M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

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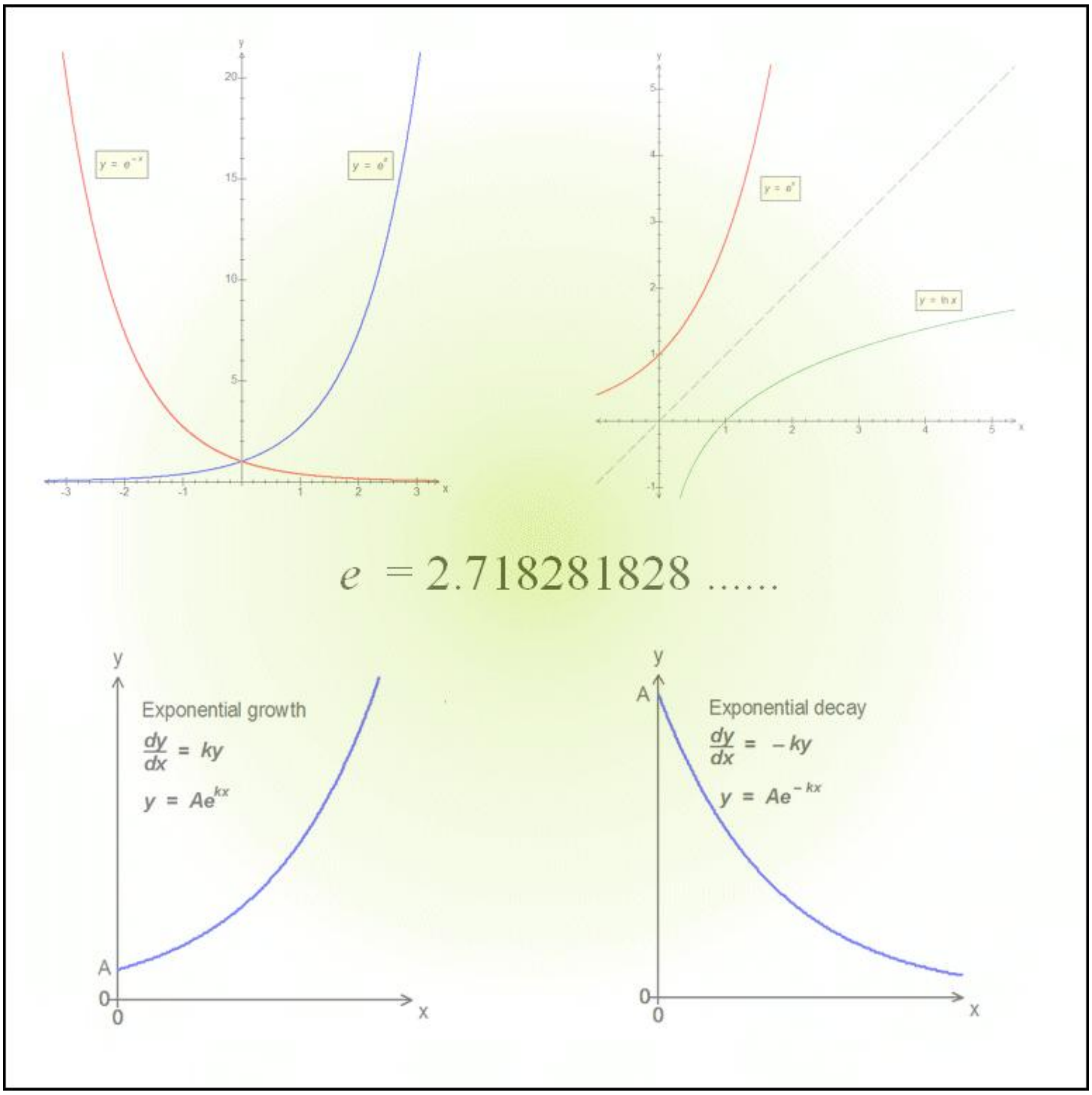
THE EXPONENTIAL AND NATURAL LOGARITHMIC FUNCTIONS: e^x, ln x



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THE EXPONENTIAL AND NATURAL LOGARITHMIC FUNCTIONS

The exponential function, e^x.

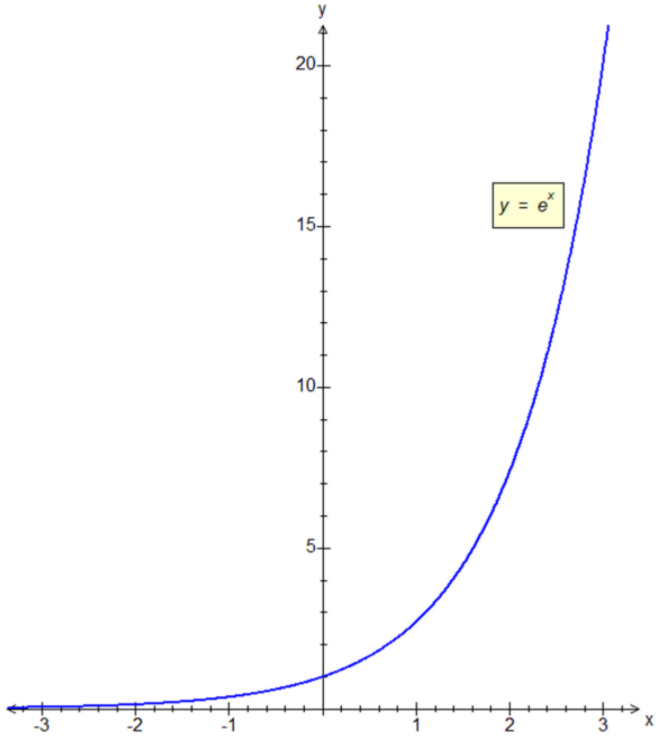
Exponential functions of the type

$f(x) = a^x$, where a is a non-zero positive constant, have already been discussed in earlier sections.

All exponential functions have graphs crossing the y-axis at the point (0, 1) as would be expected of the zero index law, but there is one particular case of the exponential function whose graph has a gradient of 1 at the point (0, 1).

This is known as ***the*** exponential function, and it is denoted by e^x where the base is an irrational number, e. The number e has an infinitely long but non-recurring decimal value, and its first few digits are 2.71828.

Like all exponential functions with a positive base greater than 1, its value becomes increasingly large and positive as x increases, and increasingly small and positive as x decreases, but it never reaches zero.



The exponential function e^x is unusual in having a gradient equal to the value of the function itself for all values of x.

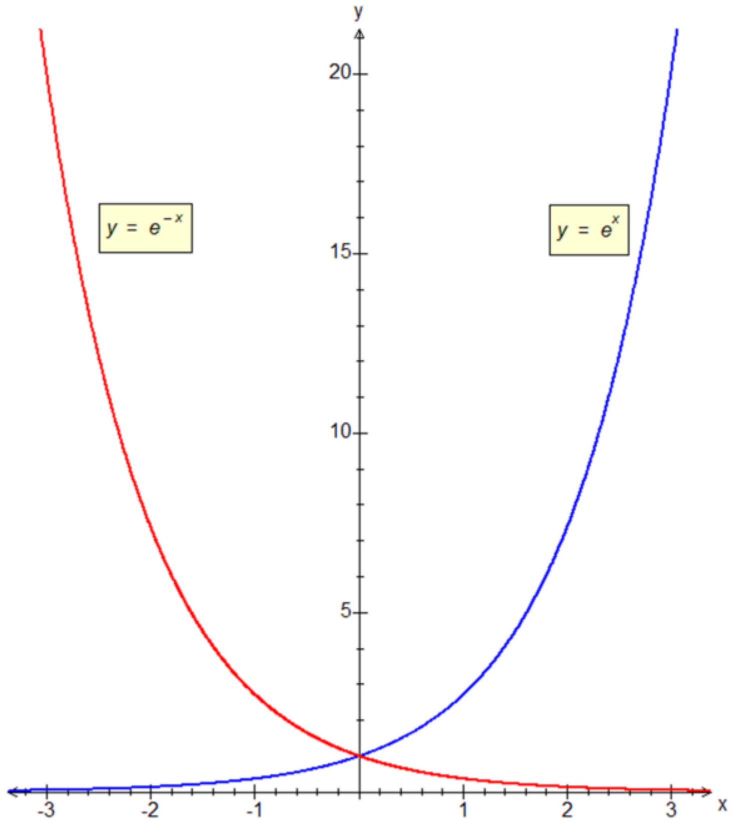
In other words, the derived function of e^x is e^x itself – a fact that will be studied in detail in further calculus sections.

The next graph shows the relationship between the functions $y = e^x$ and $y = e^{-x}$.

(Note that e^{-x} is the same as $\frac{1}{e^x}$).

The graph of e^{-x} is the same as that of e^x reflected in the y-axis.

Note how the function e^x is an increasing function for all x, and the function e^{-x} is a decreasing function for all x.



The natural logarithmic function, ln x.

An exponential function such as $f(x) = 10^x$ has a corresponding inverse function of $f^{-1}(x) = \log_{10} x$. In this example, the base of the logarithm is 10.

The inverse function corresponding to $f(x) = e^x$ is $f^{-1}(x) = \ln x$. This is called the **natural logarithm** of x, and is sometimes also written as $\log_e x$.

Although the base of natural logarithms (e) is irrational, the laws of logarithms hold exactly as they do for rational bases like 2 or 10.

Thus:

$$\ln a + \ln b = \ln ab$$

$$\ln a - \ln b = \ln \tfrac{a}{b}$$

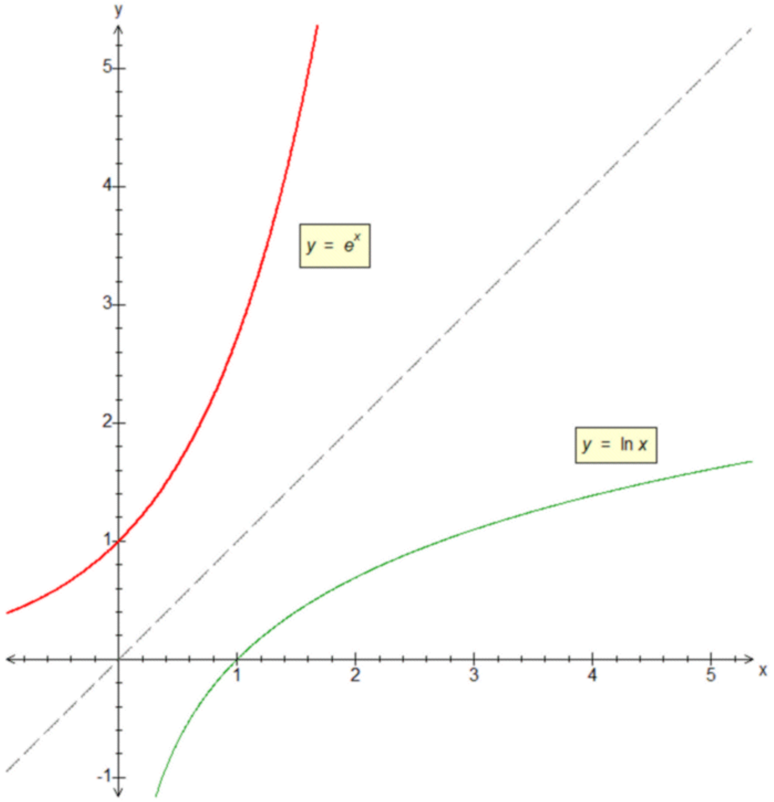
$$n \ln a = \ln a^n$$

$$\ln \tfrac{a}{b} = -\ln \tfrac{b}{a}$$

The graphs on the right show the relationship between the exponential function e^x and its inverse, the natural log function, or $\ln x$.

Each is a reflection of the other in the line $y = x$.

The functions e^x and $\ln x$ are handled just like other exponential and logarithmic functions.



Examples (1): Express as single logarithms: i) $2 \ln x + 4 \ln y$; ii) $3 \ln x - ½ \ln y$; iii) $(\ln x) - 2$

i) $2 \ln x + 4 \ln y = \ln (x^2 y^4)$; ii) $3 \ln x - ½ \ln y = \ln \left(\frac{x^3}{\sqrt{y}} \right)$; iii) $\ln x - 2 = \ln x - \ln (e^2) = \ln \left(\frac{x}{e^2} \right)$.

Examples (2): Solve the equations i) $e^x = 20$; ii) $e^x = 0.04$; iii) $\ln x = 0.7$.

We take natural logs of both sides in the first two examples to obtain

i) $\ln (e^x) = \ln 20 \Rightarrow x = \ln 20 \Rightarrow x = 2.996$ (to 3 d.p.)
ii) $\ln (e^x) = \ln 0.04 \Rightarrow x = \ln 0.04 \Rightarrow x = -3.219$ (to 3 d.p.)

In iii), we take exponents of both sides ("e to both sides") to obtain
$\ln x = 0.7 \Rightarrow x = e^{0.7} \Rightarrow x = 2.014$ (to 3 d.p.)

Remember, $e^{\ln x}$ is simply x itself, as is $\ln (e^x)$!

Sometimes care is needed, as logarithm laws specific to one base do not hold in others.

Examples (3): The values of ln 2 and ln 3 are 0.693 and 1.099 to 3 decimal places respectively.

Use this result to find i) ln 6 ; ii) ln 8 ; iii) ln 0.5 ; iv) ln 200.
Which of these results cannot be found without a calculator, given just the data in the question ?

i) $6 = 2 \times 3, \Rightarrow \ln 6 = \ln 2 + \ln 3 = 1.792$ (to 3 dp).
ii) $8 = 2^3, \Rightarrow \ln 8 = 3 \ln 2 = 2.079$ (to 3 dp).
iii) $0.5 = ½, \Rightarrow \ln (½) = -(\ln 2) = -0.693$ (to 3 dp).

iv) This question was originally set in an earlier section. There the base of the logarithm was 10, and $\log_{10}2$ was given as 0.301.

There we used the multiplication law as well as recognising a power of 10 as the multiplier:
$200 = 100 \times 2$, and since $100 = 10^2$, then $\log_{10}100 = 2$ and finally $\log_{10}200 = 2 + \log_{10}2 = 2.301$.

By using the base of 10, the logarithms of the integer powers of 10 were themselves integers, such as $\log_{10} 100 = 2$, $\log_{10} 1000 = 3$ and $\log_{10} 0.1 = -1$.

That assumption does not hold for natural logs, because we have changed the base from 10 to e.
We cannot mix up logarithms in two bases to give a result like $\ln 200 = \ln 2 + \log_{10}200$, or 0.693 + 2.

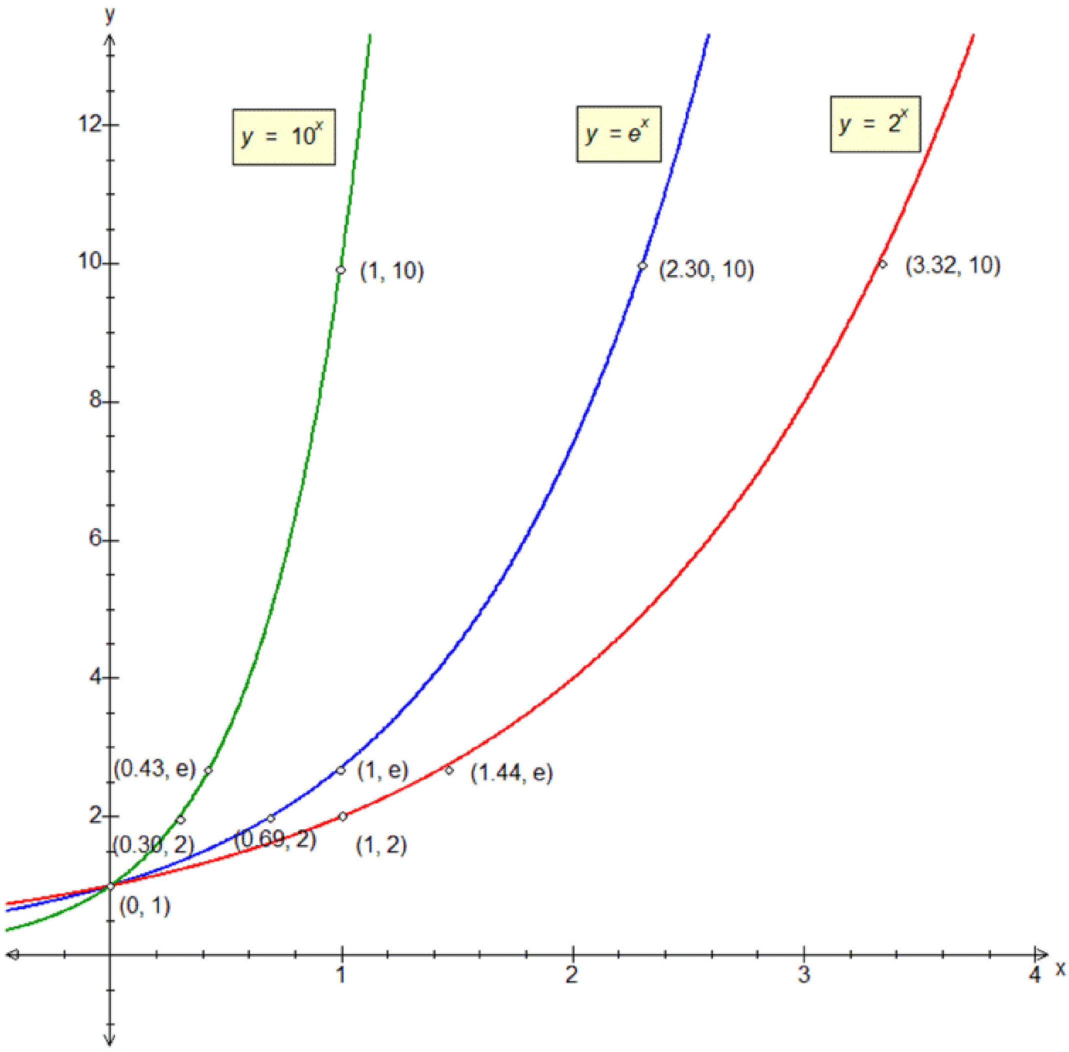
Therefore, without being given ln 100, the given data is insufficient to answer the question.

More on transformations of exponential and logarithmic graphs.

Transformations of exponential and logarithmic graphs have been covered in detail in the Core 2 document on exponential functions.

Here we show how exponential and logarithmic graphs to different bases are related to each other.

Example (6): Study the graphs of $y = e^x$, $y = 2^x$, and $y = 10^x$. How are the transformed graphs related to that of e^x ? How are the graphs of 10^x and 2^x related to each other ?



We can see that the graph of e^x is steeper than that of 2^x, but less so than that of 10^x.

In fact, the three graphs are related by a series of *x*-stretches.

Look at the points on the three graphs where $y = e$, and we can see how they are related by a series of stretches in the *x*-direction. Whereas the graph of e^x passes through $(1, e)$, the graph of 2^x passes through $(\approx 1.44, e)$ and that of 10^x passes through $(\approx 0.43, e)$.

In other words, the graph of 2^x is an *x*-stretch of the graph of e^x with a scale factor of about 1.44, and the graph of 10^x is an *x*-stretch of the graph of e^x with a scale factor of about 0.43.

Those figures appear difficult to interpret at first, but we remember that a^x is equivalent to $e^{(x \ln a)}$.

Hence $2^x = e^{x \ln 2}$ and $10^x = e^{x \ln 10}$

We can now see that the graph of 2^x can be obtained from that of e^x by an x-stretch with factor $\frac{1}{\ln 2}$ or approximately 1.44, and that of 10^x from e^x by an x-stretch with factor $\frac{1}{\ln 10}$ or approximately 0.43.

The graph of a general exponential function a^x can therefore be obtained from that of e^x by an x-stretch with factor $\frac{1}{\ln a}$.

Next, we look at the sets of points (1, 10) and (≈3.32, 10) as well as (2, ≈0.30) and (2,1).

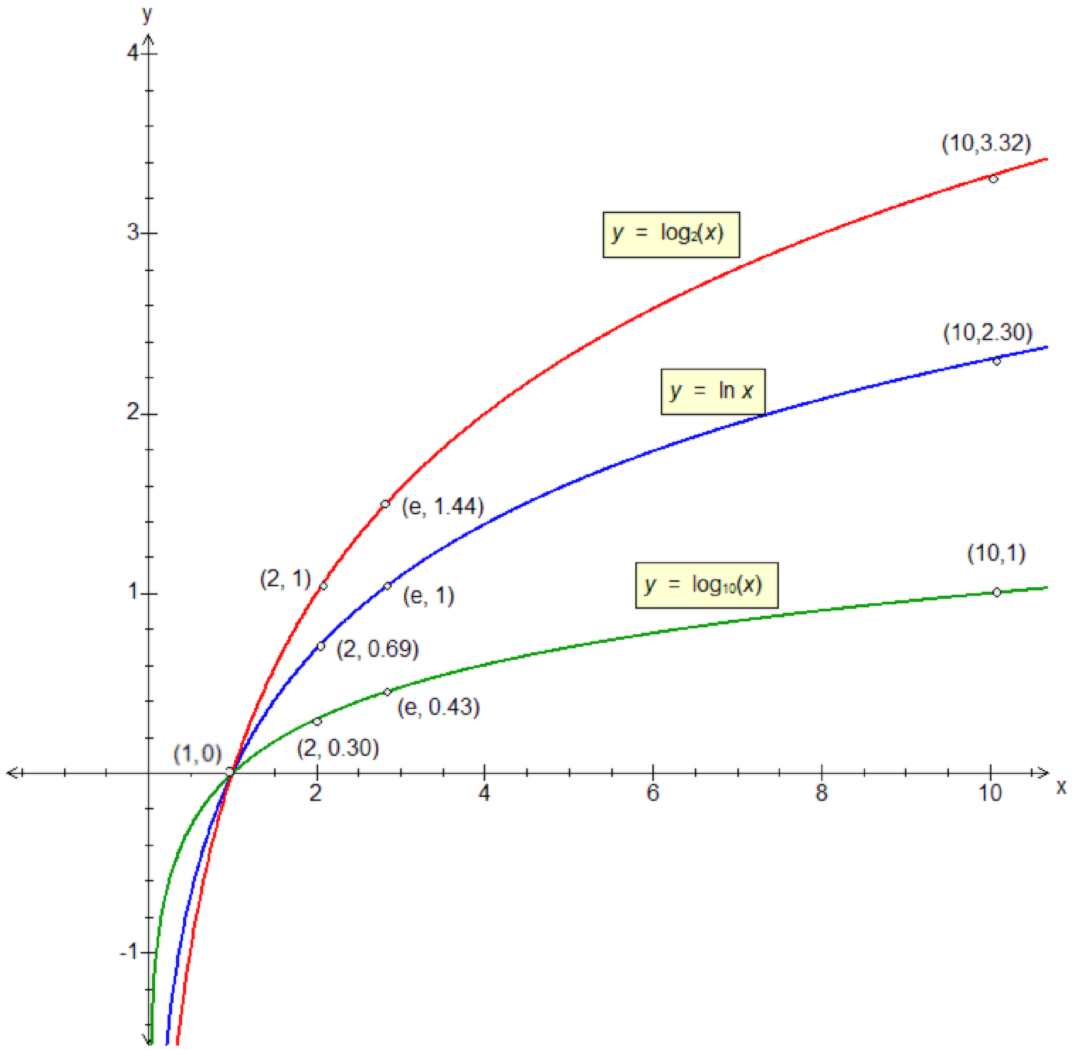
From the given details, it appears that the graph of 2^x can be obtained from that of 10^x by an x-stretch with factor 3.32, and 10^x from 2^x by an x-stretch with factor 0.30.

This can be generalised further by making use of the change of base rule.

The graph of a^x can be transformed into that b^x by an x-stretch with factor $\frac{\log a}{\log b}$. (The base of the logarithm is immaterial here, as long as it is consistent.)

The graph of 10^x is an x-stretch of the graph of 2^x, with factor $\frac{\ln 2}{\ln 10}$, or $\log_{10} 2$, about 0.30.
Conversely, the graph of 2^x is an x-stretch of the graph of 10^x, with factor $\frac{\ln 10}{\ln 2}$, or $\log_2 10$, about 3.32.

Example (7): Study the graphs of $y = \ln x$, $y = \log_2 x$, and $y = \log_{10} x$. How are the transformed graphs related to that of $\ln x$? How are the graphs of $\log_{10} x$ and $\log_2 x$ related to each other ?



We can see that the graph of $\ln x$ is steeper than that of $\log_{10} x$, but less so than that of $\log_2 x$.

In fact, the three graphs are related by a series of *y*-stretches.

Look at the points on the three graphs where $x = e$, and we can see how they are related by a series of stretches in the *y*-direction. Whereas the graph of $\ln x$ passes through $(e, 1)$, the graph of $\log_2 x$ passes through $(e, \approx 1.44)$ and that of $\log_{10} x$ passes through $(e, \approx 0.43)$.

In other words, the graph of $\log_2 x$ is a *y*-stretch of the graph of $\ln x$ with a scale factor of about 1.44, and the graph of $\log_{10} x$ is a *y*-stretch of the graph of $\ln x$ with a scale factor of about 0.43.

By checking the points(10, 1) on the graph of $\log_{10} x$ and $(10, \approx 3.32)$ on the graph of $\log_2 x$, we can deduce that the latter graph is a *y*-stretch of the former by a scale factor of about 3.32.

Similarly, by checking the points(2, 1) on the graph of $\log_2 x$ and $(2, \approx 0.30)$ on the graph of $\log_{10} x$, we can deduce that the latter graph is a *y*-stretch of the former by a scale factor of about 0.30.

These results come from the change of base rule: $\log_a x = \dfrac{\ln x}{\ln a} = \dfrac{1}{\ln a} \times \ln x$.

Hence the graph of $\log_a x$ can be obtained from that of ln *x* by a *y* –stretch of factor of $\frac{1}{\ln a}$.

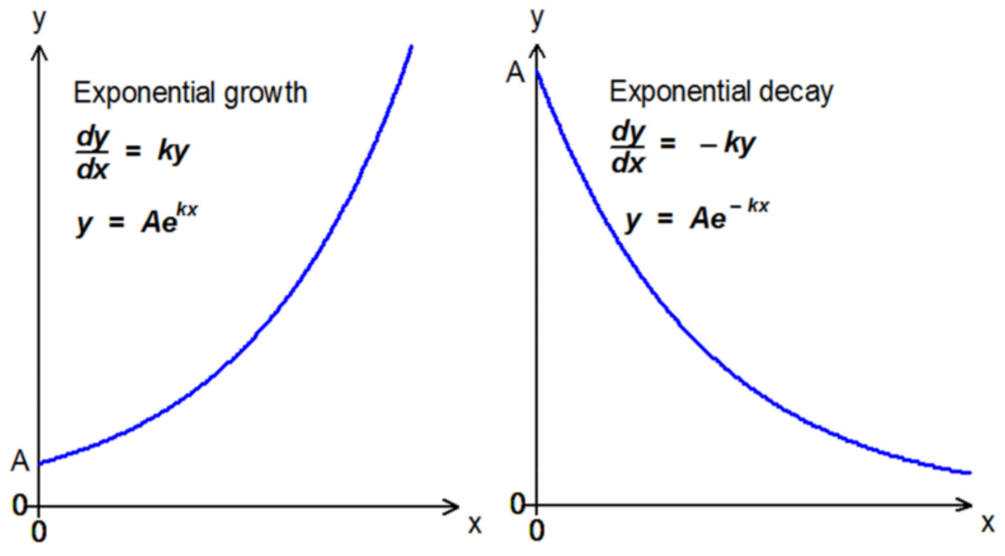
This last case can be generalised: the graph of $\log_a x$ can be transformed into that of $\log_b x$ by a *y*-stretch with factor $\frac{\log a}{\log b}$.

Exponential Growth and Decay.

(This topic will be covered in extra detail under the section on Differential Equations).

Many real-life situations can be modelled by exponential functions, especially if they are dependent on time. Examples include:

- The balance of an account paying fixed compound interest
- Depreciation of a car
- Increase of a population of bacteria
- Radioactive decay



The standard graphs modelling exponential growth and decay are shown above.

Exponential growth has the equation $y = Ae^{kx}$ and exponential decay, $y = Ae^{-kx}$.
The constant *A* represents an initial value when $x = 0$, and the *positive* constant *k* is a scale factor which transforms e^x into e^{kx} or e^{-kx}.

Because exponential growth and decay is usually dependent on *time* , the letter *t* is often used in preference to *x* to denote the variable.

Example (10): The table below records the population of a sample of bacteria over a four-day experimental period.

Time in days, t	0	1	2	3	4
Bacterial population, y	60	120	240	480	960
ln y (for part iii)	4.094	4.787	5.481	6.174	6.867

i) State the value of A, namely the bacterial population at $t = 0$

ii) Using $t = 4$, express the relationship between the population of bacteria and the elapsed time in the form $y = Ae^{kt}$.

iii) What would the graph of ln y against t look like ? How do the gradient and y-intercept relate to the original question ?

Note: The values for the bacterial population, y, can be seen at once to form a geometric progression whose first term, a, is 60 and whose common ratio is 2. We could therefore say that $y = 60 \times 2^t$.

The question however asks for a different approach !

i) The value of A, the initial population, is 60 bacteria when $t = 0$.

ii) When $t = 4$, the population of bacteria is 960, so we must solve Ae^{kt}. = 960, i.e. find k given $A = 60$ and $t = 4$.

$$Ae^{kt} = 960 \Rightarrow 60e^{4k} = 960 \Rightarrow e^{4k} = \frac{960}{60} = 16.$$

$$e^{4k} = 16 \Rightarrow 4k = \ln 16 \Rightarrow k = \frac{1}{4}\ln 16 = \ln 2 \Rightarrow k = 0.69315.$$

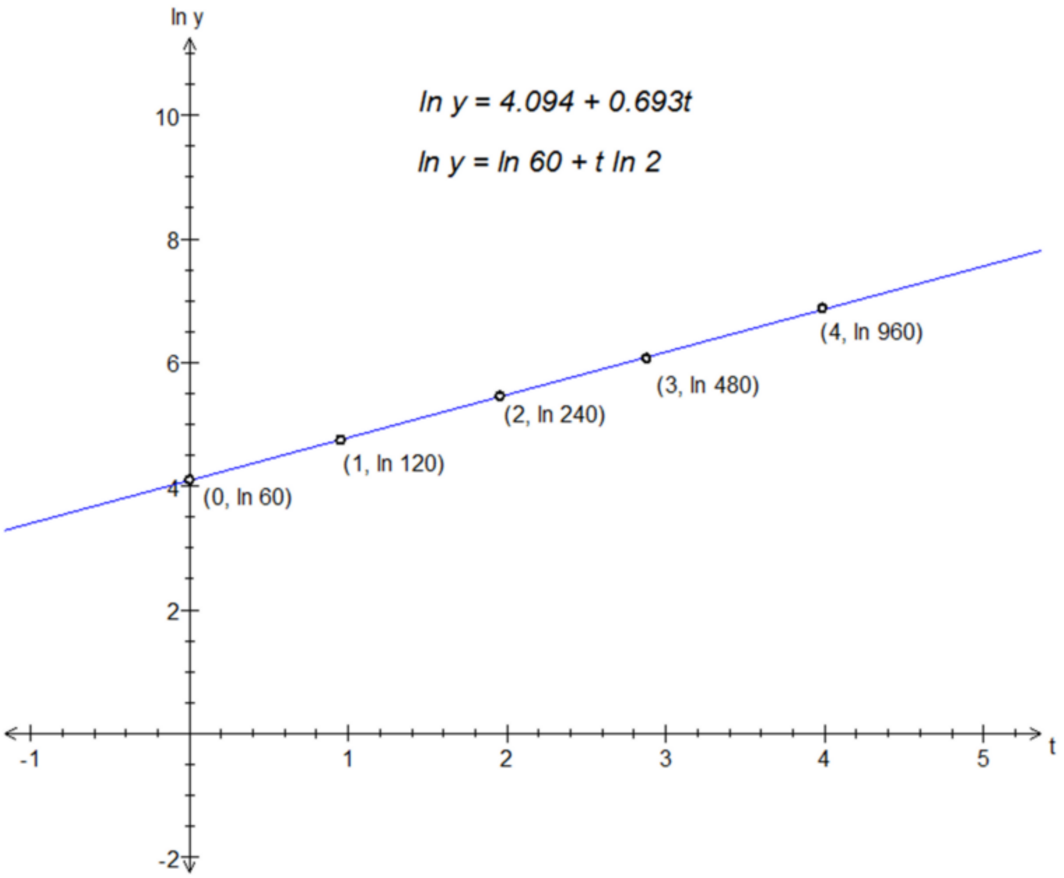
Hence $y = 60e^{0.69315t}$ or $60e^{(\ln 2)t}$

This is the same as saying $y = 60 \times 2^t$ because $e^{(\ln 2)} = 2$.

iii) Starting with the expression (using the accurate value of ln 2 rather than the approximation)

$y = 60e^{(\ln 2)t}$ and taking natural logarithms of both sides, we obtain

$\ln y = \ln 60 + t \ln 2.$



This is the equation of a straight-line graph $Y = mX + C$ where $Y = \ln y$, $C = \ln 60$, $m = \ln 2$ and $X = t$.

The gradient of the line is ln 2 and the y-intercept is at (0, ln 60).

This result can be generalised: for any graph $y = Ae^{kt}$ the equation of the graph of ln y against time is

$\ln y = \ln A + kt$ (for graphs with positive gradient, signifying growth), or
$\ln y = \ln A - kt$ (for graphs with negative gradient, signifying decay).

Example (11): A second-hand car has a value of £9170 when it is one year old and £6470 when it is three years old. It is assumed that the depreciation rate is constant with age.

i) Set up two equations of the form $y = Ae^{-kt}$ using $t = 1$ and $t = 3$.

ii) Hence find k and A. What does A represent ?

iii) What is the estimated value of the car after seven years ?

i) Substituting $t = 1$, $y = 9170$ gives $Ae^{-k} = 9170$; doing the same for $t = 3$ gives $Ae^{-3k} = 6470$.

Dividing, $\frac{Ae^{-k}}{Ae^{-3k}} = \frac{9170}{6470} = 1.4173 \Rightarrow e^{2k} = 1.4173$, and therefore $2k = 0.34876$ and $k = 0.17438$.

Substituting for k in the expression with $t = 1$, we now have $Ae^{-kt} = Ae^{-0.17438} = 9170$.

Hence, $A = \frac{9170}{e^{-0.17438}} = \frac{9170}{0.8400} = 10917$.

We could also have said that $A = 9170\, e^{0.17438} = 10917$, as dividing a quantity by e^{-x} is the same as multiplying it by e^{x}.

The quantity A represents the value of the car, namely £10917, when $t = 0$, i.e when the car is new.

The final expression for the value of the car is therefore $y = 10917e^{-0.17438t}$.

iv) To find the value of the car after seven years, we substitute $t = 7$ into the equation from the final part of ii):

$y = 10917e^{-0.17438t} \Rightarrow y = 10917e^{-1.2206} \Rightarrow y = £3221$.

The car is thus worth £3221 after seven years.

Compound interest example.

Example (12a): A building society offers a savings account with a five-year fixed interest rate equivalent of 3.6% per annum, compounded annually. How much interest would accrue on this account after five years, assuming a one-off initial deposit of £10,000 ?

After zero years, with $t = 0$, we can substitute $Ae^{kt} = 10000$, and therefore $A = 10{,}000$, as $e^{kt} = 1$ for all k when $t = 0$.

After one year, $t = 1$, Ae^{kt} = 10000 plus 3.6%, or £10,360. We thus solve

$$Ae^{k} = 10360 \Rightarrow 10000e^{k} = 10360 \Rightarrow e^{k} = \frac{10360}{10000} = 1.036.$$

Taking natural logarithms of both sides, $k = \ln 1.036 = 0.035367$

After five years , with $t = 5$, the total sum invested, or Ae^{5k}, is $10000e^{0.176836}$, or £11934.35.

The accrued interest after 5 years is therefore £1934.35.

Example (12b): A rival building society offers similar savings accounts to the previous examples with a five-year fixed interest rate equivalent of 3.6% per annum, but this time compounded at 0.9% every quarter. How much interest would accrue on that account ?

The value of A is 10000 as in Example 12(a).
After one quarter, $t = 1$, Ae^{kt} = 10000 plus 0.9%, or £10,090.

Reckoning as in (7a), $k = \ln 1.009 = 0.0089597$

After five years, or 20 quarters , with $t = 20$, the total sum invested, or Ae^{20k}, is $10000e^{0.179195}$, or £11962.54. This is the same as $10000\,(1.009^{20})$.

The accrued interest after 5 years is therefore £1962.54 – higher than if the interest were compounded only once per year .

Note - We could repeat the last example by compounding the interest over ever-smaller time intervals, such as dividing the 3.6% into twelve payments of 0.3% every month to give a final value of $10000\,(1.003^{60})$ or £11968.95.

There is in fact a limiting value where the interest is accrued continuously, but this is not in scope of the section. As a matter of interest (no pun intended !), this limiting value is £11972.17.

Example (13): The population of toads at a reserve near Southport was estimated at 520 at the start of 2002 and 740 at the start of 2007.

Use these values to i) find an exponential function to model the population growth; ii) use the function so obtained to estimate the toad population at the start of 2010; iii) estimate when the population reached 1000.

i) Using the start of 2002 as the zero point, we can substitute A = 520, as e^{kt} = 1 for all k when t = 0. Five years later, at the start of 2007, with t = 5, the population had risen to 740, so we now have to solve

$$Ae^{5k} = 740 \Rightarrow 520e^{5k} = 740 \Rightarrow e^{5k} = \frac{740}{520} = \frac{37}{26}.$$

Taking natural logarithms of both sides, $5k = \ln\left(\frac{37}{26}\right) \Rightarrow k = \frac{\ln \frac{37}{26}}{5} = 0.07056$.

The toad population can therefore be modelled by the expression $\boldsymbol{P = 520e^{0.07056t}}$ where t is the number of years since the start of 2002.

ii) There is a time difference of 8 years between the start of 2002 and the start of 2010, so we substitute t = 8 into the exponential growth equation.

The toad population at the start of 2010 is $520e^{0.07056 \times 8} \Rightarrow 520e^{0.5645}$, or 914 to the nearest integer.

iii) Here we need to find the value of t to solve $520e^{0.5645\,t}$ = 1000.

Hence, $e^{0.0.07056t} = \frac{1000}{520} = \frac{25}{13} \Rightarrow 0.07056\,t = \ln\left(\frac{25}{13}\right) \Rightarrow t = \frac{\ln(\frac{25}{13})}{0.07056}$

Therefore t = 9.27 years or 9 years 3 months, and so the toad population reached 1000 at about April 2011.

Example (14): Two collectors, Tony and David, decided to invest in antiques and collectibles in January 1982. Tony invested £1500 and David £1200. When the portfolios were valued 25 years later in January 2007, Tony's investment was valued at £18,000 and David's at £22,800.

i) Show that Tony's investment grew at an annual rate of 10.45% between his purchasing and the valuation, and also express the value of Tony's investment in the form $I = Ae^{kt}$.

ii) Find the annual growth rate of David's investment, and express it in the form $J = Ae^{kt}$.

iii) In which year did the value of David's portfolio catch up with that of Tony's, assuming a uniform growth rate over the period ?

Tony's portfolio can be modelled as $I = Ae^{kt}$ where A = 1500 (initial outlay in £) and $Ae^{25k} = 18000$ (the valuation at the end of 25 years).

Thus, $Ae^{25k} = 18000 \Rightarrow e^{25k} = \dfrac{18000}{1500} = 12 \Rightarrow e^{k} = 12^{\frac{1}{25}} = 1.104503.....$ and $k = 0.09940$.

Since the value $e^{kt} = 1.104503...$ when $t = 1$, the investment has increased by $(1.104503 - 1) \times 100$ % , or 10.45% after 1 year .

$\therefore$ Tony's portfolio can be valued as $I = 1500e^{0.09940t}$

ii) We set up a similar equation for David's portfolio, as $J = Ae^{kt}$ where A = 1200 (initial outlay in £) and $Ae^{25k} = 22800$ (the valuation at the end of 25 years).

Thus, $Ae^{25k} = 22800 \Rightarrow e^{25k} = \dfrac{22800}{1200} = 19 \Rightarrow e^{k} = 19^{\frac{1}{25}} = 1.124993.....$ and $k = 0.11778$.

$\therefore$ The annual growth rate of David's investment is 12.50% , and the equation for the value of the portfolio is $J = 1200e^{0.11778t}$

To find when David's portfolio caught up in value with Tony's, we must solve the equation in one of two forms:

Using the equations directly in the form Ae^{kt} :

$1500e^{0.09940t} = 1200e^{0.11778t}$ for t.

$$e^{0.11778t} = \left(\frac{1500}{1200}\right) e^{0.09940t} \Rightarrow \frac{e^{0.11778t}}{e^{0.09940t}} = 1.25$$

Taking natural logs (remember, dividing actual numbers means subtracting the logs),

$0.11778t - 0.09940t = \ln 1.25 \Rightarrow 0.01838t = 0.22314$,

and hence $t = \frac{0.22314}{0.01838}$ or 12.1 years.

David's portfolio therefore catches up with Tony's 12.1 years after January 1982, or early in 1994.

Using the equations in the form without *e* :

Alternatively, we could take the percentage growth values of 10.450 % for Tony and 12.499% for David and use them to solve the equation for t in this way:

Tony's investment can be expressed as $1500(1.10450)^t$ and David's as $1200(1.12499)^t$:

$1500(1.10450)^t = 1200(1.12499)^t \Rightarrow 1.12499^t = \left(\frac{1500}{1200}\right) 1.10450^t \Rightarrow \frac{1.12499^t}{1.10450^t} = 1.25$

Taking natural logs,

$t \ln 1.12499 - t \ln 1.10450 = \ln 1.25 \Rightarrow t (\ln 1.12499 - \ln 1.10450) = 0.22314$

$\Rightarrow t (0.11778 - 0.09940) = 0.22314 \Rightarrow 0.01838t = 0.22314$,

and hence $t = \frac{0.22314}{0.01838}$ or 12.1 years.

David's portfolio therefore catches up with Tony's 12.1 years after January 1982, or early in 1994.

Example (15): The radioactivity of a sample of phosphorus-32 was taken in terms of a Geiger counter reading, and the adjusted count was recorded as 1720 'hits' per minute.

The same sample was then locked away and another reading taken 30 days later, and then the adjusted count was recorded as 402 'hits' per minute.

Assuming that the number of 'hits' is proportional to the mass of the phosphorus-32 remaining, what is the daily decay constant k and hence the half-life of phosphorus-32 ? (The half-life of a radioactive substance is the time taken for the mass, and hence the activity, to decline to one-half of its original value.

As this is an exponential decay example, the activity readings R are to be modelled as $R = Ae^{-kt}$.

The reading at the start (when $t = 0$) is 1720, so $A = 1720$.
After 30 days, the reading $R = Ae^{30(-k)} = 402$.

$$Ae^{30(-k)} = 402 \Rightarrow 1720e^{30(-k)} = 402 \Rightarrow e^{30(-k)} = \frac{402}{1720}.$$

Taking natural logarithms of both sides, $30(-k) = \ln 402 - \ln 1720 \Rightarrow -k = \dfrac{\ln 402 - \ln 1720}{30}$.

$\therefore -k = -0.04845$.

The radioactive decay constant, k, is therefore 0.04845, and the radioactive count for this particular sample of phosphorus-32 can be modelled by the equation $\boldsymbol{R = 1720e^{-0.04845t}}$.

To find the half-life of phosphorus-32, we can either solve $1720e^{-0.04845t} = 860$ for t, since 860 is half of 1720, or we can simplify the question to solving $e^{-0.04845t} = \frac{1}{2}$.

We must solve $e^{-0.04845t} = \dfrac{1}{2} \Rightarrow -0.04845\,t = \ln\left(\dfrac{1}{2}\right)$ (taking logs)

$\Rightarrow 0.04845\,t = \ln 2$ (multiplying by -1 on each side) $\Rightarrow t = \dfrac{\ln 2}{0.04845}$

$\therefore$ the half-life of phosphorus-32 is $\dfrac{\ln 2}{0.04845}$ days or 14.3 days.

Note: the step of multiplying both sides by -1 is not absolutely necessary, and is only used to provide positive values in the fractional expression.

The expressions $\dfrac{\ln\left(\frac{1}{2}\right)}{-0.04845}$ and $\dfrac{\ln 2}{0.04845}$ are equivalent !