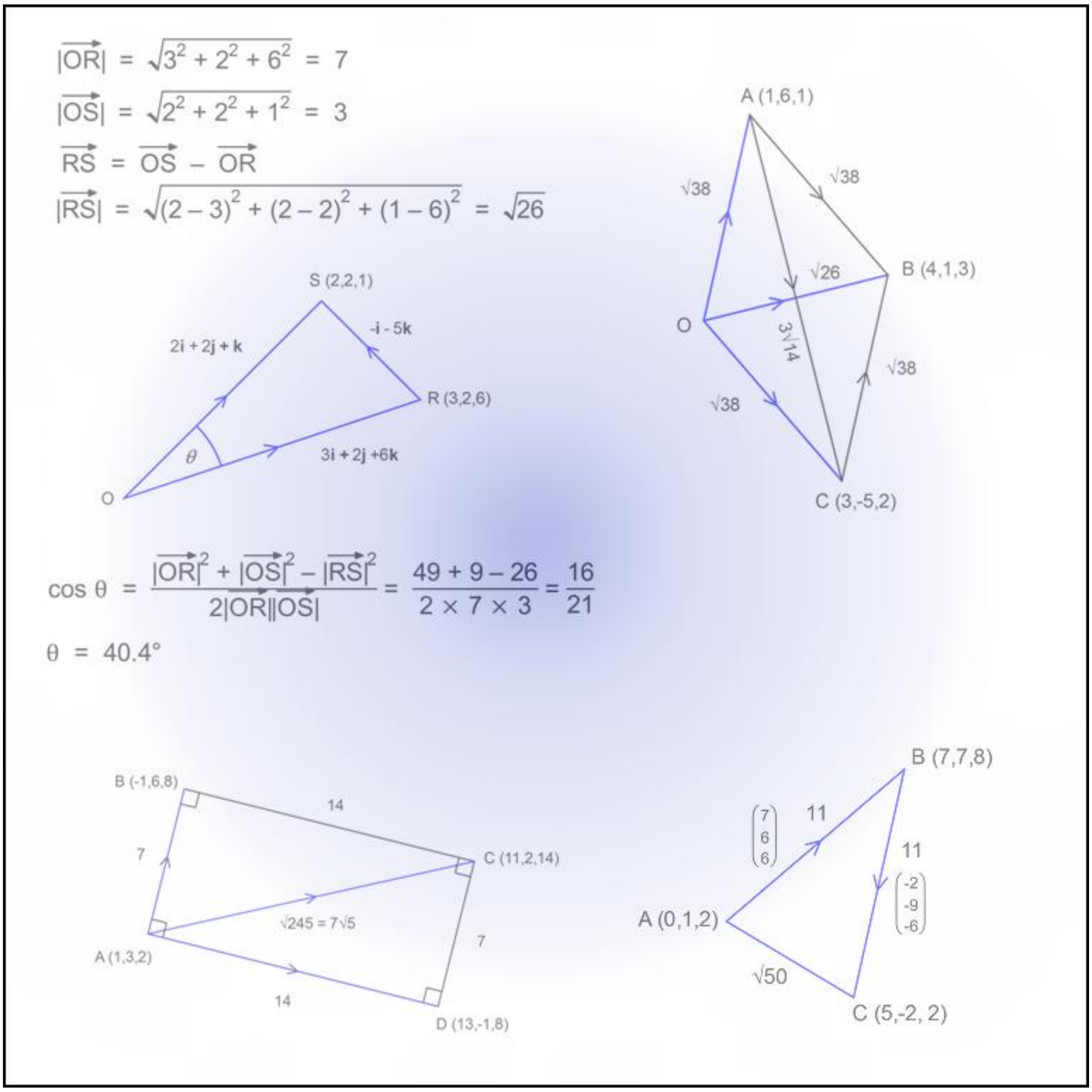
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 2

VECTORS IN THREE DIMENSIONS



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VECTORS IN THREE DIMENSIONS.

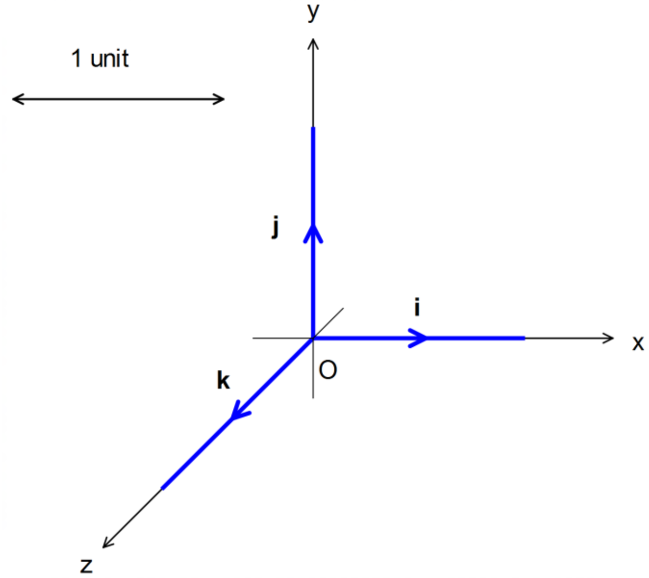
The sections on vectors at AS-Level were restricted to solving problems in the two-dimensional plane, but we can solve problems in three dimensions as well.

The two standard unit vectors in the two-dimensional x-y plane are **i** and **j** , where **i** is parallel to the x-axis and **j** is parallel to the y-axis.

To extend the idea to three dimensions, we have a third unit vector, **k** parallel to the z-axis.

In column form, the three vectors are $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Note that the positive z-axis points forwards towards the eye and "out of the paper".



The two forms are interchangeable; thus vector $\mathbf{a} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$ can be expressed as 4**i** + 3**j** -2**k**.

Vector arithmetic in three dimensions follows the same rules as in two:

Example (1): Given vectors $\mathbf{a} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$, find i) **a**+**b**, ii) **a**-**b**, iii) 2**a**, iv) 3**a**-2**b**.

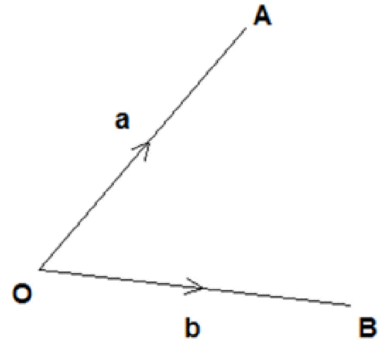
i) **a+b =** $\begin{pmatrix} 4+1 \\ 3-3 \\ -2+5 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$; ii) **a+b =** $\begin{pmatrix} 4-1 \\ 3+3 \\ -2-5 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix}$; iii) 2**a** = $2\begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ -4 \end{pmatrix}$;

iv) 3**a** – 2**b** = $3\begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 12 \\ 9 \\ -6 \end{pmatrix} - \begin{pmatrix} 2 \\ -6 \\ 10 \end{pmatrix} = \begin{pmatrix} 10 \\ 15 \\ -16 \end{pmatrix}$

Recap - "Double-letter and arrow" notation.

Vectors can be denoted by a single boldface letter, but another notation is to state their end points and write an arrow above them.

In the right-hand diagram, vector **a** joins points *O* and *A* and vector **b** joins point *O* and *B*.



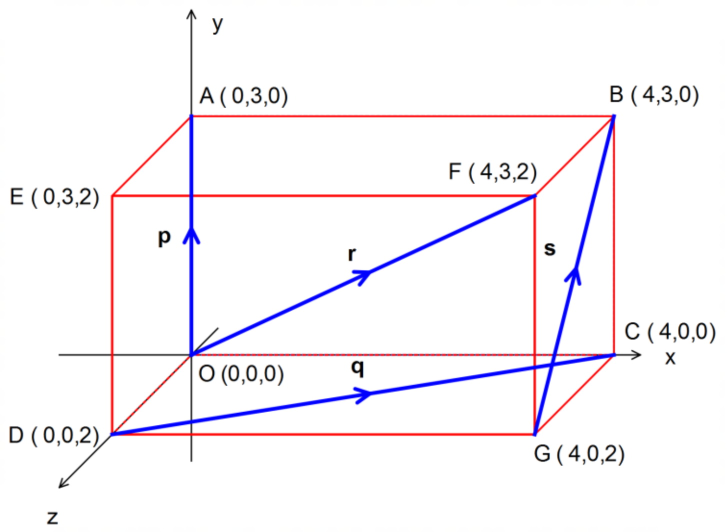
Therefore $\overrightarrow{OA}$ = **a** and $\overrightarrow{OB}$ = **b**.

The direction of the arrow is important here;
the vector $\overrightarrow{AO}$ goes in the opposite direction to $\overrightarrow{OA}$ although it has the same magnitude.

Hence $\overrightarrow{AO}$ = - $\overrightarrow{OA}$ = -**a**.

Example (1): *OABCEFGH* is a cuboid. Note that the vertex *O* at the origin is on the rear face of the cuboid.

Express the vectors **p**, **q**, **r** and **s** in two-letter, **i-j-k** component and column notation.



i) **p** = $\overrightarrow{OA}$, and because *O* is the origin

and the position vector of *A* is 3**j** or $\begin{pmatrix}0\\3\\0\end{pmatrix}$,

p = 3**j** or in column form $\begin{pmatrix}0\\3\\0\end{pmatrix}$.

ii) **q** = $\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD}$ = 4**i** – 2**k** , or in column form $\begin{pmatrix}4\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\2\end{pmatrix} = \begin{pmatrix}4\\0\\-2\end{pmatrix}$.

iii) **r** = $\overrightarrow{OF}$ = 4**i** + 3**j** + 2**k** or in column form $\begin{pmatrix}4\\3\\2\end{pmatrix}$.

iv) **s** = $\overrightarrow{GB} = \overrightarrow{OB} - \overrightarrow{OG}$ = (4**i** + 3**j**) – (4**i** + 2**k**) = 3**j** – 2**k**, or in column form $\begin{pmatrix}4\\3\\0\end{pmatrix} - \begin{pmatrix}4\\0\\2\end{pmatrix} = \begin{pmatrix}0\\3\\-2\end{pmatrix}$.

The magnitude of a two-dimensional vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is $\sqrt{a^2 + b^2}$ from Pythagoras.

In three dimensions, the magnitude of a vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is $\sqrt{a^2 + b^2 + c^2}$

Example (3): Find the magnitudes of the following vectors:

i) $\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$; ii) $\begin{pmatrix} 0.6 \\ 0 \\ 0.8 \end{pmatrix}$; iii) $\begin{pmatrix} 4 \\ -8 \\ 1 \end{pmatrix}$; iii) $\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$.

i) The magnitude of the vector $\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$ is $\sqrt{6^2 + 3^2 + 4^2} = 7$ units.

ii) The vector $\begin{pmatrix} 0.6 \\ 0 \\ 0.8 \end{pmatrix}$ has a magnitude of $\sqrt{0.6^2 + 0^2 + 0.96^2} = 1$ unit.

iii) Similarly, vector $\begin{pmatrix} 4 \\ -8 \\ 1 \end{pmatrix}$ has a magnitude of $\sqrt{4^2 + (-8)^2 + 1^2} = 9$ units.

iv) The **i**- and **j-** components of the vector are both zero, so its magnitude is $\sqrt{0^2 + 0^2 + 4^2} = 4$.

In Example 3(ii) we encountered a unit vector distinct from the standard ones of **i**, **j** and **k**.

Example (4): Find the unit vectors having the same direction as the following:

i) $\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$; ii) $\begin{pmatrix} 4 \\ -8 \\ 1 \end{pmatrix}$; iii) $\begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$; iv) $\begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$

i) The magnitude of the vector $\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$ is $\sqrt{6^2+3^2+4^2}=7$, so we divide the components by 7 to obtain the unit vector in the same direction; it is $\frac{1}{7}\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6/7 \\ 3/7 \\ 2/7 \end{pmatrix}$.

ii) The vector $\begin{pmatrix} 4 \\ -8 \\ 1 \end{pmatrix}$ has magnitude $\sqrt{4^2+(-8)^2+1^2}=9$, so we divide the components by 9 to obtain the corresponding unit vector in the same direction, i.e. $\frac{1}{9}\begin{pmatrix} 4 \\ -8 \\ 1 \end{pmatrix} = \begin{pmatrix} 4/9 \\ -8/9 \\ 1/9 \end{pmatrix}$.

These last two examples have rational components, but this is rarely the case.

iii) The magnitude of the vector $\begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$ is $\sqrt{3^2+(-2)^2+(-1)^2}=\sqrt{14}$, so we divide the components by √14 to obtain the unit vector in the same direction; it is $\frac{1}{\sqrt{14}}\begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3/\sqrt{14} \\ -2/\sqrt{14} \\ -1/\sqrt{14} \end{pmatrix}$.

iv) Since only the **j**-component of vector $\begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$ is non-zero , the unit vector with the same direction is simply $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Angles between vectors in three dimensions.

The method is the same as in two dimensions; we calculate their lengths using Pythagoras and then use the cosine rule. The only difference is that we are dealing with two-dimensional diagrams to represent three dimensions, so it is best to omit any coordinate axes to avoid confusion.

To find the angle between a vector and any of the coordinate axes, we simply use a standard unit vector to represent the axis – **i** for the *x*-axis, **j** for the *y*-axis, or **k** for the *z*-axis.

Example (5): Find the angle θ between the vector

$\overrightarrow{OA} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and the *x*-axis.

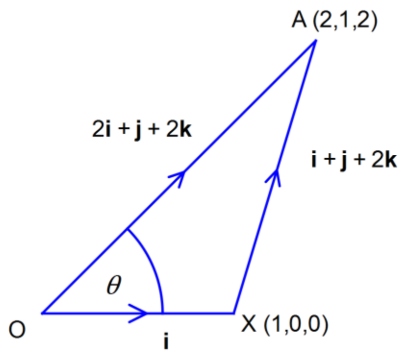
The *x*-axis is represented here by the vector $\overrightarrow{OX}$, which is equivalent to the standard unit vector **i.**

If the question had asked for the angle between $\overrightarrow{OA}$ and the *y*-axis, we would have used vector $\overrightarrow{OY}$, equivalent to the standard unit vector **j.**

The same would apply to the *z*-axis – we would use a vector equivalent to **k.**

$$|\overrightarrow{OA}| = \sqrt{2^2 + 1^2 + 2^2} = 3$$
$$|\overrightarrow{OX}| = 1$$

$$\overrightarrow{XA} = \overrightarrow{OA} - \overrightarrow{OX}$$
$$|\overrightarrow{XA}| = \sqrt{(2-1)^2 + (1-0)^2 + (2-0)^2} = \sqrt{6}$$



$$\cos\theta = \frac{|\overrightarrow{OA}|^2 + |\overrightarrow{OX}|^2 - |\overrightarrow{XA}|^2}{2|\overrightarrow{OA}||\overrightarrow{OX}|} = \frac{9 + 1 - 6}{2 \times 3 \times 1} = \frac{2}{3}$$

$$\theta = 48.2°$$

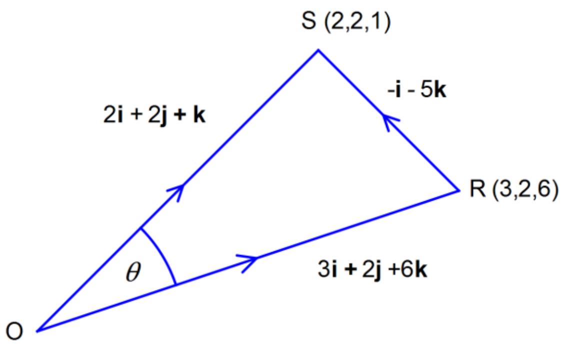
Example (6): Find the angle θ between the vectors $\overrightarrow{OR}$ = 3**i** + 2**j** + 6k and $\overrightarrow{OS}$ = 2**i** + 2**j** + **k**.

$$|\overrightarrow{OR}| = \sqrt{3^2 + 2^2 + 6^2} = 7$$

$$|\overrightarrow{OS}| = \sqrt{2^2 + 2^2 + 1^2} = 3$$

$$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR}$$

$$|\overrightarrow{RS}| = \sqrt{(2-3)^2 + (2-2)^2 + (1-6)^2} = \sqrt{26}$$



$$\cos\theta = \frac{|\overrightarrow{OR}|^2 + |\overrightarrow{OS}|^2 - |\overrightarrow{RS}|^2}{2|\overrightarrow{OR}||\overrightarrow{OS}|} = \frac{49 + 9 - 26}{2 \times 7 \times 3} = \frac{16}{21}$$

$$\theta = 40.4°$$

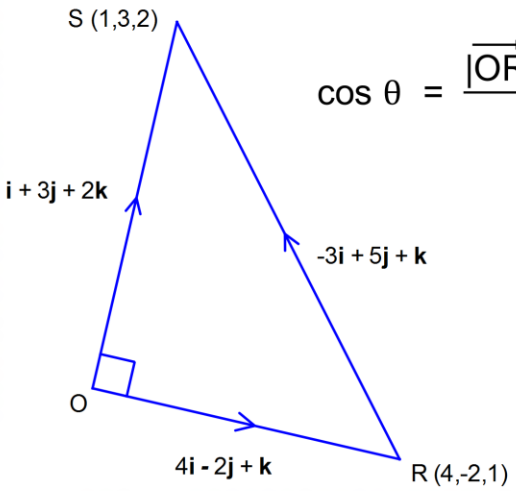
Example (7): Find the angle θ between the vectors $\overrightarrow{OR}$ = 4**i** - 2**j** + **k** and $\overrightarrow{OS}$ = **i** + 3**j** + 2**k**

$$|\overrightarrow{OR}| = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{21}$$

$$|\overrightarrow{OS}| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$$

$$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR}$$

$$|\overrightarrow{RS}| = \sqrt{(1-4)^2 + (3-(-2))^2 + (2-1)^2} = \sqrt{35}$$

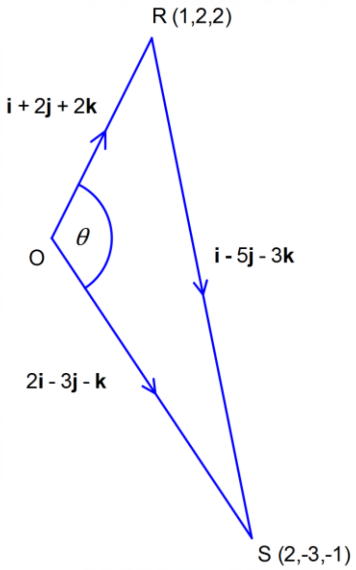


$$\cos\theta = \frac{|\overrightarrow{OR}|^2 + |\overrightarrow{OS}|^2 - |\overrightarrow{RS}|^2}{2|\overrightarrow{OR}||\overrightarrow{OS}|} = \frac{21 + 14 - 35}{2\sqrt{21}\sqrt{14}} = 0$$

$$\theta = 90°$$

We could also deduce that the angle θ = *SOR* is a right angle because the square of the length of $\overrightarrow{RS}$ is the sum of the squares of the lengths of $\overrightarrow{OR}$ and $\overrightarrow{OS}$.

Example (8): Find the angle θ between the vectors $\overrightarrow{OR}$ = **i** + 2**j** + 2**k** and $\overrightarrow{OS}$ = 2**i** - 3**j** - **k** .



$$|\overrightarrow{OR}| = \sqrt{1^2 + 2^2 + 2^2} = 3$$

$$|\overrightarrow{OS}| = \sqrt{2^2 + (-3)^2 + (-1)^2} = \sqrt{14}$$

$$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR}$$

$$|\overrightarrow{RS}| = \sqrt{(2-1)^2 + (-3-2)^2 + (-1-2)^2} = \sqrt{35}$$

$$\cos\theta = \frac{|\overrightarrow{OR}|^2 + |\overrightarrow{OS}|^2 - |\overrightarrow{RS}|^2}{2|\overrightarrow{OR}||\overrightarrow{OS}|} = \frac{9 + 14 - 35}{2 \times 3 \times \sqrt{14}} = \frac{-2}{\sqrt{14}}$$

$$\theta = 122.3°$$

The angle between the vectors is obtuse here, given its negative cosine.

Example (9): In a triangle *ABC*, point *A* has position vector $\begin{pmatrix}0\\1\\2\end{pmatrix}$, $\overrightarrow{AB} = \begin{pmatrix}7\\6\\6\end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix}-2\\-9\\-6\end{pmatrix}$,

i) find the coordinates of *B* and *C*,
ii) show that triangle *ABC* is isosceles.
iii) show that cos *ABC* = $\frac{96}{121}$.
iv) find the area of the triangle.

i) To find the coordinates of *B*, we add the position vector of *A* , i.e. $\begin{pmatrix}0\\1\\2\end{pmatrix} + \begin{pmatrix}7\\6\\6\end{pmatrix} = \begin{pmatrix}7\\7\\8\end{pmatrix}$.

Hence *A* = (0, 1, 2) and *B* = (7, 7, 8).

The coordinates of *C* are found in the same way: $\begin{pmatrix}7\\7\\8\end{pmatrix} + \begin{pmatrix}-2\\-9\\-6\end{pmatrix} = \begin{pmatrix}5\\-2\\2\end{pmatrix}$, so *C*= (5, -2, 2)

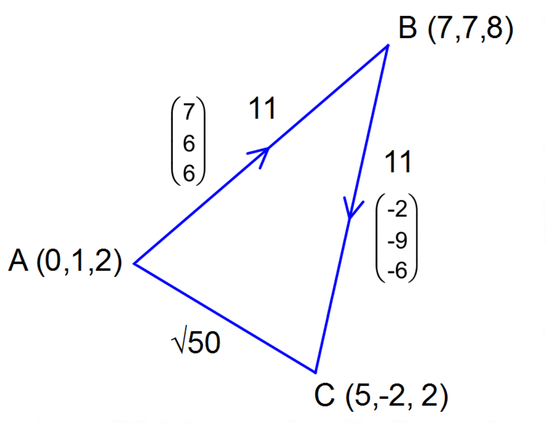
ii) The length of *AB*, i.e. $|\overrightarrow{AB}|$, = $\sqrt{7^2+6^2+6^2} = \sqrt{121} = 11$.

In the same way, $|\overrightarrow{BC}|$, = $\sqrt{(-2)^2+(-9)^2+(-6)^2} = \sqrt{121} = 11$.

Hence the lengths *AB* and *BC* are equal.

But $|\overrightarrow{AC}| = \sqrt{(5-0)^2+(-2-1)^2+(2\_(-2))^2} = \sqrt{50}$.

The two sides *AB* and *BC* are equal in length, but the length of *AC* is different. Hence triangle *ABC* is isosceles.



iii) By the cosine rule,

$$\cos ABC = \frac{(AB)^2+(BC)^2-(AC)^2}{2(AB)(BC)}.$$

$$= \frac{121+121-50}{2\times 11\times 11} = \frac{192}{242} = \frac{96}{121}.$$

iv) *ABC* = $\cos^{-1}\left(\frac{96}{121}\right)$ = 37.5 °, so the area

of triangle *ABC* = ½ (*AB*)(*BC*) sin *ABC* =

$\frac{1}{2}(11\times 11\times \sin 37.5°) = 36.8$ square units.

Example (10): The vertices of a quadrilateral *OABC* have the following position vectors:

O = **0;** *A* = **i** + 6**j** + **k**; *B* = 4**i** + **j** + 3**k**; *C* = 3**i** – 5**j** + 2**k**.

i) Show that *OABC* is a rhombus.
ii) Show that *OABC* is not a square.

i) Because the position vector of *O* is the zero vector, we can immediately say that

$\overrightarrow{OA}$ = **i** + 6**j** + **k** and $\overrightarrow{OC}$ = 3**i** – 5**j** + 2**k** , where *OA* and *OC* are adjacent sides of the quadrilateral.

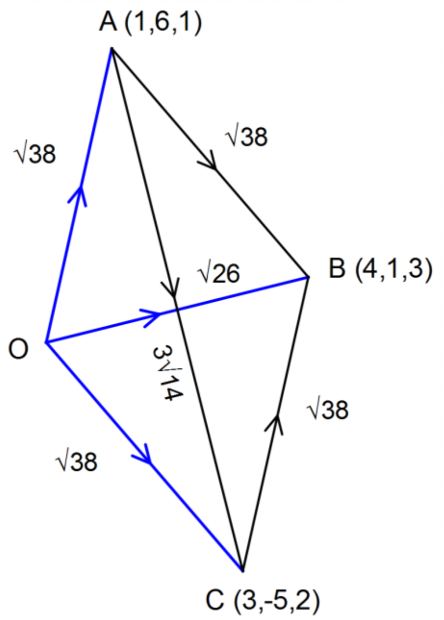
The length $|\overrightarrow{OA}| = \sqrt{1^2+6^2+1^2} = \sqrt{38}$; length $|\overrightarrow{OC}|$, $= \sqrt{3^2+(-5)^2+2^2} = \sqrt{38}$.

Hence the adjacent sides *OA* and *OC* are equal.

To prove that *OABC* is a rhombus, we could either calculate the lengths of *OB* and *BC* and show that all four sides are equal, or we could show that *OA* and *CB* are equal and parallel, as are *OC* and *AB*.

Since $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, its vector is (4**i** + **j** + 3**k**) – (**i** + 6**j** + **k**) = 3**i** – 5**j** + 2**k**.
Hence $\overrightarrow{AB} = \overrightarrow{OC}$, so its length is also $\sqrt{3^2+(-5)^2+2^2} = \sqrt{38}$.
- 16**j**
Also, $\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC}$, so its vector is

(4**i** + **j** + 3**k**) – (3**i** – 5**j** + 2**k**) = **i** + 6**j** + **k**
.
Thus $\overrightarrow{CB} = \overrightarrow{OA}$, with length $\sqrt{1^2+6^2+1^2} = \sqrt{38}$.



All the sides of *OABC* are equal, and both pairs of opposite sides are parallel, so *OABC* is a rhombus**.**

ii) The diagonals of a square are equal in length, but those of a rhombus are not.

The diagonal *OB* has length $\sqrt{4^2+1^2+3^2} = \sqrt{26}$.

To find the length of the other diagonal *AC*, we reckon

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

and its vector is (3**i** – 5**j** + 2**k**) – (**i** + 6**j** + **k**) = 2**i** - 11**j** + **k**.

The length is $\sqrt{2^2+(-11)^2+1^2} = \sqrt{126} = 3\sqrt{14}$.
The diagonals *OB* and *DC* are unequal in length, so *OABC* is not a square.

Example (11): The vertices of a quadrilateral *ABCD* have the following position vectors:

$$A = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}; B = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}; C = \begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix}; D = \begin{pmatrix} 8 \\ 3 \\ 6 \end{pmatrix}.$$

Show that *ABCD* is a parallelogram, but not a rectangle.

i) A sketch would show that *AB* and *AD* are two adjacent sides, and that *AC* is a diagonal.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}; \quad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix};$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} 8 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}.$$

We then work out *BC* and *DC* :

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$, so sides *BC* and *AD* are equal and parallel.

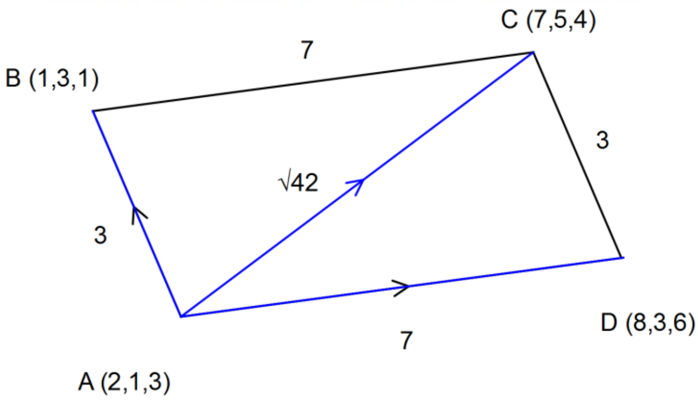
$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = \begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 8 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$, so sides *DC* and *AB* are equal and parallel.

ABCD is therefore at least a parallelogram, so we use Pythagoras to check if the triangle *ABC* is right-angled, i.e.

$|\overrightarrow{AC}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2$.

Now $|\overrightarrow{AC}|^2 = 5^2 + 4^2 + 1^2 = 42$,
$|\overrightarrow{AB}|^2 = (-1)^2 + 2^2 + 2^2 = 9$,
and $|\overrightarrow{BC}|^2 = 6^2 + 2^2 + 3^2 = 49$.

For *ABCD* to be a rectangle, angle *ABC* must be a right angle, and therefore
$|\overrightarrow{AC}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2$



Now, $|\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 = 58$, but $|\overrightarrow{AC}|^2 = 42$, so triangle *ABC* cannot be right-angled, and hence *ABCD* is not a rectangle.

Example (12): The vertices of a quadrilateral *ABCD* have the following position vectors:

$A = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$; $B = \begin{pmatrix} -1 \\ 6 \\ 8 \end{pmatrix}$; $C = \begin{pmatrix} 11 \\ 2 \\ 14 \end{pmatrix}$; $D = \begin{pmatrix} 13 \\ -1 \\ 8 \end{pmatrix}$.

i) Show that *ABCD* is a rectangle, but not a square.
ii) Find the area of the rectangle.

i) A sketch would show that *AB* and *AD* are two adjacent sides, and that *AC* is a diagonal.

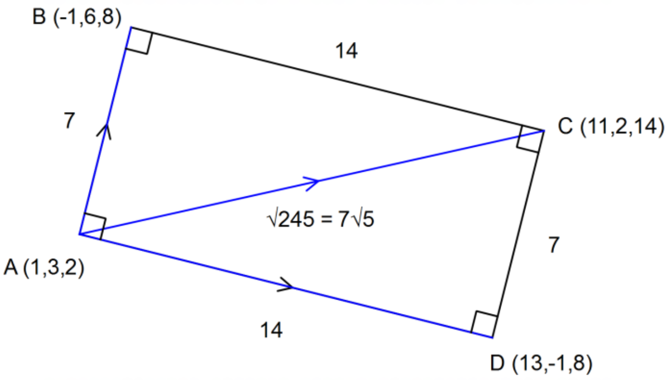
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -1 \\ 6 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 6 \end{pmatrix}; \quad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 11 \\ 2 \\ 14 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \\ 12 \end{pmatrix};$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} 13 \\ -1 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix}.$$

We then work out *BC* and *DC* :

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 11 \\ 2 \\ 14 \end{pmatrix} - \begin{pmatrix} -1 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix}$$, so sides *BC* and *AD* are equal and parallel.

$$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = \begin{pmatrix} 11 \\ 2 \\ 14 \end{pmatrix} - \begin{pmatrix} 13 \\ -1 \\ 8 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 6 \end{pmatrix}$$, so sides *DC* and *AB* are equal and parallel.



ABCD is therefore at least a parallelogram, so we apply Pythagoras in reverse to show that the triangle *ABD* is right-angled, i.e.

$|\overrightarrow{AC}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2$.

Now $|\overrightarrow{AC}|^2 = 7^2 + 14^2 = 245$,

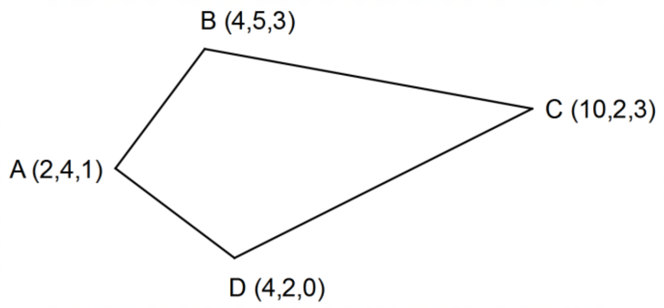
$|\overrightarrow{AB}|^2 = 7^2 = 49$ and $|\overrightarrow{BC}|^2 = 14^2 = 196$.

Hence $|\overrightarrow{AC}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{AD}|^2$,
angle *BAD* = 90°, and *ABCD* is a rectangle.

Because the lengths of *AB* and *AD* are different, *ABCD* cannot be a square.

ii) The area of the rectangle is 14 × 7, or 98, square units.

Example (13): The diagram below shows a quadrilateral *ABCD*.



i) Show, using Pythagoras, that angle *BAD* is a right angle.

ii) Show that *ABCD* is a kite.

iii) Find the area of the kite.

iv) The point *C* is moved to *C'* such that *ABC'D* is a square.

a) Find the position vector of *C'*.
b) Find the area of the resulting square.

v) The point *A* is moved to *A'* such that *A'BCD* is a rhombus.

a)Find the position vector of *A'*. (*C* is the point (14, 7) again).
b) Find the area of the rhombus.

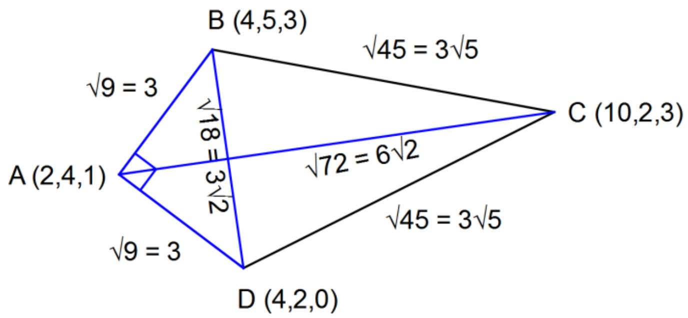
i) Using Pythagoras,

$(AB)^2 = (4-2)^2 + (5-4)^2 + (3-1)^2 = 2^2 + 1^2 + 2^2 = 9$
$(AD)^2 = (4-2)^2 + (2-4)^2 + (0-1)^2 = 2^2 + (-2)^2 + (-1)^2 = 9$
$(DB)^2 = (4-4)^2 + (5-2)^2 + (3-0)^2 = 0^2 + 3^2 + 3^2 = 18$
Since $(DB)^2 = (AB)^2 + (AD)^2$, the triangle *BAD* is right-angled.

ii) A kite has two adjacent pairs of sides equal, and from the last part, $AB = AD = \sqrt{9} = 3$.
We work out the lengths of *BC* and *DC* in the same way:

$(BC)^2 = (10-4)^2 + (2-5)^2 + (3-3)^2 = 6^2 + (-3)^2 + 0^2 = 45$
$(DC)^2 = (10-4)^2 + (2-2)^2 + (3-0)^2 = 6^2 + 0^2 + 3^2 = 45$

$BC = DC = \sqrt{45} = 3\sqrt{5}$, so both pairs of adjacent sides of *ABCD* are equal, therefore *ABCD* is a kite.



The area of a kite (like that of a rhombus) is half the product of the diagonals, so we need to find the length of *AC*

By Pythagoras, $(AC)^2 =$

$(10-2)^2 + (2-4)^2 + (3-1)^2 = 64 + 4 + 4 = 72$,
so $AC = \sqrt{72}$ or $6\sqrt{2}$.

As $(BD)^2 = 18$, $BD = \sqrt{18}$ or $3\sqrt{2}$.

The area of the kite is therefore

$\frac{1}{2}\left(6\sqrt{2}\right)\left(3\sqrt{2}\right) = 18$ square units.

iv) a) From part i), we know that angle *BAD* is a right angle, and that lengths *AB* and *AD* are equal.
So for *ABC'D* to be a square, the vectors *BC'* and *AD* must be equal and parallel, as must *AB* and *DC'*.

Let the position vector of *C*' be $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

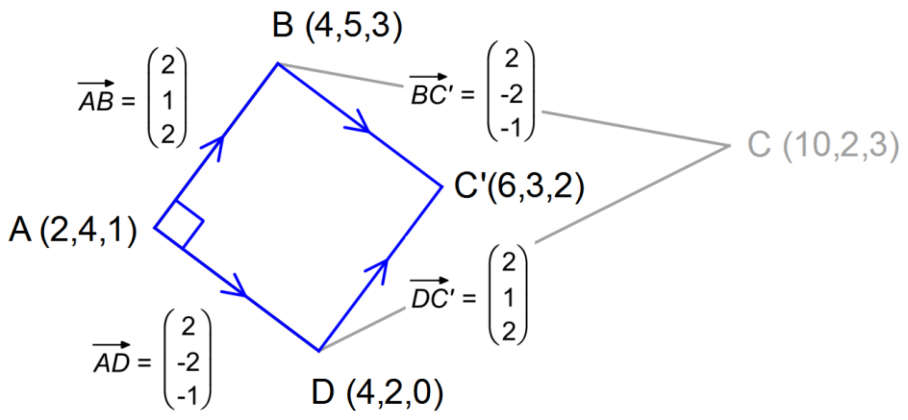
Now $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$, and $\overrightarrow{BC'} = \overrightarrow{OC'} - \overrightarrow{OB} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} x-4 \\ y-5 \\ z-3 \end{pmatrix}$.

Thus $\begin{pmatrix} x-4 \\ y-5 \\ z-3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$, hence the position vector of *C*' , $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $= \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$ for *ABC'D* to be a square.

Using *AB* and *DC'* would lead to the same result, since if three sides of a quadrilateral are equal and two of them form a parallel pair, then the fourth side is equal to the other three, as well as parallel to the" unmatched "side.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, and $\overrightarrow{DC'} = \overrightarrow{OC'} - \overrightarrow{OD} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} x-4 \\ y-2 \\ z \end{pmatrix}$.

Thus $\begin{pmatrix} x-4 \\ y-2 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, hence the position vector of *C*' , $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $= \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$ for *ABC'D* to be a square.



b) We have worked out in part ii) that $AB = 3$, so the area of the square *ABC'D* is 9 square units.

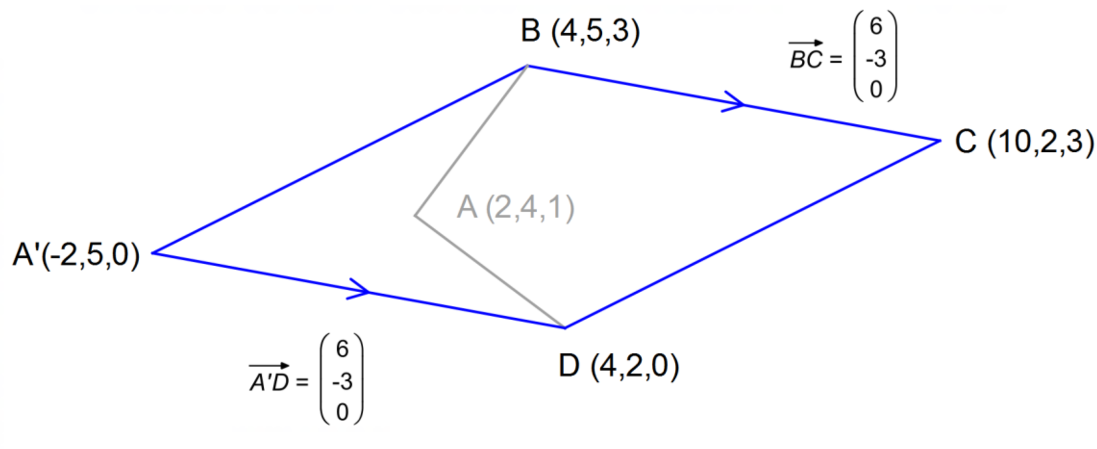
v) Unlike angle *BAD*, *BCD* is not a right angle, but the process of finding the direction vector of *A'* is identical.

The lengths of *AB* and *AD* are equal, so for *A'BCD* to be a rhombus, the vectors *BC* and *A'D* must be equal and parallel, as must *A'B* and *DC.*

Again, let the position vector of *A*' be $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and choose to work with vectors $\overrightarrow{A'D}$ and $\overrightarrow{BC}$.

Now $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 10 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix}$, so $\overrightarrow{A'D} = \overrightarrow{OD} - \overrightarrow{OA'} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4-x \\ 2-y \\ -z \end{pmatrix}$,

Thus $\begin{pmatrix} 4-x \\ 2-y \\ -z \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix}$, hence the position vector of *A'* , $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $= \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}$ for *A'BCD* to be a rhombus.



The area of a rhombus is half the product of the diagonals, so we need to find the length of *A'C*

By Pythagoras, $(A'C)^2 = (10-(-2))^2 + (2-5)^2 + (3-0)^2 = 12^2 + (-3)^2 + 3^2 = 162$, so $A'C = \sqrt{162}$ or $9\sqrt{2}$.

As $(BD)^2 = 18$ from part i), $BD = \sqrt{18}$ or $3\sqrt{2}$.

Hence the area of the rhombus is $\frac{1}{2}\left(9\sqrt{2}\right)\left(3\sqrt{2}\right) = 27$ square units.