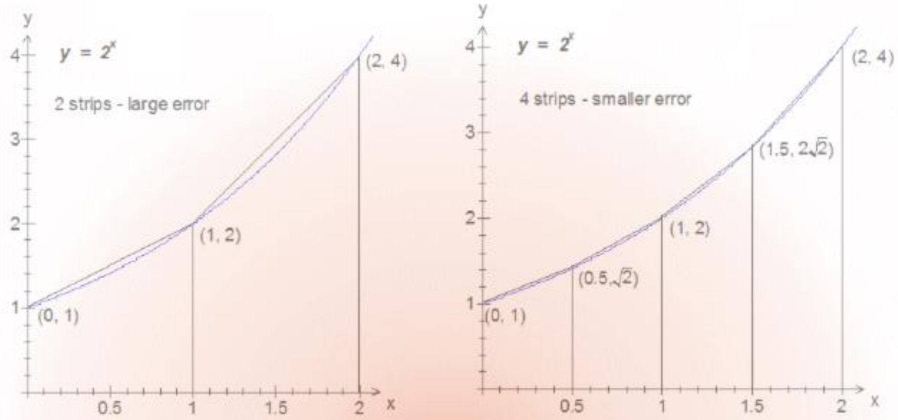
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

NUMERICAL INTEGRATION - TRAPEZIUM RULE



$y = 2^x$ $b\text{-}a = 2$ $h = 0.5$ $n = 4$

n	0	1	2	3	4
x_n	0	0.5	1	1.5	2
y_n	1	$\sqrt{2}$	2	$2\sqrt{2}$	4

							Totals
First / last y	1				4	Sum × 1	5
All other y		$\sqrt{2}$	2	$2\sqrt{2}$		Sum × 2	$4 + 6\sqrt{2}$
						G/T	$9 + 6\sqrt{2}$
						× ½h	4.371

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NUMERICAL INTEGRATION – THE TRAPEZIUM RULE.

Often, a function may prove difficult or impossible to integrate analytically. Definite integrals can however be obtained by numerical approximation.

In the diagram below, the value of $I = \int_a^b f(x)dx$ represents the area under the graph below of $y = f(x)$ between the points $x = a$ and $x = b$.

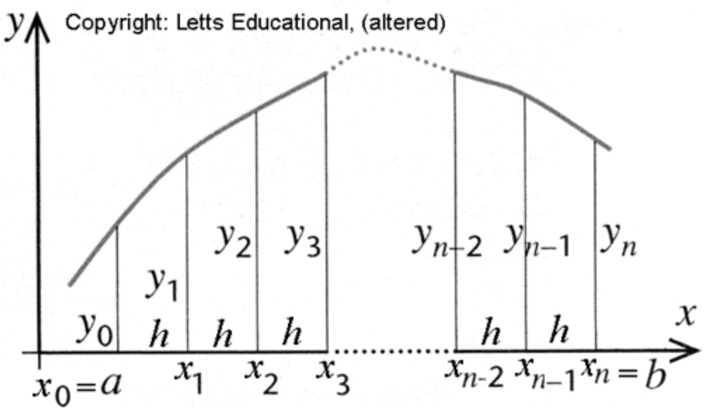
This area can be approximated using the **trapezium rule.** This divides the area into a number of strips of equal width, treats each strip as a trapezium, and adds together the areas of all the trapezia.

The diagram shows that the width of each strip, and hence the height of each trapezium (turned on its side) has a value *h*. Because *x* takes values from *a* to *b*, it follows that $h = \frac{b-a}{n}$ where *n* is the number of strips.

Moving from left to right, the first strip has an area of $\frac{1}{2}h(y_0 + y_1)$, the second one an area of $\frac{1}{2}h(y_1 + y_2)$, the third $\frac{1}{2}h(y_2 + y_3)$ and so on until the last strip whose area is $\frac{1}{2}h(y_{n-1} + y_n)$.

Note how each *y*-value is counted double except the first and the last.
Taking out $\frac{1}{2}h$ as a factor, the total area *I* can be given by the trapezium rule as

$$I = \tfrac{1}{2}h(y_0 + 2(y_1 + y_2 + ...y_{n-1}) + y_n).$$



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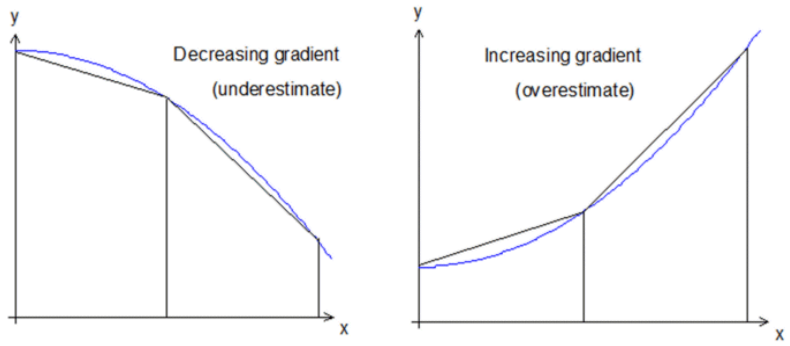
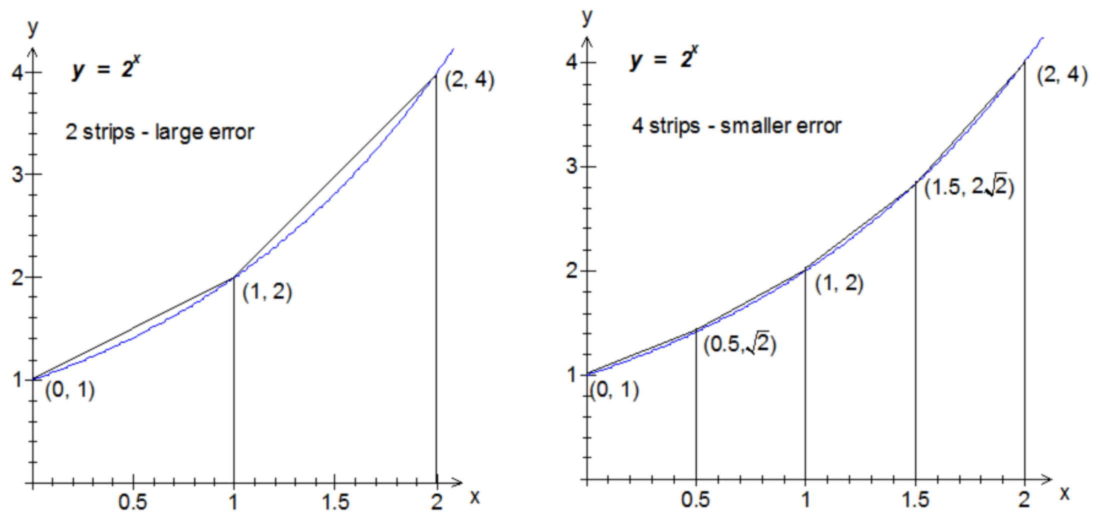
The trapezium rule usually requires quite a large number of strips to give a fairly good approximation to the area under a curve. If the gradient of the function to be integrated is decreasing, then the computed integral will be an underestimate; if the gradient is increasing, the computed value will be an overestimate.



In the left-hand diagram, the curve lies *above* the upper edges of the trapezia, so the error in using the trapezium rule is one of underestimation.

In the right-hand diagram, the reverse occurs. The curve lies *below* the upper edges of the trapezia, so the error is one of overestimation.

Example (1): Use the trapezium rule with 4 strips to estimate the value of $\int_0^2 2^x\,dx$. Is the estimate going to be too large or too small ?



As can be seen in the above diagram, there is a large error in the example on the left, which is the result of applying the trapezium rule with only two strips. Because the upper edges of the trapezia are above the curve, the error will take the form of an over-estimate. The example on the right shows the working with four strips. The error is less pronounced here, so using the trapezium rule gives an area estimate of

$$I = \tfrac{1}{2}h(y_0 + 2(y_1 + y_2 + ...y_{n-1}) + y_n)$$

where the number of strips n = 4, the interval $b - a$ = 2, the width of a single strip $h = \frac{b-a}{n}$ = 0.5,

and the values of y (ordinates) are are 1, √2, 2, 2√2 and 4. (Values left as surds)

The estimated area under the curve is thus ½ (0.5(1 + 2(√2+ 2 + 2√2) + 4)) or **4.37 square units**.

The true value for the integral is 4.328 square units, suggesting an error of about 1% in the estimate.

Compare this with the result for two strips (h = 1), which is ½ (1 + 2(2) + 4) or **4.5 square units –** giving an error of about 4%.

It is best to tabulate the values for ease of checking; also, work to one more decimal place than is required for the final answer, or use exact forms. The first and last ordinates (y-values) are added together in one row, but all the others are doubled after adding.

For homework examples, such tables can be created rapidly using a spreadsheet program. Examination questions will usually be limited to 4 strips due to time issues.

The working looks like this in tabular form :

$\boldsymbol{y = 2^x}$ $\boldsymbol{b\text{-}a = 2}$ $\boldsymbol{h = 0.5}$ $\boldsymbol{n = 4}$

n	0	1	2	3	4
x_n	**0**	**0.5**	**1**	**1.5**	**2**
y_n	**1**	**√2**	**2**	**2√2**	**4**

							Totals
First / last y	1				4	Sum × 1	5
All other y		√2	2	2√2		Sum × 2	4 + 6√2
						G/T	**9 + 6√2**
						× ½*h*	**4.371**

Count the end values singly ; double all others.

The estimated value of $\int_0^2 2^x\,dx$ is 4.37 to 3 significant figures.

In linear form: $\int_0^2 2^x\,dx \approx$ ½ (0.5(1 + 2(√2+ 2 + 2√2) + 4)) or 4.37.

Example (2): Use the trapezium rule with 8 strips to estimate $\int_0^2 2^x\,dx$ to 3 significant figures. .

The previous result was about 1% in error and was not correct to one decimal place. Increasing the number of strips to 8 would reduce the error considerably.

We will again use the rule $I = \frac{1}{2}h(y_0 + 2(y_1 + y_2 + ...y_{n-1}) + y_n)$

but now there are 8 strips (n = 8), the interval $b - a$ = 2, and so the width of a strip, h, is now 0.25.

Using $I = \frac{1}{2}h(y_0 + 2(y_1 + y_2 + ...y_{n-1}) + y_n)$, the results are:

y* = 2*ˣ   ***b-a* = 2**   ***h* = 0.25**   ***n* = 8**

n	0	1	2	3	4	5	6	7	8
x_n	**0**	**0.25**	**0.5**	**0.75**	**1**	**1.25**	**1.5**	**1.75**	**2**
y_n	**1**	**1.1892**	**1.4142**	**1.6818**	**2**	**2.3784**	**2.8284**	**3.3636**	**4**

											Totals
First / last *y*	1								4	Sum × 1	5
All other *y*		1.1892	1.4142	1.6818	2	2.3784	2.8284	3.3636		Sum × 2	29.7112
										G/T	**34.7112**
										× ½*h*	**4.339**

The value of $\int_0^2 2^x\,dx$ is therefore estimated at **4.34** to 3 significant figures.

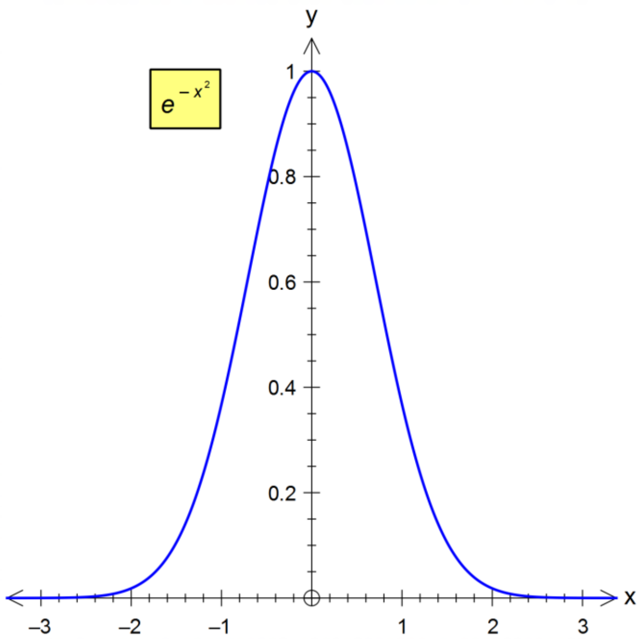
The true value for the integral is 4.3281 square units, and therefore the error has been reduced to just one unit in the second decimal place, or about 0.2%.

We have used decimal approximations here using 4 decimal places but again we could have used exact calculator values.

Working in linear form:

$\int_0^2 2^x\,dx$ ≈ ½ (0.25(1 + 2(1.1892 + 1.4142 + 1.6818 + 2 + 2.3784 + 2.8284 + 3.3636) + 4)) or **4.34**.

Example (3): The graph of the function $y = e^{-x^2}$ is shown on the right.



i) Use the trapezium rule with 4 strips to estimate $\int_{-2}^{2} e^{-x^2}\,dx$ to 3 significant figures.

ii) By means of a diagram, show why it is not readily possible to deduce if the value in i) is an overestimate or an underestimate .

iii) Repeat part i) using 8 strips.

$\boldsymbol{y = e^{-x^2}}$ **b-a = 4** ***h*** **= 1** ***n*** **= 4**

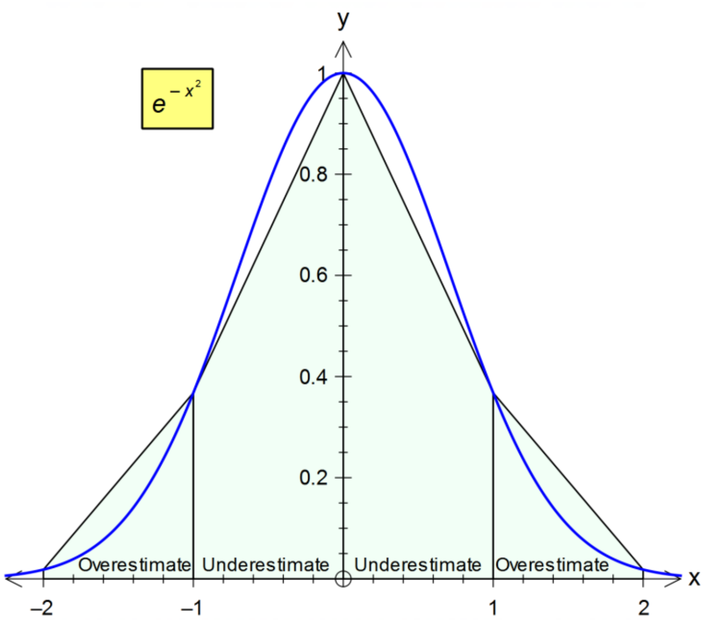
n	0	1	2	3	4
x_n	**-2**	**-1**	**0**	**1**	**2**
y_n	**0.0183**	**0.3679**	**1**	**0.3679**	**0.0183**

							Totals
First / last *y*	0.0183				0.0183	Sum × 1	0.0366
All other *y*		0.3679	1	0.3679		Sum × 2	3.4715
						G/T	**3.5081**
						× ½***h***	**1.754**

In linear form: $\int_{-2}^{2} e^{-x^2}\,dx \approx$

½ (0.5(0.0183 + 2(0.3679 +1 + 0.3679) + 0.0183)) or 1.75.

The estimated value of $\int_{-2}^{2} e^{-x^2}\,dx$ is 1.75 to 3 significant figures.



ii) The areas of the two middle strips are underestimates since the upper edges of the strips lie below the curve.

Conversely, the areas of the two outer strips are overestimates since the upper edges lie above the curve.

In other words, we cannot tell how the pairs of errors "cancel out", so it is not possible to deduce if the estimate is more than the actual integral, or less than it. .

Example (3) (continued)

iii) The working with 8 strips is as follows:

$y = e^{-x^2}$ **b-a = 4** **h = 0.5** **n = 8**

n	0	1	2	3	4	5	6	7	8
x_n	**-2**	**-1.5**	**-1**	**-0.5**	**0**	**0.5**	**1**	**1.5**	**2**
y_n	**0.0183**	**0.1054**	**0.3679**	**0.7788**	**1**	**0.7788**	**0.3679**	**0.1054**	**0.0183**

											Totals
First / last y	0.0183								0.0183	Sum × 1	0.0366
All other y		0.1054	0.3679	0.7788	1	0.7788	0.3679	0.1054		Sum × 2	7.0083
										G/T	**7.0449**
										× ½***h***	**1.761**

In linear form: $\int_{-2}^{2} e^{-x^2}\,dx \approx$

½ (0.5(0.0183 + 2(0.1054 + 0.3679 + 0.7788 + 1 + 0.7788 + 0.3679 + 0.1054) + 0.0183)) or **1.76**.

The estimated value of $\int_{-2}^{2} e^{-x^2}\,dx$ is 1.76 to 3 significant figures.

Example (4): Use the trapezium rule with 5 strips to estimate $\int_0^1 \frac{1}{1+x^2}\,dx$ to three decimal places.

$y = \frac{1}{1+x^2}$ **b-a = 1** **h = 0.2** **n = 5**

n	0	1	2	3	4	5
x_n	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**
y_n	**1**	**0.9615**	**0.8621**	**0.7353**	**0.6098**	**0.5**

								Totals
First / last y	1					0.5	Sum × 1	1.5
All other y		0.9615	0.8621	0.7353	0.6098		Sum × 2	6.3373
							G/T	**7.8373**
							× ½*h*	**0.7837**

The value of $\int_0^1 \frac{1}{1+x^2}\,dx$ is approximately 0.784 to three decimal places.

(The true value is $\frac{\pi}{4}$ or 0.7854 to four decimal places).

Linear form: $\int_0^1 \frac{1}{1+x^2}\,dx \approx$ ½ (0.2(1 + 2(0.9615 + 0.8621 + 0.7353 + 0.6098) + 0.5)) or **0.784**.

Example (5): Use the trapezium rule with 5 strips to estimate $\int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\,dx$ to three decimal places.

$y = \frac{1}{\sqrt{1-x^2}}$ **b-a = 0.5** **h = 0.1** **n = 5**

n	0	1	2	3	4	5
x_n	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
y_n	**1**	**1.0050**	**1.0206**	**1.0483**	**1.0911**	**1.1547**

								Totals
First / last y	1					1.1547	Sum × 1	2.1547
All other y		1.0050	1.0206	1.0483	1.0911		Sum × 2	8.3301
							G/T	**10.4848**
							× ½***h***	**0.5242**

The value of $\int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\,dx$ is approximately 0.524 to three decimal places.

(The true value is $\frac{\pi}{6}$ or 0.5236 to four decimal places, suggesting a smaller relative error in this example).

Linear form:

$\int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\,dx$ ≈ ½ (0.1(1 + 2(1.0050 + 1.0206 + 1.0483 + 1.0911) + 1.1547)) or **0.524**.

Example (5): Use the trapezium rule with 4 strips to estimate the value of $\int_0^4 x\sqrt{(x^2+9)}\,dx$ to one decimal place.

The number of strips n = 4, the interval is $b - a$ = 4, and so the width of a single strip, h, is 1.

(We have used exact surds when writing down the y-values).

$y = x\sqrt{(x^2+9)}$ **b-a = 4** **h = 1** **n = 4**

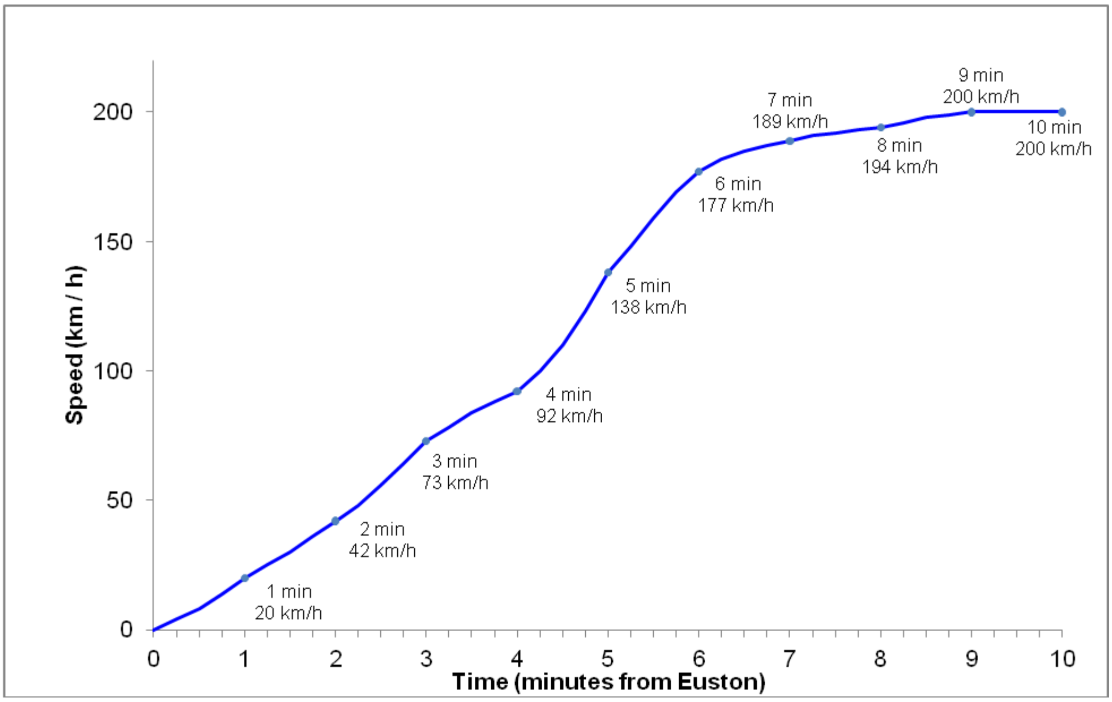
n	0	1	2	3	4
x_n	**0**	**1**	**2**	**3**	**4**
y_n	**0**	**√10**	**2√13**	**3√18**	**20**

							Totals
First / last y	0				20	Sum × 1	20
All other y		√10	2√13	3√18		Sum × 2	46.20
						G/T	**66.20**
						× ½*h*	**33.1**

Linear form: $\int_0^4 x\sqrt{(x^2+9)}\,dx$ ≈ ½ (1(0 + 2(√10 + 2√13 + 3√18) + 20)) or **33.1**.

The previous examples were purely mathematical in nature, but the method can equally be applied in real-life situations. Two such examples follow overleaf.

Example (6): The travel graph below shows the speed (in km/h) of a train leaving London's Euston station, over a ten-minute time interval. (The area under the curve represents the distance travelled.)



Use the trapezium rule with 10 strips to find the distance covered by the train during this 10-minute interval, to the nearest 0.1 km. Remember to divide *h* by 60 due to the use of km/h as the unit of speed.

$h = 1$; $n = 10$

n	0	1	2	3	4	5	6	7	8	9	10
x_n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
y_n	**0**	**20**	**42**	**73**	**92**	**138**	**177**	**189**	**194**	**200**	**200**

													Totals
First/last *y*	0										200	Sum × 1	200
All other *y*		20	42	73	92	138	177	189	194	200		Sum × 2	2250
												G/T	**2450**
												× ½(*h*/60)	**20.42**

Count the end values singly ; double all others.
Estimated integral = ½(*h*/60) (200 + 2250) = 20.42 to 4 s.f.

∴ The train has covered **20.42 km** in 10 minutes, using the trapezium rule with 10 strips

(The value of *h* had to be divided by 60 because the speeds were quoted in kilometres per hour, but timings quoted in minutes).

Linear form:

Distance travelled $\approx \left(\frac{1}{2}\right)\left(\frac{1}{60}\right)(0+2(20+42+73+92+138+177+189+194+200)+200)$ km
= 20.42 km.

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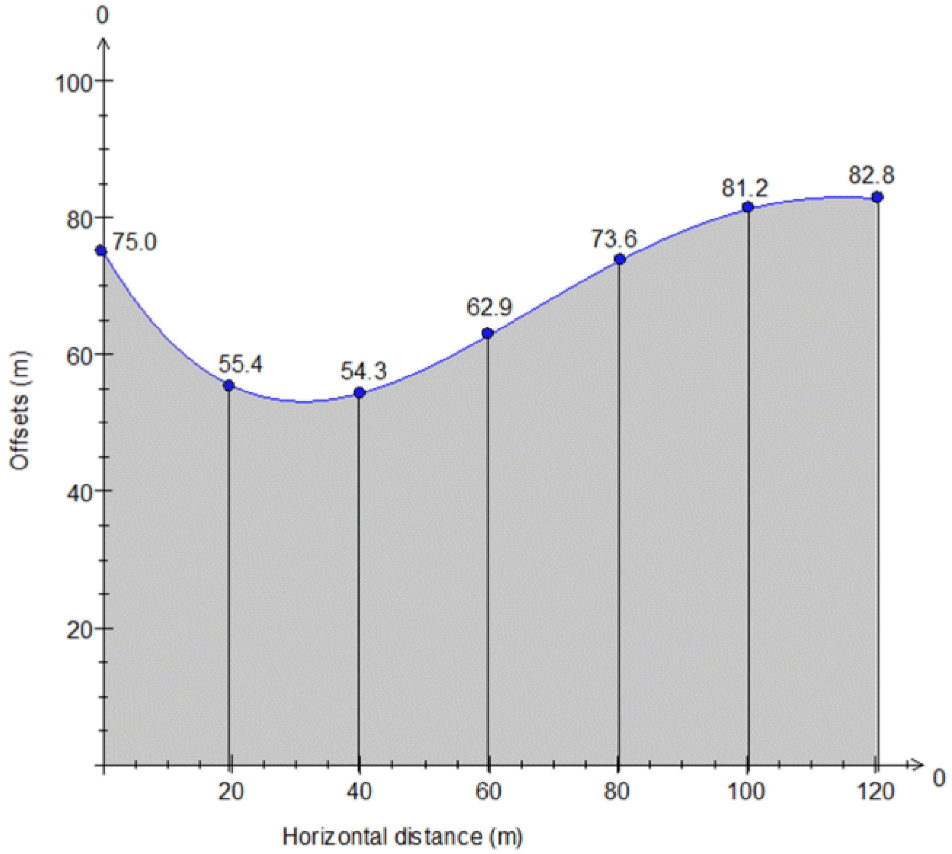
Example (7): Estimate the area of the plot of land below, divided into 20m-wide strips. The perpendicular offset distances from the baseline are also shown here.

Use the trapezium rule with 6 strips to estimate the area in hectares, to 3 significant figures. (1 hectare = 10,000 m^2).



Here, $h = 20$, $n = 6$ and the y-values are the offsets.

n	0	1	2	3	4	5	6
x_n	**0**	**20**	**40**	**60**	**80**	**100**	**120**
y_n	**75.0**	**55.4**	**54.3**	**62.9**	**73.6**	**81.2**	**82.8**

									Totals
First/last y	75.0						82.8	Sum × 1	157.8
All other y		20	42	73	92	138		Sum × 2	654.8
								G/T	**812.6**
								× ½h	**8126**

Linear form: Area $\approx \left(\frac{1}{2}\right)(20)(75.0 + 2(55.4 + 54.3 + 62.9 + 73.6 + 81.2) + 82.8)$ or 8126 m^2.

$\therefore$ Estimated plot area = **8130 m^2** or **0.813 hectares** to 3 s.f., using the trapezium rule with 6 strips.