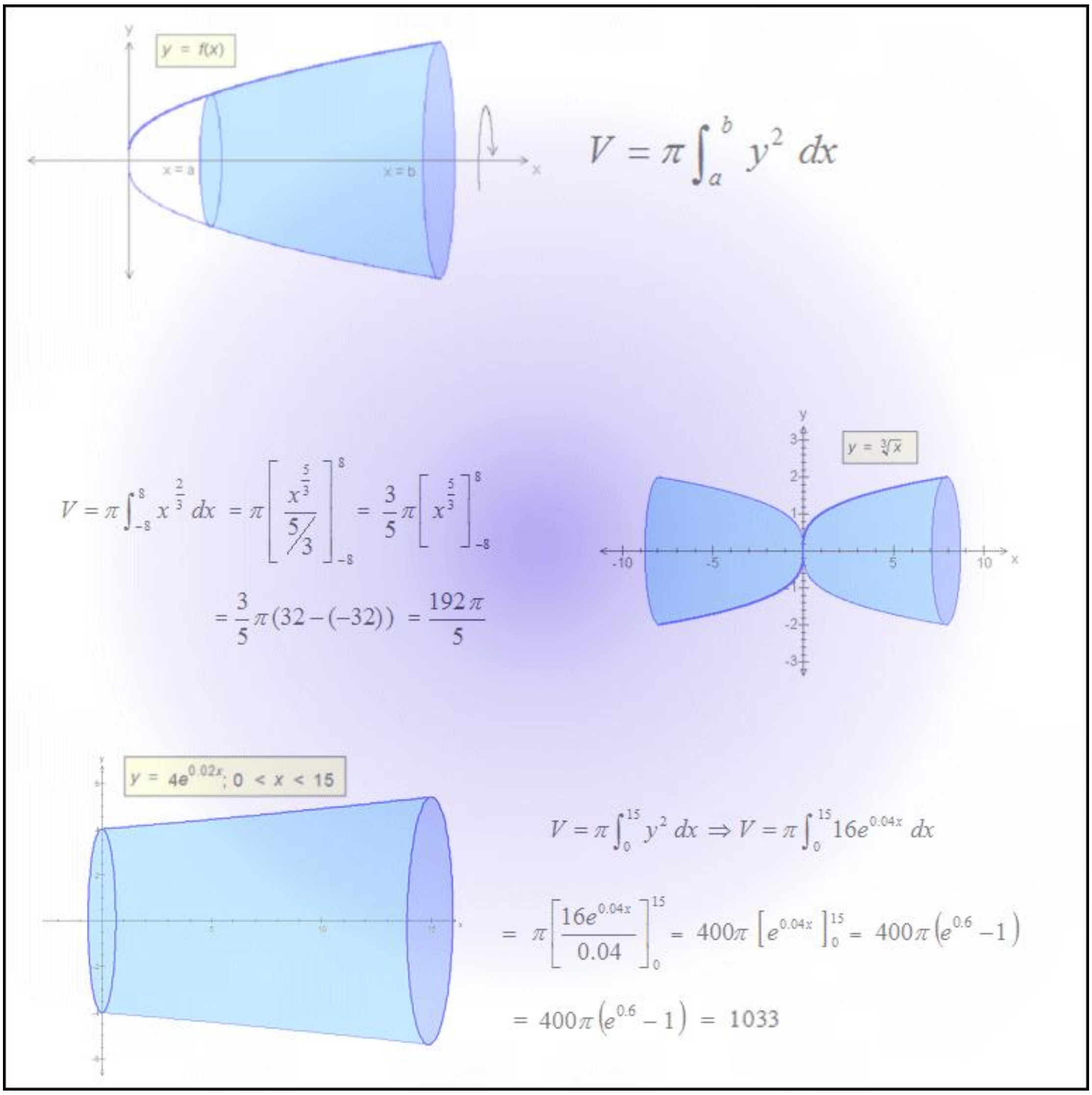
M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

AQA : **C3** Edexcel: **C4** OCR: **C3** OCR MEI: **C4**

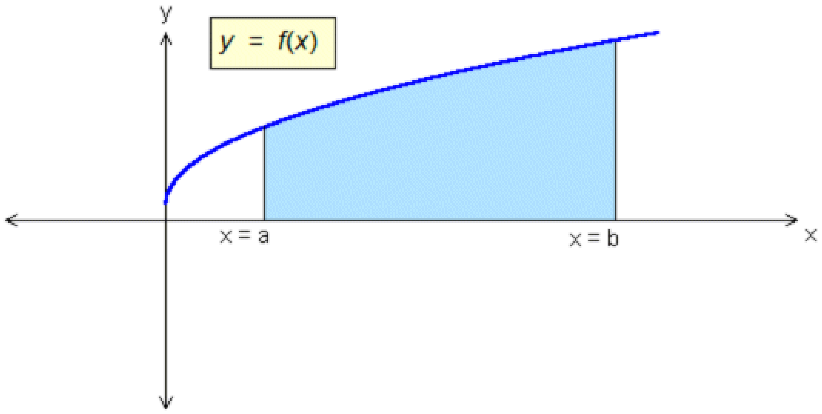
INTEGRATION - VOLUMES OF REVOLUTION



Version : 2.3 Date: 03-04-2013

Volumes of Revolution.

This is an extension of the method for finding areas under a curve.

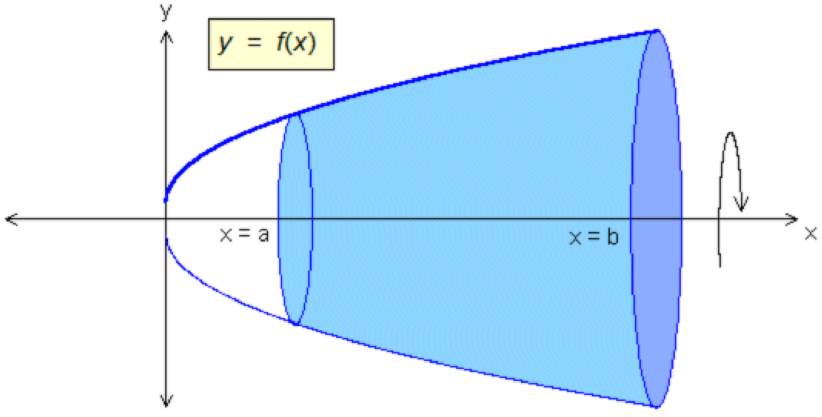


The shaded region above is bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$. Its area is given as $\int_a^b f(x)dx$.

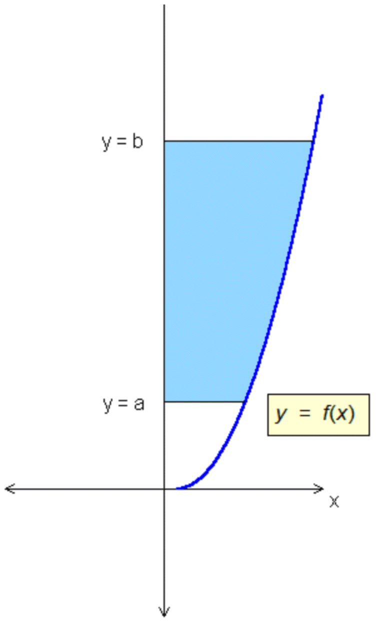
Rotating this area completely about the x-axis describes a solid figure as shown below. The volume of this figure is the **volume of revolution**. This volume is given by the formula

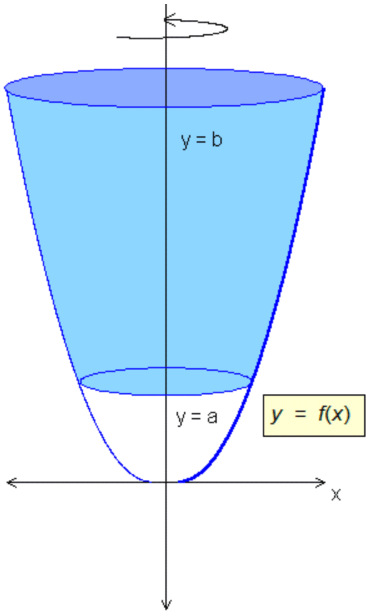
$V = \int_a^b \pi y^2 \, dx$. Some versions of the formula have π outside, as $V = \pi \int_a^b y^2 \, dx$.

(This document favours using π outside the integral sign.)



The following applies to rotation about the *y*-axis: here, the region to be integrated is is bounded by the curve $y = f(x)$, the *y*-axis and the lines $y = a$ and $y = b$.





This time the formula for the volume is given by

$$V = \int_a^b \pi x^2 \, dy \quad (\text{or } V = \pi \int_a^b x^2 \, dy).$$

Example (1):

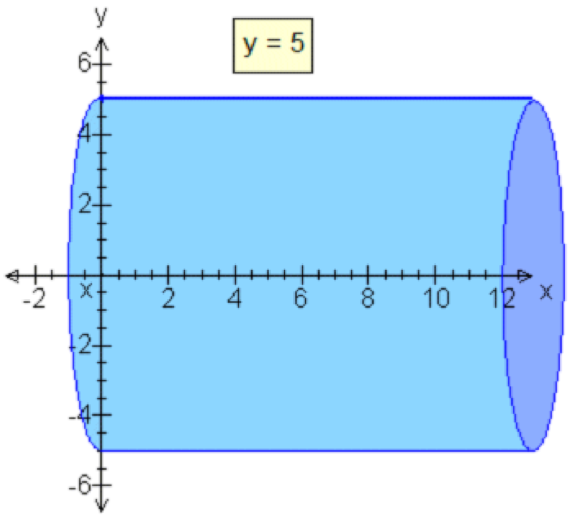
i) Find the volume of revolution of the region bounded by the *x*-axis, the line $y = 5$, and the lines $x = 0$ and $x = 12$. Leave the result in terms of π.

ii) Repeat part i), but set $y = r$ and the upper limit of the integrand to *h*.

i) Since $y = 5$ is a constant function, the volume formula is simply $V = \pi \int_0^{12} (5^2)\, dx$

$= \pi [25x]_0^{12} = 300\pi$.

The solid of revolution here is a cylinder of height 12 units and radius 5 units.



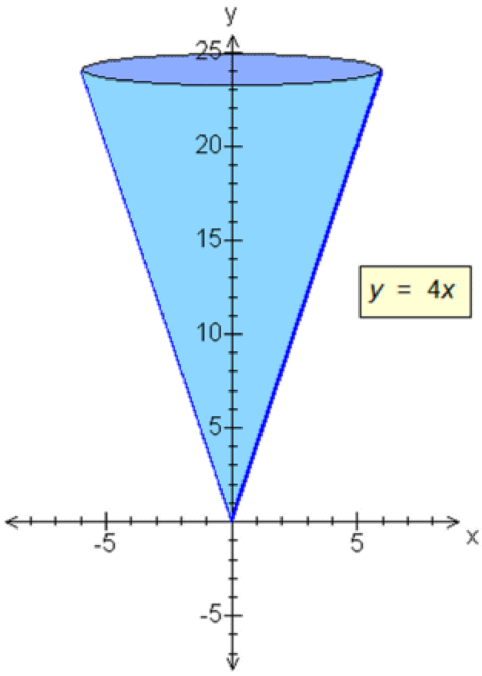
When $y = r$ and the upper *x*-limit = *h*, we have the volume formula

$$V = \pi \int_0^h r^2 \, dx = \pi [r^2 x]_0^h = \pi r^2 h.$$

This is the standard formula for the volume of a cylinder of radius *r* and height *h*.

Example (2): Find the volume of revolution of the region bounded by the y-axis, the line $y = 4x$, and the lines $y = 0$ and $y = 24$. Leave the result in terms of π.

This time, the solid of revolution is a cone of height 24 units and base radius of 6 units (here one-quarter the height).



Since $y = 4x$, $x = \frac{y}{4}$, and the formula for the volume becomes $V = \pi \int_0^{24} \frac{y^2}{16}\, dy.$

The volume of the generated solid is

$$\pi \left[\frac{y^3}{48} \right]_0^{24} = 288\pi.$$

Again this formula can be generalised; the y-limits can be set to 0 and h and the equation of the sloping line to $y = \frac{hx}{r}$ (or the equivalent, $x = \frac{ry}{h}$) .

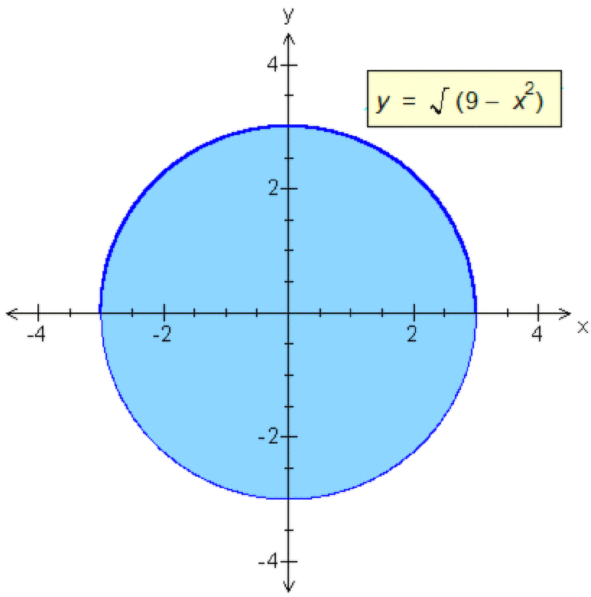
The equation for the volume therefore becomes

$$V = \pi \int_0^h \frac{r^2 y^2}{h^2}\, dy = \pi \left[\frac{r^2}{h^2} \left(\frac{y^3}{3} \right) \right]_0^h = \pi \left(\frac{r^2}{h^2} \times \frac{h^3}{3} \right) = \frac{1}{3} \pi r^2 h .$$

This is the standard formula for the volume of a cone of base radius r and vertical height h.

Example (3): Find the volume of revolution of the region bounded by the x-axis, the curve

$y = \sqrt{(9 - x^2)}$, and the lines x = -3 and x = 3. Leave the result in terms of π.



The equation of the curve is a rearrangement of $x^2 + y^2 = 9$, and therefore its graph is a semicircle.

The resulting solid of revolution is a sphere of radius 3 units (not well shown in diagram).
The volume of the generated sphere is

$$V = \pi \int_{-3}^{3} (9 - x^2)\, dx \;=\; \pi \left[9x - \frac{x^3}{3} \right]_{-3}^{3}$$

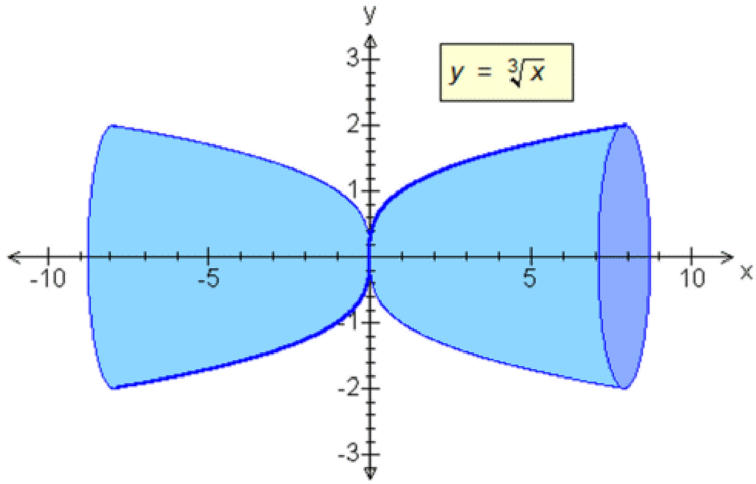
$$= \pi\big[(27 - 9) - \big((-27) - (-9)\big)\big] = 36\pi\,.$$

This formula can be generalised for any sphere of radius r units - here the region of integration will be bounded by the x-axis, the curve $y = \sqrt{(r^2 - x^2)}$, and the lines $x = -r$ and $x = r$.

The volume will thus be $V = \pi \int_{-r}^{r} (r^2 - x^2)\, dx \;=\; \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r}$

$$\pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left((-r)^3 - \frac{(-r)^3}{3}\right)\right] \;=\; \pi\left[\left(\frac{2r^3}{3}\right) + \left(\frac{2r^3}{3}\right)\right] = \frac{4}{3}\pi r^3\,.$$

Example (4): Find the volume of revolution of the region bounded by the *x*-axis, the curve $y = \sqrt[3]{x}$, and the lines *x* = -8 and *x* = 8. Leave the result in terms of π.



The volume is given as $V = \pi \int_{-8}^{8} x^{\frac{2}{3}}\, dx = \pi \left[\frac{x^{\frac{5}{3}}}{5/3} \right]_{-8}^{8} = \frac{3}{5}\pi \left[x^{\frac{5}{3}} \right]_{-8}^{8}$

$= \frac{3}{5}\pi(32-(-32)) = \frac{192\pi}{5}$ cubic units.

Although the graph of the function $y = \sqrt[3]{x}$ is partly below the *x*-axis and partly above it for the limits of the integral, the problem of negative areas is not an issue when applied to finding volumes of revolution.

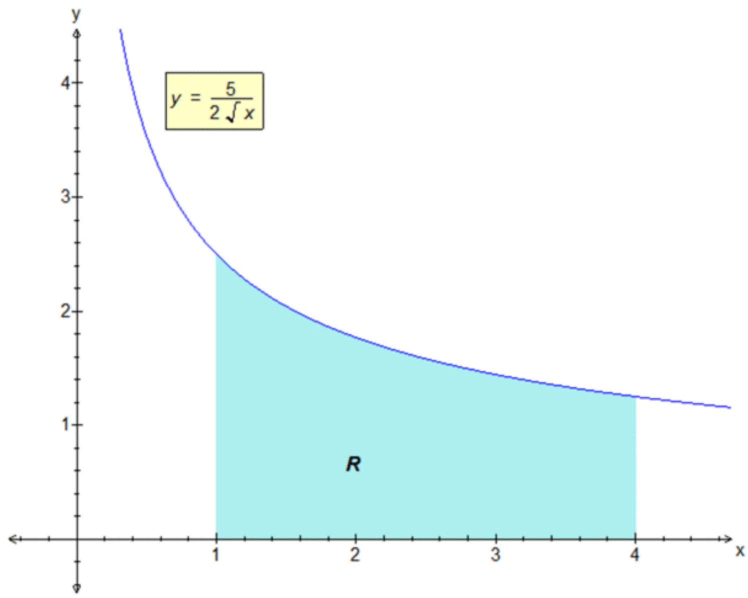
Sometimes a question might include working out both areas and volumes:

Example (5):

i) Find the area of region R, which is bounded by the x-axis, the lines $x = 1$, $x = 4$ and the curve $y = \dfrac{5}{2\sqrt{x}}$.

ii) The region R is rotated fully about the x – axis.

Find the exact volume of revolution of the generated solid, expressing the result in terms of π and ln 2.



i) The area of region R is $\int_1^4 \frac{5}{2\sqrt{x}}\,dx$ or $\int_1^4 \frac{5}{2}x^{-\frac{1}{2}}\,dx$,

or $\left[5\sqrt{x}\right]_1^4 = \left[(5\sqrt{4} - 5\sqrt{1})\right]$ = 5 square units.

ii) The volume of revolution is $V = \pi\int_1^4 \left(\frac{5}{2\sqrt{x}}\right)^2 dx = \pi\int_1^4 \frac{25}{4x}\,dx = \frac{25}{4}\pi\int_1^4 \frac{1}{x}\,dx$.

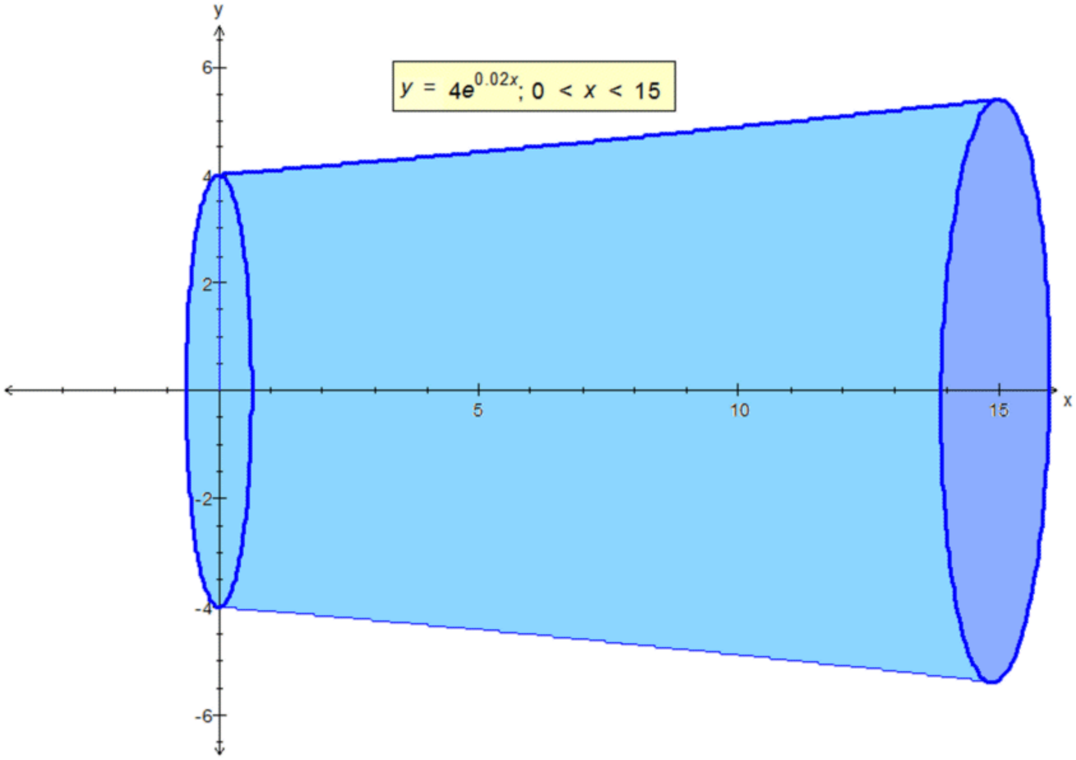
This evaluates to $\frac{25}{4}\pi\left[\ln x\right]_1^4 = \frac{25}{4}\pi\left[(\ln 4 - \ln 1)\right] = \frac{25}{4}\pi(\ln 4)$.

Since the question asks for the result in terms of ln 2, we use ln 4 = 2 ln 2,

and hence the volume of the generated solid is $\frac{25}{2}\pi(\ln 2)$ cubic units.

Example (6): A drinking glass is 16 cm tall, inclusive of a solid base 1cm thick (i.e. the effective height is 15cm.).

Its interior curved surface is generated by the equation $y = 4e^{0.02x}$, followed by a revolution about the *x*-axis. (Diagram below excludes solid base.)



i) Show that the inside diameter of the glass is 8 cm at the base, and 10.8 cm at the top.

ii) Find the volume of revolution of the resulting solid, i.e. the capacity of the glass, to the nearest cm^3.

i) At $x = 0$, $y = 4$, so the inside diameter of the top of the glass is twice that, i.e. 8 cm.
At $x = 15$, $y = 4e^{0.3}$ = 5.4, so the inside radius of the top of the glass is 5.4 cm, and hence the diameter is twice that, or 10.8 cm.

ii) The volume of revolution of the resulting solid is $V = \pi \int_0^{15} y^2 \, dx \Rightarrow V = \pi \int_0^{15} 16e^{0.04x} \, dx$.

This integrates to $\pi \left[\frac{16e^{0.04x}}{0.04} \right]_0^{15}$ = $400\pi \left[e^{0.04x} \right]_0^{15}$ = $400\pi \left(e^{0.6} - 1 \right)$.

The volume of revolution is $400\pi \left(e^{0.6} - 1 \right)$ cm^3 or 1033 cm^3 to the nearest cm^3.

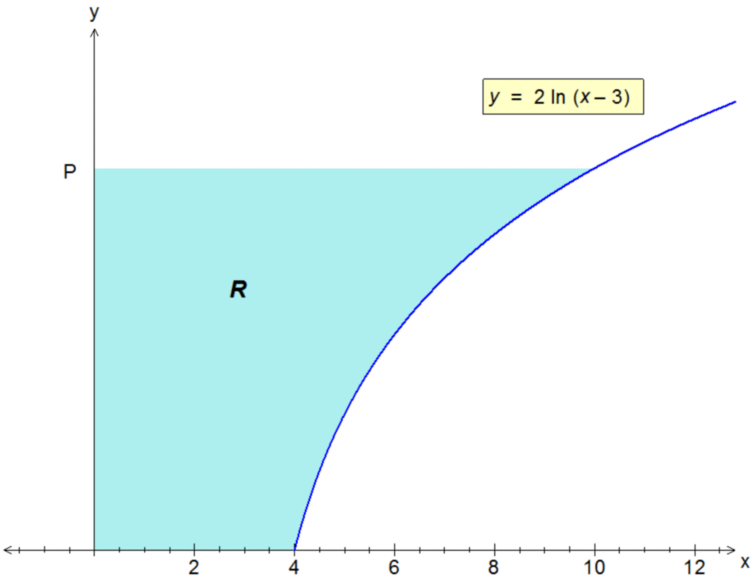
Example (7):

The diagram on the left shows the curve with equation $y = 2\ln(x - 3)$. The point P has coordinates $(0, p)$, and the shaded region R is bounded by the curve, the axes, and the line $y = p$.

The units on the axes are in centimetres, and the region R is rotated completely about the ***y*-axis** to form a solid.

i) Show that the volume, V cm^3, of the solid formed is given by

$$V = \pi\left(e^p + 12e^{\frac{p}{2}} + 9p - 13\right).$$



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i) Because we are dealing with a rotation about the y-axis, the first step is to rewrite $y = 2\ln(x-3)$ with x as the subject.

Taking exponents of both sides, $e^y = e^{(2\ln(x-3))} \Rightarrow e^y = e^{\ln((x-3)^2)} \Rightarrow e^y = (x-3)^2$.

(Remember, for any function $f(x)$, $e^{\ln(f(x))} = f(x)$ and also $\ln e^{(f(x))} = f(x)$ because e^x and $\ln x$ are inverses of each other !)

We then take square roots: $(x-3) = e^{\frac{y}{2}} \Rightarrow x = 3 + e^{\frac{y}{2}}$.

Having made x the subject, we can now use the volume formula.

$$V = \pi\int_0^p x^2\,dy \quad\Rightarrow\quad V = \pi\int_0^p \left(3 + e^{\frac{y}{2}}\right)^2 dy.$$

Expanding, $V = \pi\int_0^p 9 + 6e^{\frac{y}{2}} + e^y\,dy$, and integrating, $V = \pi\left[9y + 12e^{\frac{y}{2}} + e^y\right]_0^p$.

$$\Rightarrow\quad V = \pi\left[\left(9p + 12e^{\frac{p}{2}} + e^p\right) - (0 + 12 + 1)\right] \Rightarrow V = \pi\left[\left(e^p + 12e^{\frac{p}{2}} + 9p - 13\right)\right].$$