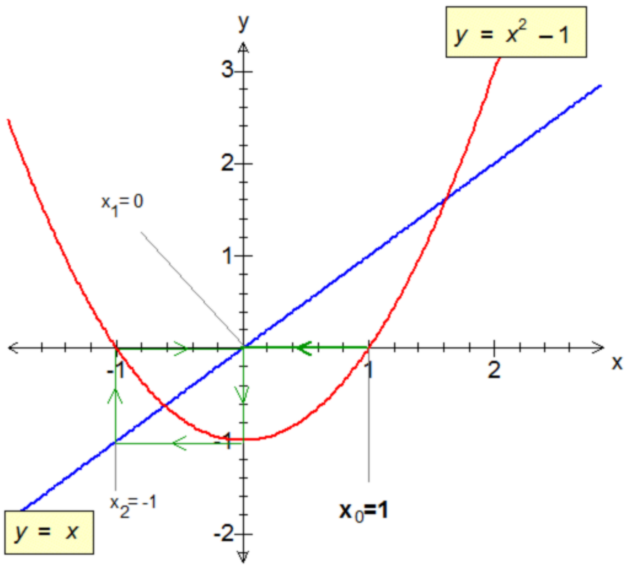
Using formula (3c): $x_{n+1} = x_n^{\ 2} - 1$

This formula will be found to be unsatisfactory for iteration, but will highlight several of the problems that might be encountered when choosing an inappropriate formula.

We will try and find the positive root of the equation by starting with $x_0 = 1$.

Calculator: press 1 and = keys, followed by entering $Ans^2 - 1$ and then repeatedly pressing the = key.

n	x_n
0	1
1	0
2	-1
3	0
4	-1



By choosing $x_0 = 1$, the formula appears to end up oscillating between 0 and -1 without giving a root.

Start from point (1, 0) on the *x*-axis, we are already on the curve, so we move across to the line $y = x$ to give the next value of *x*, namely x_1 or 0.

Moving down to the curve, and across to the line $y = x$ again, gives the next value of *x*, x_2 or –1.

Moving up to the curve, and across to the line $y = x$ again, gives the next value of *x*, x_3 or 0.

Moving down to the curve, and across to the line $y = x$ again, gives the next value of *x*, x_4 or –1.

The iteration has become stuck in an infinite loop !

Trying to find the positive root using $x_0 = 2$ is also no help !

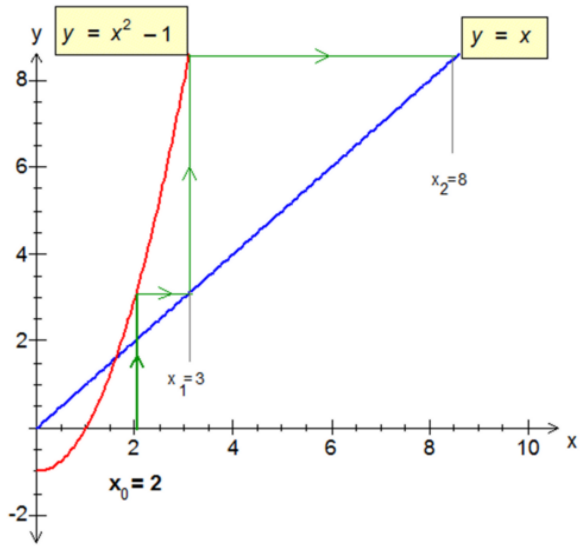
n	x_n
0	2
1	3
2	8
3	63
4	3968

The iteration 'blows up' by diverging further and further from the solution !

We start from point (2, 0) on the *x*-axis, so we move up to the curve and then across to the line $y = x$ to give the next value of *x*, namely x_1 or 3.

Moving up to the curve, and across to the line $y = x$ again, gives the next value of *x*, x_2 or 8.

It can clearly be seen that we have a 'diverging staircase' scenario, where the subsequent values of *x* become indefinitely large.



Finally we try and find the negative root using a value other than 0 or –1, say x_0 = -0.5.

We already know that using a starting value of 0 or –1 would lead us into an infinite loop, so we try a different starting value.

n	x_n	***n***	x_n
0	-0.5	**5**	-0.8802
1	-0.75	**6**	-0.2253
2	-0.4375	**7**	-0.9492
3	-0.8086	**8**	-0.0990
4	-0.3462	**9**	-0-9902

Start from point (-0.5, 0) on the *x*-axis (corresponding to x_0).

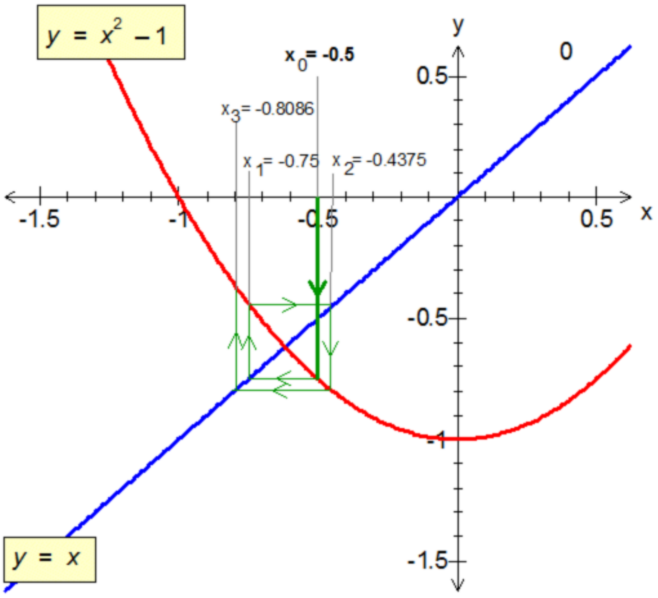
Move down to the curve, and then across to the line $y = x$.
This gives x_1, or –0.75.

Move up to the curve, and across to the line $y = x$ again.
This gives x_2 , i.e. –0.4375.

Move down to the curve, and then across to the line $y = x$.
This gives x_3, or –0.8086.

We can see that, although we have a ‘cobweb’ diagram here, successive iterations extend *outwards* away from the root, so that the process will not converge.

(In fact, it will end up oscillating between 0
and –1).



Some examination questions give a suggested formula for you.

Example (4): Verify that the formula $x_{n+1} = \dfrac{x_n^{\,2}+1}{2x_n-1}$ is a rearrangement of $x^2 - x - 1 = 0$.

Substitute x_0 =2 and -1, and hence find the roots of the equation to 6 decimal places. What do you notice about the convergence ?

Removing the subscripts and rearranging the iteration formula we have $x = \dfrac{x^2+1}{2x-1}$

$\Rightarrow x(2x-1) = x^2 + 1 \Rightarrow 2x^2 - x = x^2 + 1 \Rightarrow 2x^2 - x - x^2 - 1 = 0 \Rightarrow \mathbf{x^2 - x - 1 = 0}$.

Key 2 and =, followed by entering $\dfrac{Ans^2+1}{2(Ans)-1}$ and then repeatedly pressing the = key.

Repeat the above, but with start value of -1.

n	x_n
0	2
1	1.666667
2	1.619048
3	1.618034
4	1.618034

n	x_n
0	-1
1	-0.666667
2	-0.619048
3	-0.618034
4	-0.618034

This rearrangement converges far more rapidly than the other two, giving 6-decimal place values of 1.618034 and –0.618034.

Example (5): Find the roots of $e^x - 5x - 4 = 0$ by choosing suitable iteration formulae, to 4 decimal places. Use 3 and –1 as the starting values.

Rearrange as $e^x = 5x + 4$, and by taking logs of both sides, we have $x = \ln(5x + 4)$ and hence $x_{n+1} = \ln(5x_n + 4)$

Key 3 and = , followed by entering $\ln(5Ans + 4))$ and then repeatedly pressing the = key.

n	x_n	**n**	x_n
0	3	**5**	2.9244
1	2.9444	**6**	2.9244
2	2.9297	**7**	2.9243
3	2.9258	**8**	2.9243
4	2.9247	**9**	2.9243

This rearrangement falls over at once when starting with $x = -1$, as the first step would attempt to find ln (–1).

We can also rearrange the original equation as

$5x = e^x - 4 \Rightarrow x = \frac{e^x - 4}{5}$ and thus $x_{n+1} = \frac{e^{x_n} - 4}{5}$.

Key -1 and = , followed by entering $\frac{e^{Ans} - 4}{5}$ and then repeatedly pressing the = key.

The first root was found with a starting value of 3, so try finding the root using a start value of -1:

n	x_n	**n**	x_n
0	-1	**5**	-0.7008
1	-0.7264	**6**	-0.7008
2	-0.7033		
3	-0.7010		
4	-0.7008		

$\therefore$ the roots of $e^x - 5x - 4 = 0$ are 2.9243 and –0.7008 to 4 decimal places.

Example (6) : i) Show that the equation $9x^2 - \tan^{-1} x = 0$ where x is in radians and non-zero, produces the iteration formula $x_{n+1} = \frac{1}{3}\sqrt{\tan^{-1}(x_n)}$.

ii) Use a starting value of $x = 0.1$ to solve the equation $9x^2 - \tan^{-1} x = 0$ correct to 3 decimal places.

i) We rearrange the original equation as $9x^2 = \tan^{-1} x$, and taking square roots of both sides,

$$3x = \sqrt{\tan^{-1}(x)} \Rightarrow x = \frac{1}{3}\sqrt{\tan^{-1}(x)}$$

The final iteration formula is $x_{n+1} = \frac{1}{3}\sqrt{\tan^{-1}(x_n)}$.

Key 0.1 and = , followed by entering $\frac{1}{3}\sqrt{\tan^{-1}(Ans)}$ and then repeatedly pressing the = key.

The iteration results are :

n	x_n	**n**	x_n
0	0.1	**5**	0.1103
1	0.1052	**6**	0.1105
2	0.1079	**7**	0.1106
3	0.1093	**8**	0.1106
4	0.1100	**9**	0.1106

The non-zero solution of $9x^2 - \tan^{-1} x = 0$ is 0.111 to 3 decimal places.

Checking the suitability of an iteration formula.

Sometimes, an iterative formula might be unsatisfactory because a) convergence is too slow, b) the iteration becomes 'stuck' in an infinite loop, c) the iteration 'blows up' or d) the iteration 'falls over'.

The full study of iterative formulae requires difficult analysis beyond the scope of A-level, and therefore an examination question would include a 'ready-tweaked' formula, as in the next examples.

Example (7): i) Show that the iterative formula $x_{n+1} = \dfrac{x_n^{\ 3} - 6}{2x_n(x_n - 2)}$ can be obtained by rearranging the equation $x^3 - 4x^2 + 6 = 0$.

ii) Hence, find a root of $x^3 - 4x^2 + 6 = 0$, using a starting value of 3.

i) We begin by writing the formula as $x = \left(\dfrac{x^3 - 6}{2x(x-2)}\right) \Rightarrow \ 2x^2(x-2) = x^3 - 6$

$$\Rightarrow \ 2x^3 - 4x^2 - x^3 + 6 = 0 \Rightarrow \ x^3 - 4x^2 + 6 = 0$$

ii) Results of the iteration using the formula $x_{n+1} = \dfrac{x_n^{\ 3} - 6}{2x_n(x_n - 2)}$

Key 3 and = , followed by entering $\dfrac{Ans^3 - 6}{2Ans(Ans - 2)}$ and then repeatedly pressing the = key.

n	x_n
1	3
2	3.5
3	3.5119
4	3.5138
5	3.5141

This iteration formula has converged to the root between 3 and 4, and its value is 3.5141 to 4 decimal places.

Example (8a): i) Show that the iterative formula $x_{n+1} = \frac{1}{2}\sqrt{x_n^{\ 3} + 6}$ can also be obtained by rearranging the equation $x^3 - 4x^2 + 6 = 0$.

ii) Hence, find a root of $x^3 - 4x^2 + 6 = 0$, using a starting value of 2.

i) Rearranging, $x^3 - 4x^2 + 6 = 0 \Rightarrow 4x^2 = x^3 + 6 \Rightarrow 2x = \sqrt{x^3 + 6} \Rightarrow x = \frac{1}{2}\sqrt{x^3 + 6}$

for an iterative formula of $x_{n+1} = \frac{1}{2}\sqrt{x_n^{\ 3} + 6}$.

Key 2 and =, then $\frac{1}{2}\sqrt{Ans^3 + 6}$ and = key ;

n	x_n	**n**	x_n	**n**	x_n
0	2	**5**	1.6206	**21**	1.5720
1	1.8708	**...**	....		
2	1.7712	**10**	1.5756		
3	1.6997	**...**	....		
4	1.6516....	**20**	1.5720		

This iteration formula converges to the root between 1 and 2, and its value is 1.5720 to 4 decimal places, but the convergence is a bit slow.

Example(8b): i) Show that the iterative formula $x_{n+1} = \frac{1}{2}\left(\frac{6}{x_n^{\ 2}} + 3x_n - 4\right)$ can be obtained by rearranging the equation $x^3 - 4x^2 + 6 = 0$.

ii) Hence, find the root of $x^3 - 4x^2 + 6 = 0$ lying between 1 and 2, using a starting value of 2. How does this compare to the result from Example (8a) ?

In Example (8a), we used a different iterative formula and obtained a convergence to 1.5720, but the process was rather slow, taking 20 iterations to achieve accuracy to 4 decimal places.

i) We begin with $x = \frac{1}{2}\left(\frac{6}{x^2} + 3x - 4\right) \Rightarrow 2x = \frac{6}{x^2} + 3x - 4$

$$\Rightarrow -x = \frac{6}{x^2} - 4 \Rightarrow x = 4 - \frac{6}{x^2} \Rightarrow x^3 = 4x^2 - 6 \Rightarrow x^3 - 4x^2 + 6 = 0$$

ii) Results: Key 2 and =, then $\frac{1}{2}\left(\frac{6}{Ans^2} + 3Ans - 4\right)$, followed by repeated clicks of the = key.

n	x_n
0	2
1	1.75
2	1.6046
3	1.5721
4	1.5720
5	1.5720

This iterative formula converges to the root of 1.5720 much more rapidly than the iterative formula of $x_{n+1} = \frac{1}{2}\sqrt{x_n^{\ 3} + 6}$ from Example (8a), which took over 20 iterations.

Example(9): i) Show that the iterative formula $x_{n+1} = \frac{1}{10}\left(4 - \frac{6}{{x_n}^2} + 9x_n\right)$ can be obtained by rearranging the equation $x^3 - 4x^2 + 6 = 0$.

ii) Hence, find the negative root of $x^3 - 4x^2 + 6 = 0$, using a starting value of -1.

i) We begin by writing the formula as $x = \frac{1}{10}\left(4 - \frac{6}{x^2} + 9x\right) \Rightarrow 10x = 4 - \frac{6}{x^2} + 9x$

$$\Rightarrow x = 4 - \frac{6}{x^2} \Rightarrow x^3 = 4x^2 - 6 \Rightarrow x^3 - 4x^2 + 6 = 0$$

ii) The results of the formula iteration are as follows:

Key -1- and =, then $\frac{1}{10}\left(4 - \frac{6}{Ans^2} + 9Ans\right)$, then = (repeat).

n	x_n
0	-1
1	-1.1
2	-1.0859
3	-1.0861
4	-1.0861

The negative root of $x^3 - 4x^2 + 6 = 0$ = -1.0861 to 4 decimal places.

Example (10): (Omnibus exam-style question)

i) A function is defined as $f(x) = x - \sin x - 1$. (**Note:** x is measured in radians !)
Show that the equation $f(x) = 0$ has a solution in the range $1.9 < x < 2.0$.

ii) Form an iteration formula from the equation $f(x) = 0$.

iii) Starting with $x_0 = 2$ for the iteration, use this formula to find the values of x_1, x_2 and x_3.

iv) Hence find the solution of the equation $f(x) = 0$ to four decimal places.

i) Substituting $x = 1.9$ into the equation gives $1.9 - \sin(1.9^c) - 1 = -0.046$.
Substituting $x = 2.0$ into the equation gives $2.0 - \sin(2.0^c) - 1 = 0.091$.

Given that $f(x) = x - \sin x - 1$ is a continuous function, the change of sign in the value of the function between $x = 1.9$ and $x = 2.0$ shows that there is a root in the interval, probably closer to 1.9.

ii) The expression $x - \sin x - 1$ can be rewritten as $x = \sin x + 1$, from which the iteration formula of $x_{n+1} = \sin x_n + 1$ can be derived.

iii) Using the iteration formula of $x_{n+1} = \sin x_n + 1$, and a starting value of $x_0 = 2$, the next three iterations give:

Key 2 and = , then $\sin(Ans) + 1$, then = (repeat).

$x_1 = 1.9093$ $x_2 = 1.9433$ $x_3 = 1.9314$.

iv) Further iterations give:

$x_4 = 1.9357$ $x_5 = 1.9342$ $x_6 = 1.9347$ $x_7 = 1.9345$ $x_8 = 1.9346$

The iteration converges to the root of $f(x) = x - \sin x - 1 = 0$, or 1.9346 to four decimal places.