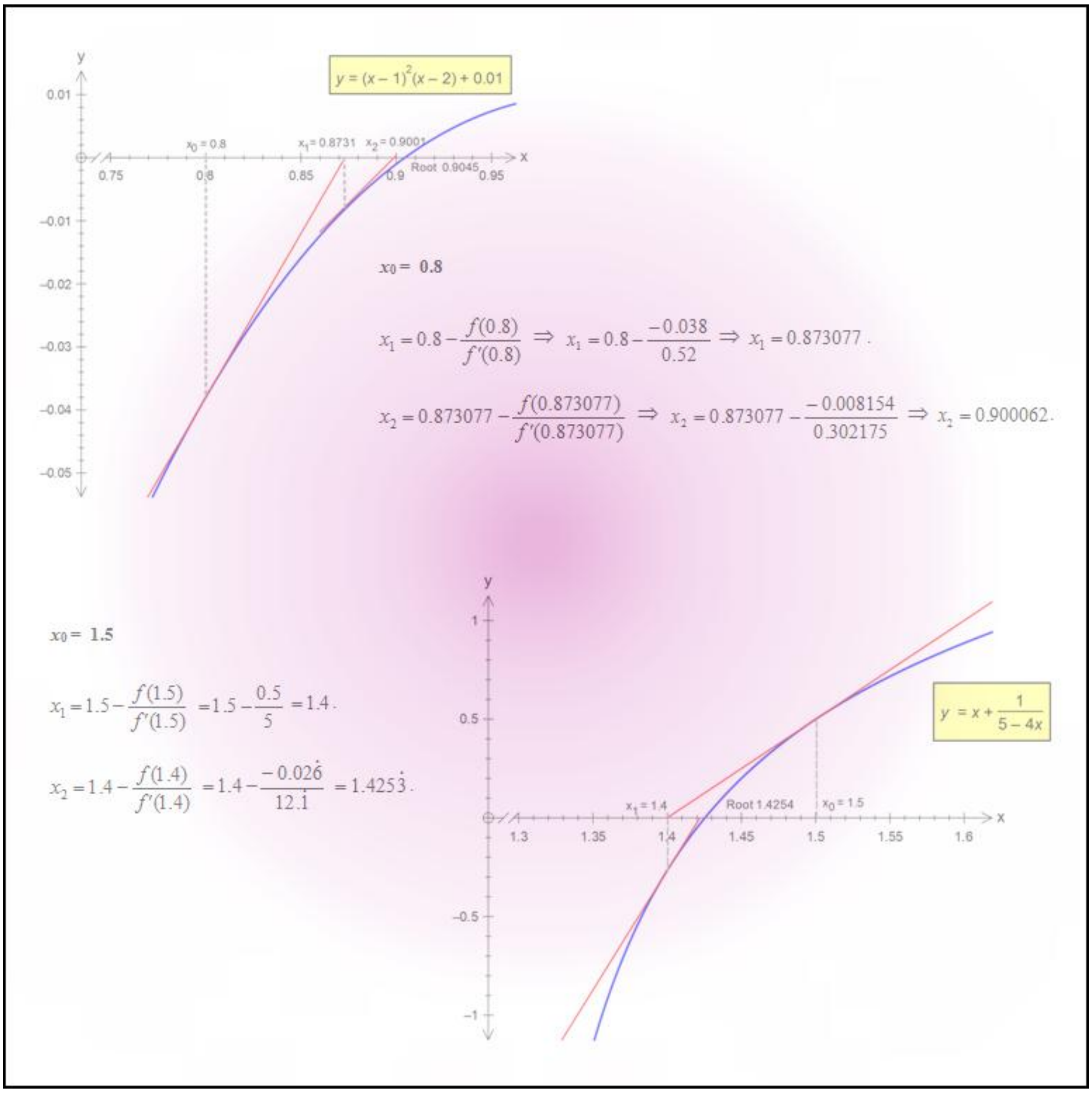
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

THE NEWTON-RAPHSON METHOD FOR SOLVING EQUATIONS



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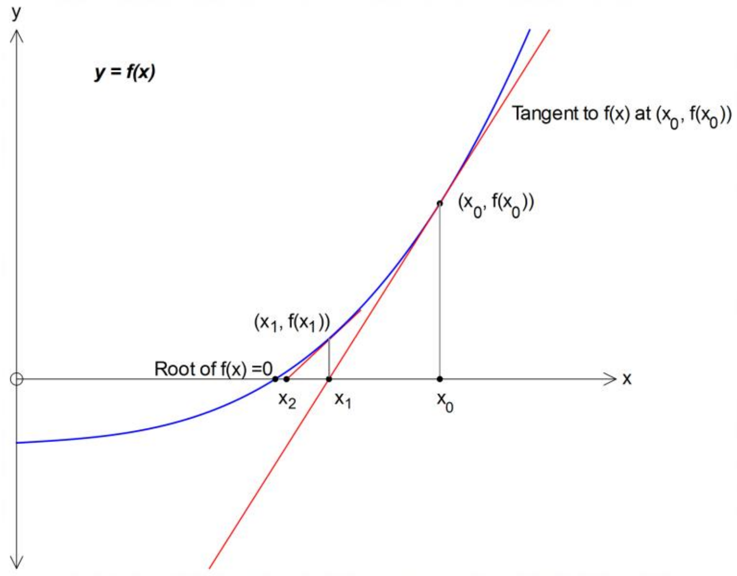
The Newton-Raphson method.

This is another iterative method of finding roots of an equation.



Take the equation $f(x) = 0$ with an unknown root α, in other words $f(\alpha) = 0$.

We begin with an initial estimated value for the root, labelling it x_0. As can be seen on the graph, this estimate is not particularly close to the true root, but we can draw the tangent at the point $(x_0, f(x_0))$ until it meets the x-axis at $(x_1, 0)$ where x_1 is a better approximation to the root.

If we draw another tangent to the curve, this time at $(x_1, f(x_1))$, then that tangent will meet the x-axis at $(x_2, 0)$ - even closer to the root. We can then continue until we have the desired accuracy.

The original tangent to $f(x)$ passes through the point $(x_0, f(x_0))$ and its gradient is $f'(x_0)$.

Its equation is of the form $y - y_0 = m(x - x_0)$ where $y_0 = f(x_0)$ and $m = f'(x_0)$.

Thus $y - f(x_0) = f'(x_0)(x - x_0)$, and as $y = 0$ at the root, we can rearrange the equation as $f'(x_0)(x - x_0) = - f(x_0)$

Dividing both sides by $f'(x_0)$ we have $x - x_0 = -\dfrac{f(x_0)}{f'(x_0)}$, and $x = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

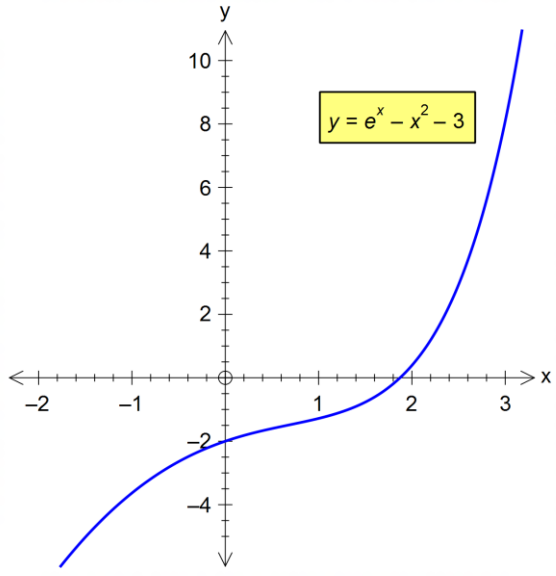
This last result can be written as an iterative formula, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

This is the **Newton-Raphson method**.

The method usually converges quickly to the required root, but it might fail in exceptional cases, such as when $f'(x)$ is close to zero or if there are discontinuities in the curve.

(On the occasions when the method fails, a better first approximation will usually give convergence.)

Example (1): Find the root of $e^x - x^2 - 3 = 0$, using a starting value, x_0, of 2. Use three iterations of the Newton-Raphson method and give the result to 4 decimal places.



This equation has a root fairly close to 2. (See graph)

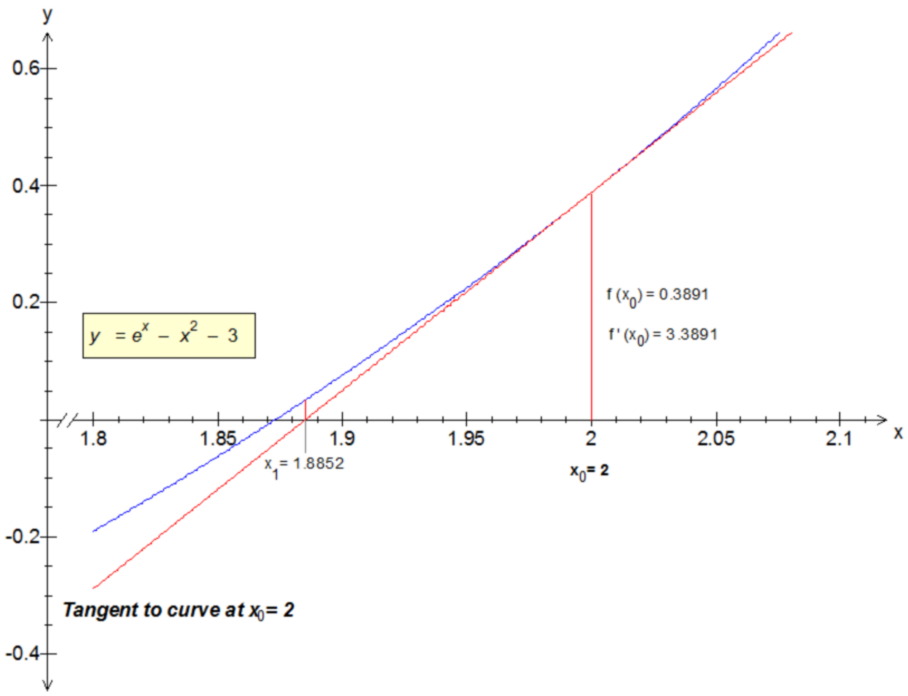
We need to differentiate the function first

$f(x) = e^x - x^2 - 3$
$f'(x) = e^x - 2x$

Substituting for $x_0 = 2$, we have

$$x_1 = 2 - \frac{f(2)}{f'(2)}$$

$$\Rightarrow \; x_1 = 2 - \frac{e^2 - 7}{e^2 - 4} \Rightarrow \; x_1 = 2 - \frac{0.389056}{3.389056} \Rightarrow \; x_1 = 1.885202.$$



The diagram illustrates the method geometrically. By continuing the tangent to the curve until it meets the x-axis, we will have an improved estimate of the actual root – here it is $x_1 = 1.8852$ to 4 d.p.

If we draw another tangent to the curve, this time at $(x_1, f(x_1))$, then that tangent will meet the x-axis even closer to the root.

We can then repeat the iterations as often as needed for the required accuracy..

Repeated iterations give

$$x_2 = 1.885202 - \frac{f(1.885202)}{f'(1.885202)} \Rightarrow x_2 = 1.885202 - \frac{0.033699}{2.817282} \Rightarrow x_2 = 1.873241.$$

$$x_3 = 1.873241 - \frac{f(1.873241)}{f'(1.873241)} \Rightarrow x_3 = 1.873241 - \frac{0.000326}{2.762876} \Rightarrow x_3 = 1.873123.$$

The root of the equation is 1.8731 to 4 decimal places.

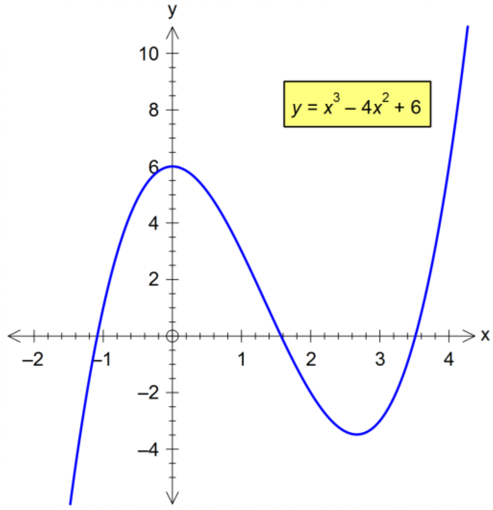
(Since $f(1.87305) = -0.0002$ and $f(1.873123) = 0.000001$, there is no danger of the result creeping down to 1.8730).

We use the *Ans* key on the calculator to perform the iterations rapidly.

$f(x) = e^x - x^2 - 3$ and $f'(x) = e^x - 2x$, so we can work as

Key 2 and = , then $Ans - \left(\frac{e^{Ans} - Ans^2 - 3}{e^{Ans} - 2Ans}\right)$, then repeat = until the desired accuracy is reached.

Example (2): Use the Newton-Raphson method with three iterations to find to 4 decimal places the three roots of the equation $x^3 - 4x^2 + 6 = 0$.



Use starting values of x = -1.1, 1.6 and 3.5.

The derivative of $f(x) = x^3 - 4x^2 + 6$ is $f'(x) = 3x^2 - 8x$.

x_0 = -1.1

Key : -1.1 and =, then $Ans - \left(\frac{Ans^3 - 4Ans^2 + 6}{3Ans^2 - 8Ans}\right)$, then = twice .

$$x_1 = -1.1 - \frac{f(-1.1)}{f'(-1.1)} \Rightarrow x_1 = -1.1 - \frac{-0.171}{12.43}$$

$\Rightarrow x_1 = -1.086243$.

$$x_2 = -1.086243 - \frac{f(-1.086243)}{f'(-1.086243)} \Rightarrow x_2 = -1.086243 - \frac{-0.00138}{12.22971} \Rightarrow x_2 = -1.086130.$$

The third iteration gives x_3 = -1.086130.

x_0 = 1.6

Key : 1.6 and =, then $Ans - \left(\frac{Ans^3 - 4Ans^2 + 6}{3Ans^2 - 8Ans}\right)$, then = twice.

$$x_1 = 1.6 - \frac{f(1.6)}{f'(1.6)} \Rightarrow x_1 = 1.6 - \frac{-0.144}{-5.12} \Rightarrow x_1 = 1.571875.$$

$$x_2 = 1.571875 - \frac{f(1.571875)}{f'(1.571875)} \Rightarrow x_2 = 1.571875 - \frac{0.000611}{-5.16263} \Rightarrow x_2 = 1.571993.$$

The third iteration gives x_3 = 1.571993.

x_0 = 3.5

Key : 3.5 and =, then $Ans - \left(\frac{Ans^3 - 4Ans^2 + 6}{3Ans^2 - 8Ans}\right)$, then = twice .

$$x_1 = 3.5 - \frac{f(3.5)}{f'(3.5)} \Rightarrow x_1 = 3.5 - \frac{-0.125}{8.75} \Rightarrow x_1 = 3.514286.$$

$$x_2 = 3.514286 - \frac{f(3.514286)}{f'(3.514286)} \Rightarrow x_2 = 3.514286 - \frac{0.001329}{8.936327} \Rightarrow x_2 = 3.514137.$$

The third iteration gives x_3 = 3.514137.

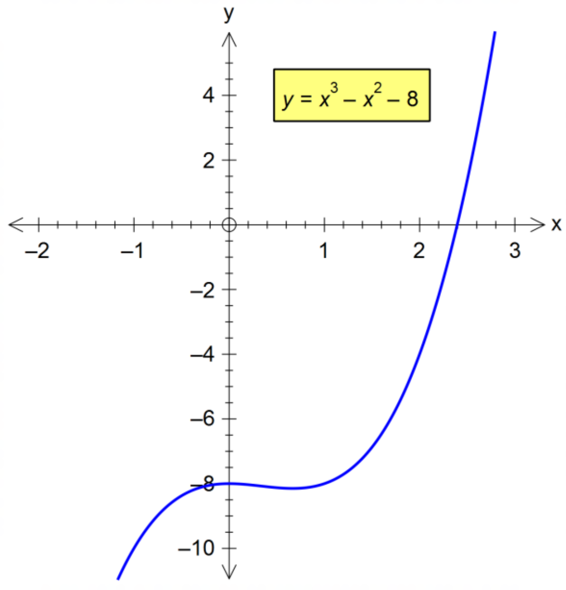
$\therefore$ The roots of $x^3 - 4x^2 + 6 = 0$ are -1.0861, 1.5720 and 3.5141 to 4 decimal places.

In actual fact, the values are also correct to six places after the third iteration.

Example (3): The equation of $f(x) = x^3 - x^2 - 8 = 0$ has a single root between 2 and 3.

i) Use three iterations of the Newton-Raphson method and a starting value, x_0, of 2.5, to calculate this root to 4 decimal places.

ii) Use a suitable lower bounding value of x to show that the result in part i) is indeed correct to 4 decimal places



i) The derivative of $f(x) = x^3 - x^2 - 8$ is $f'(x) = 3x^2 - 2x$.

$\mathbf{x_0 = 2.5}$

Key : 2.5 and =, then $Ans - \left(\dfrac{Ans^3 - Ans^2 - 8}{3Ans^2 - 2Ans}\right)$, then = three times .

$$x_1 = 2.5 - \frac{f(2.5)}{f'(2.5)} \Rightarrow x_1 = 2.5 - \frac{1.375}{13.75} \Rightarrow x_1 = 2.4.$$

$$x_2 = 2.4 - \frac{f(2.4)}{f'(2.4)} \Rightarrow x_2 = 2.4 - \frac{0.064}{12.48} \Rightarrow x_2 = 2.394872.$$

$$x_3 = 2.394872 - \frac{f(2.394872)}{f'(2.394872)} \Rightarrow x_3 = 2.394872 - \frac{0.000163}{12.416489} \Rightarrow x_3 = 2.394859.$$

The third iteration gives $x_3 = 2.394859$.

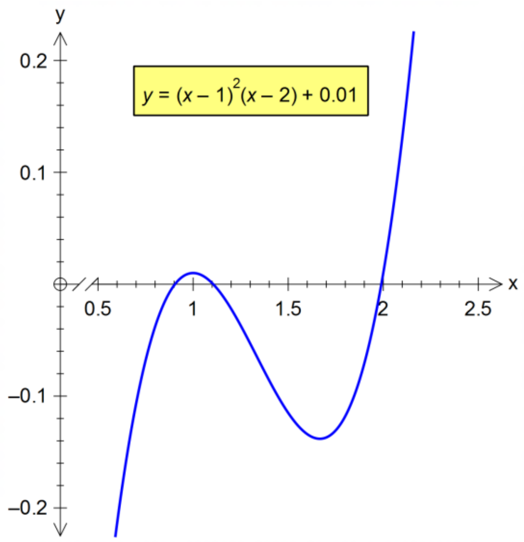
The root of $f(x) = x^3 - x^2 - 8 = 0$ is thus 2.3949 to 4 decimal places.

ii) Since $f(2.39485) = -0.0001$ and $f(2.394872) = 0.000163$, there is a change of sign in $f(x)$ between $x = 2.39485$ and $x = 2.394872$.

There is therefore no danger of the result 'converging down' to a value that rounds down to 2.3948.

When Newton-Raphson goes wrong.

Example (4): The equation of $f(x) = (x-1)^2(x-2) + 0.01 = 0$ has three roots. One root is close to 2, and the other two are close to 1.



i) Use the Newton-Raphson method with a starting value, x_0, of 2, to find the root close to 2. Continue iterating until the result is correct to 4 decimal places.

ii) Use the Newton-Raphson method with a starting value, x_0, of 0.8. Continue iterating until the result is correct to 4 decimal places.

iii) Repeat part ii) with $x_0 = 1.2$.

iv) What happens if we use a starting value of 0.996 ?

v) What happens if we use a starting value of 1 ? (Hint: work out $f(1)$ and $f'(1)$ separately.)

i) The expanded form of the same equation is $f(x) = x^3 - 4x^2 + 5x - 1.99 = 0$.

Differentiating, $f'(x) = 3x^2 - 8x + 5$.

$\mathbf{x_0 = 2}$

Key : 2 and = , then $Ans - \left(\dfrac{Ans^3 - 4Ans^2 + 5Ans - 1.99}{3Ans^2 - 8Ans + 5}\right)$, then = as required

$$x_1 = 2 - \frac{f(2)}{f'(2)} \Rightarrow x_1 = 2 - \frac{0.01}{1} \Rightarrow x_1 = 1.99.$$

$$x_2 = 1.99 - \frac{f(1.99)}{f'(1.99)} \Rightarrow x_2 = 1.99 - \frac{0.000199}{0.9603} \Rightarrow x_2 = 1.989793.$$

The next iteration is $x_3 = 1.989793$.

One root of $x^3 - 4x^2 + 5x - 1.99 = 0$ is $x = 1.9898$ to 4 decimal places.

ii) $\mathbf{x_0 = 0.8}$

Key : 0.8 and = , then $Ans - \left(\dfrac{Ans^3 - 4Ans^2 + 5Ans - 1.99}{3Ans^2 - 8Ans + 5}\right)$, then = as required

$$x_1 = 0.8 - \frac{f(0.8)}{f'(0.8)} \Rightarrow x_1 = 0.8 - \frac{-0.038}{0.52} \Rightarrow x_1 = 0.873077.$$

$$x_2 = 0.873077 - \frac{f(0.873077)}{f'(0.873077)} \Rightarrow x_2 = 0.873077 - \frac{-0.008154}{0.302175} \Rightarrow x_2 = 0.900062.$$

Subsequent iterations are $x_3 = 0.904351$, $x_4 = 0.904460$, $x_5 = 0.904460$.

Another root of $x^3 - 4x^2 + 5x - 1.99 = 0$ is $x = 0.9045$ to 4 decimal places.

iii) $\mathbf{x_0 = 1.2}$

Key : 1.2 and = , then $Ans - \left(\frac{Ans^3 - 4Ans^2 + 5Ans - 1.99}{3Ans^2 - 8Ans + 5} \right)$, then = as required

$$x_1 = 1.2 - \frac{f(1.2)}{f'(1.2)} \Rightarrow x_1 = 1.2 - \frac{-0.022}{-0.28} \Rightarrow x_1 = 1.121429.$$

$$x_2 = 1.121429 - \frac{f(1.121429)}{f'(1.121429)} \Rightarrow x_2 = 1.121429 - \frac{-0.002954}{-0.198622} \Rightarrow x_2 = 1.106554.$$

Subsequent iterations are $x_3 = 1.105750$, $x_4 = 1.105747$, $x_5 = 1.105747$.

The third root of $x^3 - 4x^2 + 5x - 1.99 = 0$ is $x = 1.1057$ to 4 decimal places.

iv) $\mathbf{x_0 = 0.996}$

Key : 0.996 and = , then $Ans - \left(\frac{Ans^3 - 4Ans^2 + 5Ans - 1.99}{3Ans^2 - 8Ans + 5} \right)$, then = as required

$$x_1 = 0.996 - \frac{f(0.996)}{f'(0.996)} \Rightarrow x_1 = 0.996 - \frac{-0.0099839}{-0.008048} \Rightarrow x_1 = -0.244549.$$

$$x_2 = -0.244549 - \frac{f(-0.244549)}{f'(-0.244549)} \Rightarrow x_2 = -0.244549 - \frac{-3.466585}{7.135802} \Rightarrow x_2 = 0.241253.$$

Subsequent iterations are $x_3 = 0.550231$, $x_4 = 0.738278$, ... $x_9 = 0.904460$.

The iteration appears to 'blow up' at the start, as the 'improved' estimate to the root at x_1 is much further away from the true root than the first one of 0.996. The process *does* eventually converge to the true root, but takes rather a lot of iterations to get there.

v) $\mathbf{x_0 = 1}$

If we use a starting value of $x_0 = 1$, we run into a problem.

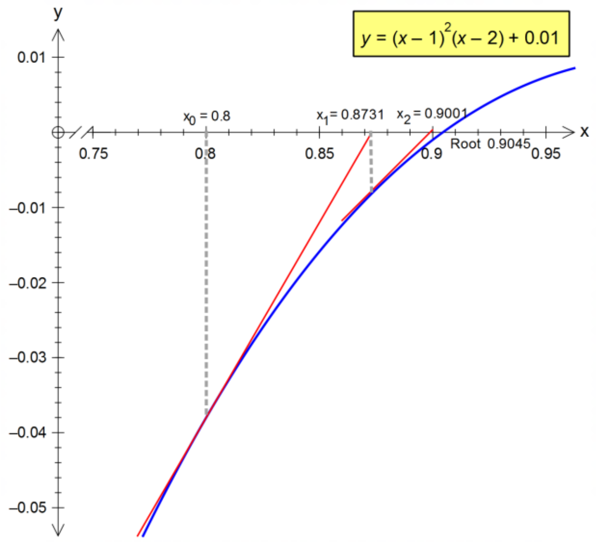
$$x_1 = 1 - \frac{f(1)}{f'(1)} \Rightarrow x_1 = 1 - \frac{0.01}{0}.$$ Division by zero !

The iteration 'falls over' because $f'(1) = 0$, i.e. the function has a stationary point at $x = 1$.

The diagram on the right shows the result of the 'well-behaved' iteration in Example (4(ii)) with the starting value of 0.8.

Each tangent to the curve ends up meeting the *x*-axis at values closer to the root in each case.

The iterations converge to the root at 0.9045 throughout the process.

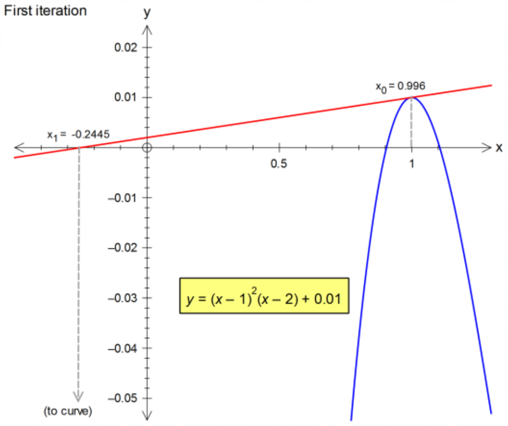


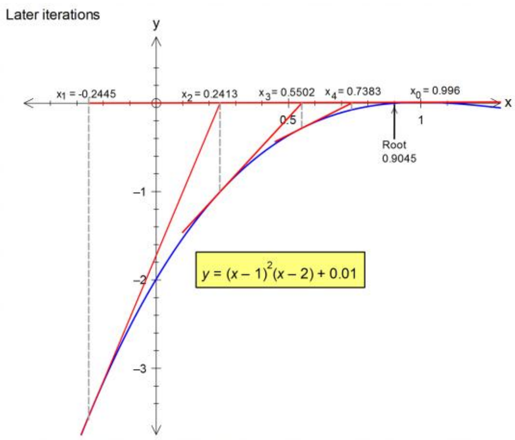
The diagrams below tell a different story when we choose an unsuitable value of $x_0 = 0.996$ in part (iv).

Because the tangent to the curve where $x = 0.996$ is almost horizontal, this tangent crosses the *x*-axis a long way from the root, and the process takes more steps than with the starting value of 0.8.

(The graph axes for the first iteration are on different scales from that for the subsequent ones).

The first graph shows how the 'improved' estimate for the root is far worse than the original, and the second one shows the 'recovery'.



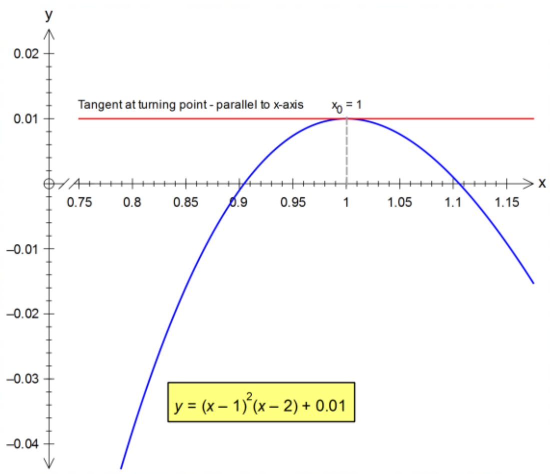


The final diagram shows what happens if we choose a starting value which coincides with a turning point on the graph – here $x_0 = 1$.

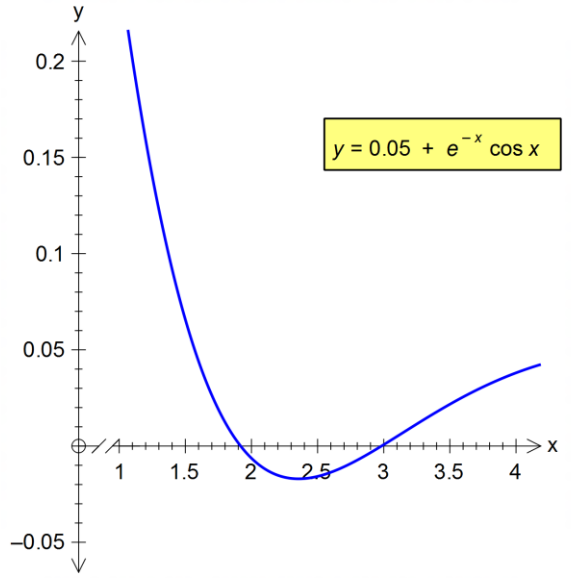
The Newton-Raphson iteration 'falls over' with a 'division by zero' error .

The gradient of the tangent at the turning point is zero, so the tangent is now horizontal, i.e. parallel to the *x*-axis.

Since the tangent never intersects the *x*-axis, we cannot continue the iterative process and find any further estimates for the root.



Example (5): The equation of $f(x) = 0.05 + e^{-x} \cos x = 0$ has a root close to 2.
Starting with $x_0 = 2$, use the Newton-Raphson method to find this root to 4 decimal places.
(Remember to work in radians !)



$f'(x) = -e^{-x} \cos x + e^{-x}(-\sin x) = -e^{-x}(\cos x + \sin x)$. (Product rule)

$\mathbf{x_0 = 2}$

Key : 2 and = , then $Ans - \left(\dfrac{0.05 + e^{-Ans} \cos(Ans)}{-e^{-Ans}(\cos(Ans) + \sin(Ans))} \right)$, then = as required

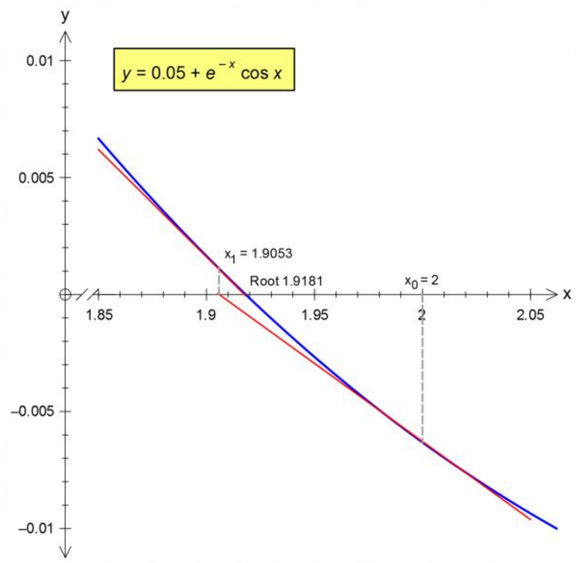
$$x_1 = 2 - \frac{f(2)}{f'(2)} \Rightarrow x_1 = 2 - \frac{-0.006319}{-0.066741} \Rightarrow x_1 = 1.905315.$$

$$x_2 = 1.905315 - \frac{f(1.905315)}{f'(1.905315)}$$
$$\Rightarrow x_2 = 1.905315 - \frac{0.001155}{-0.091684}$$
$$\Rightarrow x_2 = 1.917910.$$

Subsequent iterations are $x_3 = 1.918161$ and $x_4 = 1.918161$.

The required root of $f(x) = 0.05 + e^{-x} \cos x = 0$ is 1.9181.

The iteration is well-behaved here, with rapid convergence to the root.



Example (5a): Repeat Example (5), but with a starting value of $x_0 = 2.5$.

$f(x) = 0.05 + e^{-x} \cos x$
$f'(x) = -e^{-x} \cos x + e^{-x}(-\sin x) = -e^{-x}(\cos x + \sin x)$. (Product rule)

$\mathbf{x_0 = 2.5}$

Key : 2.5 and =, then $Ans - \left(\frac{0.05 + e^{-Ans} \cos(Ans)}{-e^{-Ans}(\cos(Ans) + \sin(Ans))} \right)$, then = as required

$$x_1 = 2.5 - \frac{f(2.5)}{f'(2.5)} \Rightarrow x_1 = 2.5 - \frac{-0.015762}{0.016636} \Rightarrow x_1 = 3.447439.$$

$$x_2 = 3.447439 - \frac{f(3.447439)}{f'(3.447439)} \Rightarrow x_2 = 3.447439 - \frac{0.019650}{0.039933} \Rightarrow x_2 = 2.995367.$$

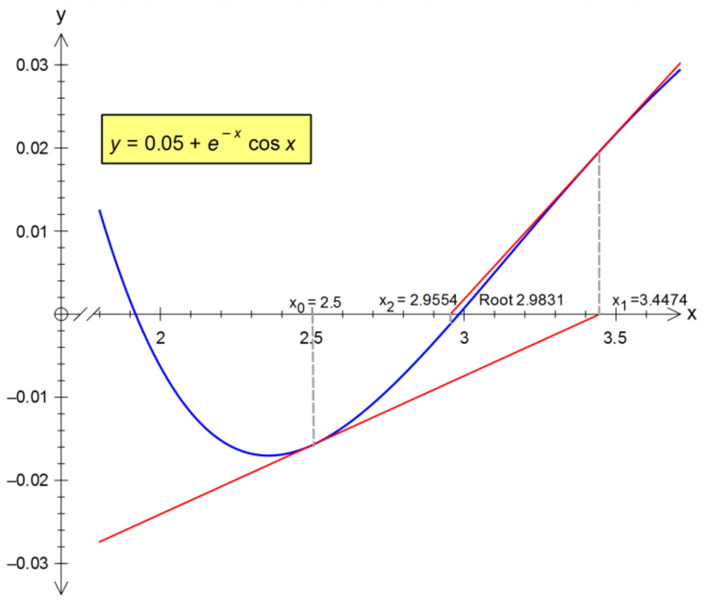
Subsequent iterations are $x_3 = 2.983292$, $x_4 = 2.983123$ and $x_5 = 2.983123$.

The iteration appears to diverge and then converge, but to a different root from the one in the question.

The starting value, x_0, namely 2.5, is on the 'wrong' side of the stationary point at $x = \frac{3\pi}{4}$, or $x = 2.3562$.

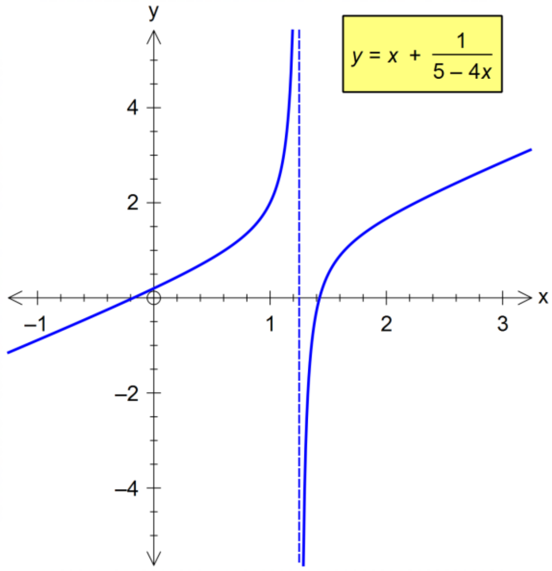
As a result, the 'improved ' estimate of x_1 or 3.4474, diverges away from the required root in the opposite direction.

Convergence is restored with the next iteration, but the process ends up homing in on the *other* root of the equation, i.e. $x = 2.9831$.



Example(6): A function is defined as

$$f(x) = x + \frac{1}{5-4x}, \quad x \neq \frac{5}{4}.$$



The equation $f(x) = 0$ has a root between 1 and 2.
Starting with $x_0 = 1.5$, use the Newton-Raphson method to find this root to 4 decimal places.

Rewrite the function as $f(x) = x + (5 - 4x)^{-1}$.
The derivative is $f'(x) = 1 + 4(5 - 4x)^{-2}$. (Chain rule).

$\mathbf{x_0 = 1.5}$

Key : 1.5 and =, then

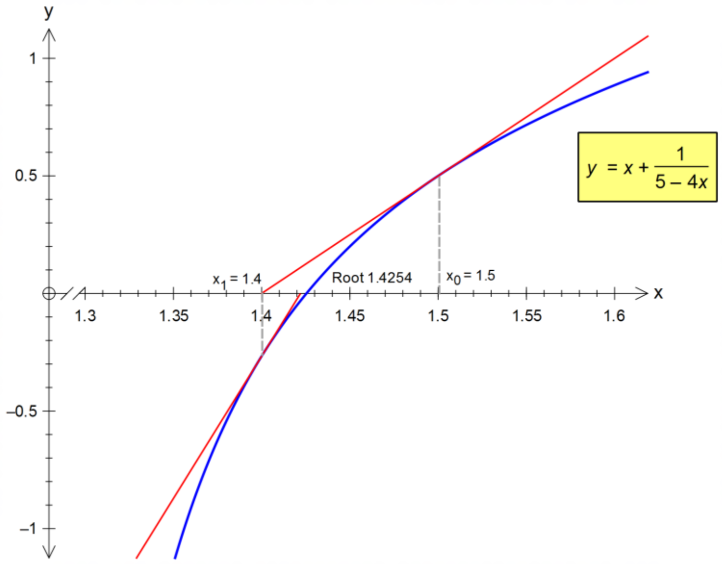
$Ans - \left(\frac{Ans + (5-4Ans)^{-1}}{1+4(5-4Ans)^{-2}}\right)$, then = as required

$$x_1 = 1.5 - \frac{f(1.5)}{f'(1.5)} \Rightarrow x_1 = 1.5 - \frac{0.5}{5} \Rightarrow x_1 = 1.4.$$

$$x_2 = 1.4 - \frac{f(1.4)}{f'(1.4)} \Rightarrow x_2 = 1.4 - \frac{-0.02\dot{6}}{12.\dot{1}} \Rightarrow x_2 = 1.425\dot{3}.$$

Subsequent iterations are $x_3 = 1.425390$, $x_4 = 1.425391$ and $x_5 = 1.425391$.

The iteration is well-behaved here, converging to the required root of $f(x) = 0$, namely 1.4254.



Example (6a): Repeat Example (6), but with a starting value of $x_0 = 2$.

$f(x) = x + (5 - 4x)^{-1}$, $x \neq \frac{5}{4}$.

$f'(x) = 1 + 4(5 - 4x)^{-2}$, $x \neq \frac{5}{4}$. (Chain rule).

$\mathbf{x_0 = 2}$

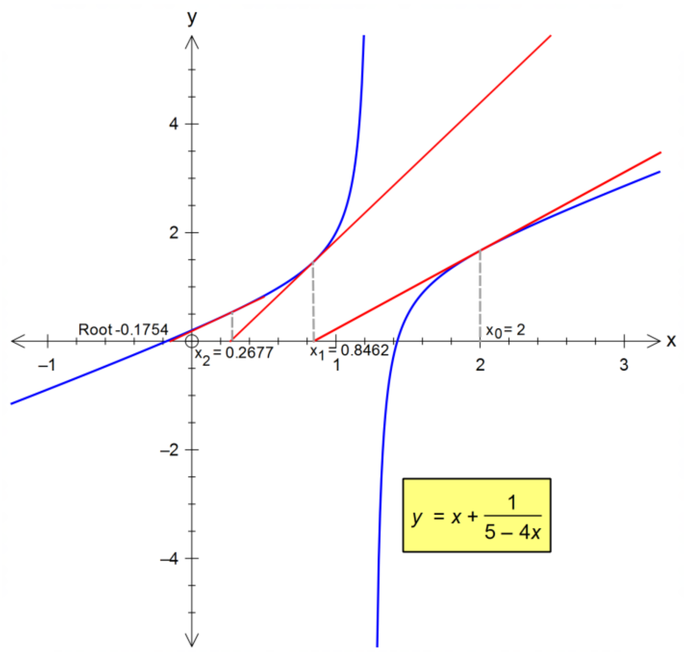
$$x_1 = 2 - \frac{f(2)}{f'(2)} \Rightarrow x_2 = 2 - \frac{1.\dot{6}}{1.\dot{4}} \Rightarrow x_1 = 0.\dot{8}4615\dot{3}.$$

$$x_2 = 0.\dot{8}4615\dot{3} - \frac{f(0.\dot{8}4615\dot{3})}{f'(0.\dot{8}4615\dot{3})} \Rightarrow x_2 = 0.\dot{8}4615\dot{3} - \frac{1.\dot{4}6520\dot{1}}{2.532880} \Rightarrow x_2 = 0.267681.$$

Subsequent iterations are $x_3 = -0.147851$, $x_4 = -0.175327$, $x_5 = -0.175391$ and $x_6 = -0.175391$.

Note the discontinuity in the graph here.

The first iteration ends up with x_1 on the wrong side of the discontinuity, and therefore on the wrong 'limb' of the graph, and later iterations converge to the negative root of the equation.



Example (6b): Repeat Example (6), but with a starting value of $x_0 = 1.65$.

$f(x) = x + (5-4x)^{-1}, \ x \neq \frac{5}{4}$.

$f'(x) = 1 + 4(5-4x)^{-2}, \ x \neq \frac{5}{4}$. (Chain rule).

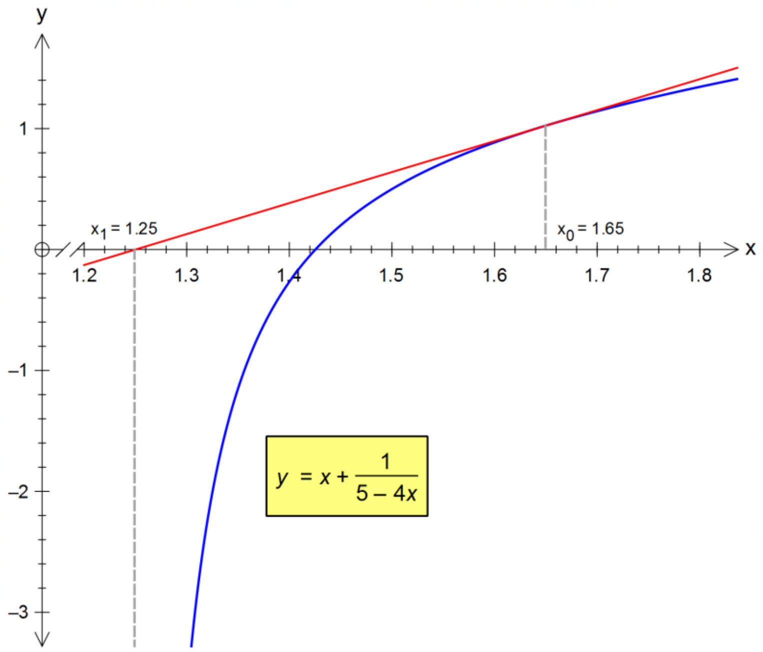
$\mathbf{x_0 = 1.65}$

$$x_1 = 1.65 - \frac{f(1.65)}{f'(1.65)} \Rightarrow x_1 = 1.65 - \frac{1.025}{2.5625} \Rightarrow x_1 = 1.25 \text{ , or } x_1 = \frac{5}{4}.$$

$$x_2 = 1.25 - \frac{f(1.25)}{f'(1.25)} \Rightarrow x_1 = 1.25 - \frac{(undefined)}{(undefined)}.$$

The iteration 'falls over'.

The value of x_1 happens to be the one value for which both $f(x)$ and $f'(x)$ are undefined. As a result, the perpendicular to the x-axis at 1.25 never meets the curve, coinciding with the asymptote, in other words, the discontinuity in the graph.



Perpendicular at x = 1.25 coincides with asymptote and cannot meet curve.
Iteration fails.