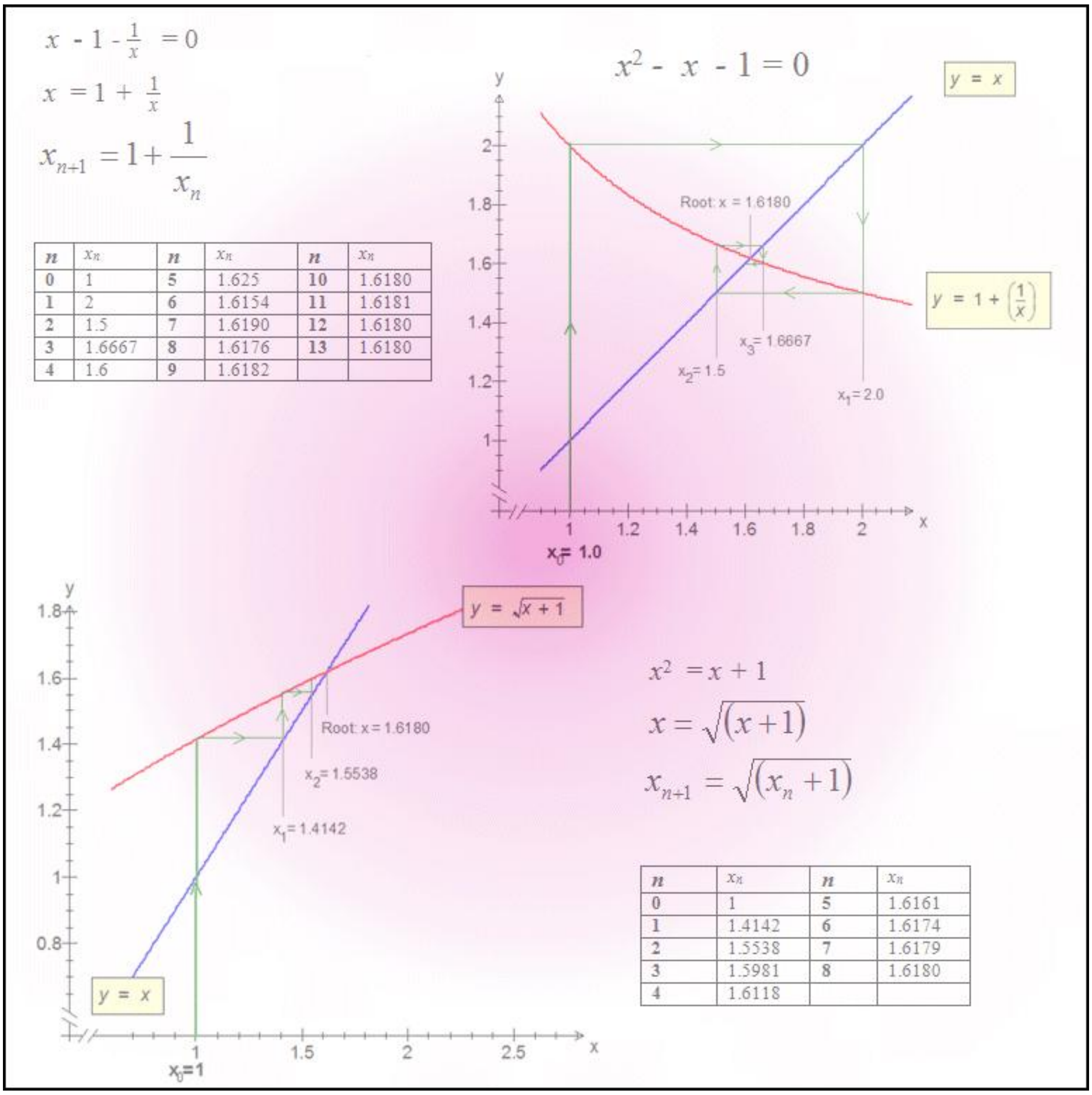
M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

AQA : **C3** Edexcel: **C3** OCR: **C3**

NUMERICAL METHODS FOR SOLVING EQUATIONS



Version : 4.3 Date: 01-11-2014

NUMERICAL METHODS.

Overview .

Most equations cannot be solved algebraically to give an exact answer, and therefore we have to resort to numerical approximations.

There are four methods discussed at A-level, but only the two in bold are widely used.

- **The decimal search (All)**
- **Formula iteration (All, except OCR MEI)**
- The bisection method
- The Newton-Raphson method

The first two methods involve trapping the root of $f(x) = 0$ between two values a and b such that $f(a)$ and $f(b)$ have different signs, and that f is continuous between a and b.

The second two methods rely on choosing a single 'starter' approximation to the root, with the intention of improving the accuracy with each successive step.

Continuous Functions – change of sign.

If the graph of a function has no breaks within an interval $x = a$ to $x = b$, then the function is said to be **continuous** on the interval. For example, all polynomial functions are continuous everywhere, but the reciprocal function $1/x$ is not continuous on any interval which contains the value $x = 0$.

- If a function $f(x)$ is continuous between $x = a$ and $x = b$, and if $f(a)$ and $f(b)$ have different signs, then the interval from a to b will contain a root of $f(x) = 0$.

Example (1): Show that $f(x) = x^3 - 4x^2 + 6 = 0$ has a root between 1 and 2.

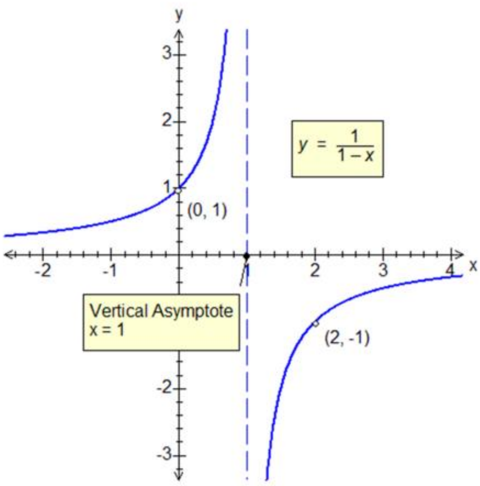
Substituting for x gives $f(1) = 3$ and $f(2) = -2$; there is a change of sign between $x = 1$ and $x = 2$. . The function is continuous, therefore there is a root between 1 and 2.

Example (1a): The function $f(x) = \dfrac{1}{1-x}$ takes a value of 1 when $x = 0$, and -1 when $x = 2$.

There is a sign change in the interval $0 < x < 2$, but it does not signify a root here !

This is because the interval in question includes a discontinuity at $x = 1$.

In fact the equation $f(x) = 0$ has no solution.



<u>The Decimal Search.</u>

This is a 'trial and improvement' method used to trap the root of an equation, familiar from GCSE.

Example (2): Find the particular root of the equation from Example (1), correct to 2 decimal places.

Using $f(x) = x^3 - 4x^2 + 6 = 0$ in Example (1), we know that $f(x)$ takes a zero value between 1 and 2.

We therefore work out $f(1.1)$, $f(1.2)$ .... up to $f(1.9)$ and find where the sign changes. (Spreadsheet programs and graphic calculators are a great help here !)

x	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
$f(x)$	2.491	1.968	1.437	0.904	0.375	-0.144	*-0.647*	*-1.128*	*-1.581*

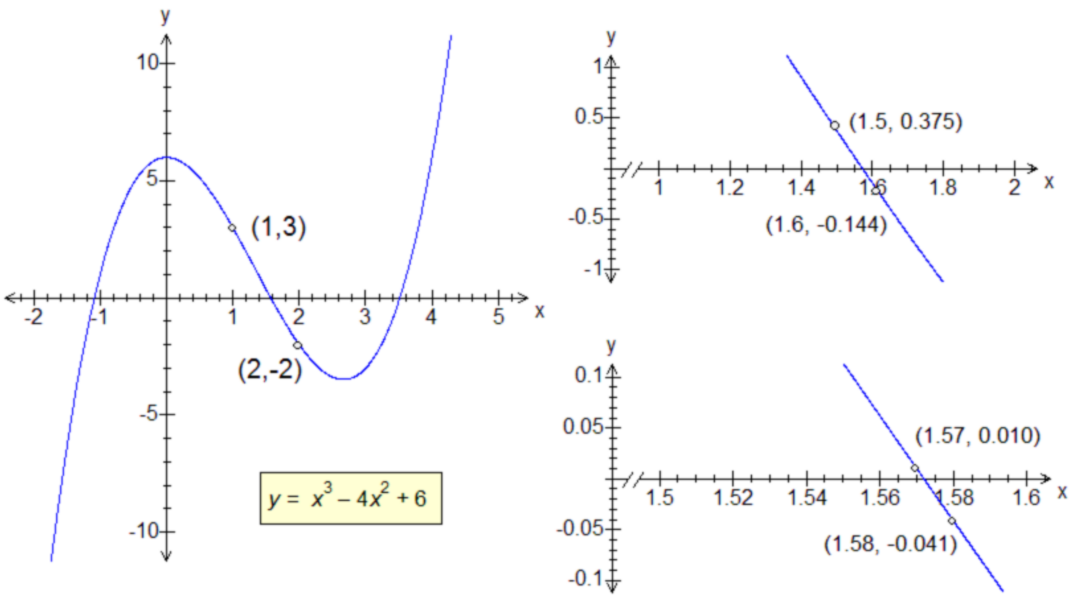
The change of sign is now known to occur between when x is between 1.5 and 1.6. (Values in italics need not have been calculated – we could have stopped at the change of sign)

We therefore repeat the previous step, calculating $f(1.51)$, $f(1.52)$ .... up to $f(1.59)$.

x	1.51	1.52	1.53	1.54	1.55	1.56	1.57	1.58	1.59
$f(x)$	0.323	0.270	0.218	0.166	0.114	0.062	0.010	-0.041	*-0.093*

The equation changes sign between 1.57 and 1.58, but we still need to find which of those two values is closer to the root, correct to two decimal places.

We therefore take the value of the function at the midpoint of those two values, namely $f(1.575)$. This value is –0.016, which means that the change of sign occurs between $x = 1.57$ and $x = 1.575$. The correct value of the root to 2 decimal places is therefore **1.57** and not 1.58.



Formula iteration.

With this method, we start with an approximation to a root and improve its accuracy by substituting into a formula. We can then repeat this process until we have the desired level of accuracy.

To create a simple iterative formula, we rearrange the equation so that *x* is expressed as a function of itself. From there, we can turn it into an iterative formula. The example below will be given in considerable illustrated detail, but there is no need to memorise it for examination questions.

Example (3): Solve the equation $x^2 - x - 1 = 0$ by creating an iteration formula, and use 1 and –0.5 as the starting values for *x*. Stop the iteration when the result has converged to 4 decimal places.

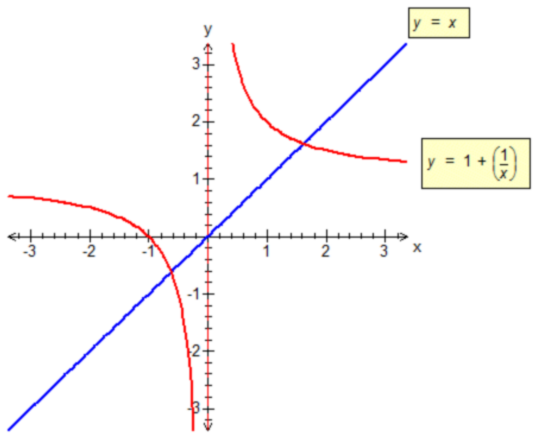
(Although this is a quadratic, which can be solved using the general formula, we are choosing this example to illustrate the method. Its solutions are 1.618 and –0.618 to 3 decimal places.)

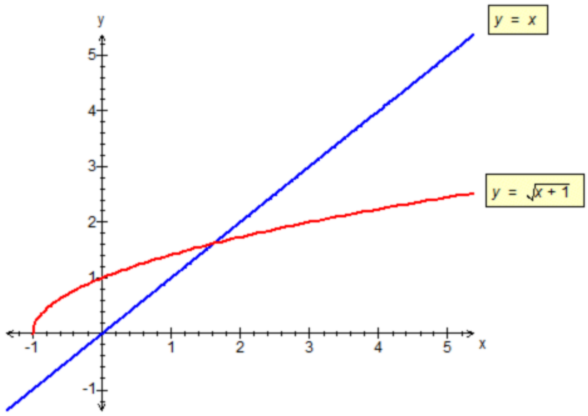
The equation can be rearranged in several ways.

One way is to re-express it (dividing by *x*) as
$x - 1 - \frac{1}{x} = 0 \Rightarrow x = 1 + \frac{1}{x}$.
This can be turned into an iterative formula by subscripting *x* on each side as follows:

$$x_{n+1} = 1 + \frac{1}{x_n}$$ **(Formula 3a)**

The roots of the original equation correspond to the *x*-coordinates of the points of intersection between the line $y = x$ and the curve $y = 1 + \frac{1}{x}$.

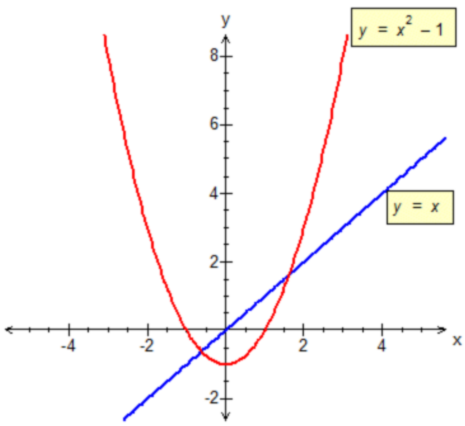




Another rearrangement is $x^2 = x + 1 \Rightarrow$
$x = \sqrt{(x+1)}$.
The corresponding iterative formula is

$$x_{n+1} = \sqrt{(x_n + 1)}$$ **(Formula 3b)**

The positive root of the original equation corresponds to the *x*- coordinate of the point of intersection between the line $y = x$ and the curve
$y = \sqrt{(x+1)}$.



A third one is to re-express it as $x = x^2 - 1$.
The derived iterative formula is

$$x_{n+1} = x_n^{\ 2} - 1$$ **(Formula 3c)**

The roots of the original equation correspond to the *x*-coordinates of the points of intersection between the line $y = x$ and the curve $y = x^2 - 1$.

Using formula (3a): $x_{n+1} = 1 + \frac{1}{x_n}$

(Calculator hint for most makes: Key in 1, = , and then 1, +, (, 1, ÷, Ans ,) , = to produce each term in the sequence).

n	x_n	**n**	x_n	**n**	x_n
0	1	**5**	1.625	**10**	1.6180
1	2	**6**	1.6154	**11**	1.6181
2	1.5	**7**	1.6190	**12**	1.6180
3	1.6667	**8**	1.6176	**13**	1.6180
4	1.6	**9**	1.6182		

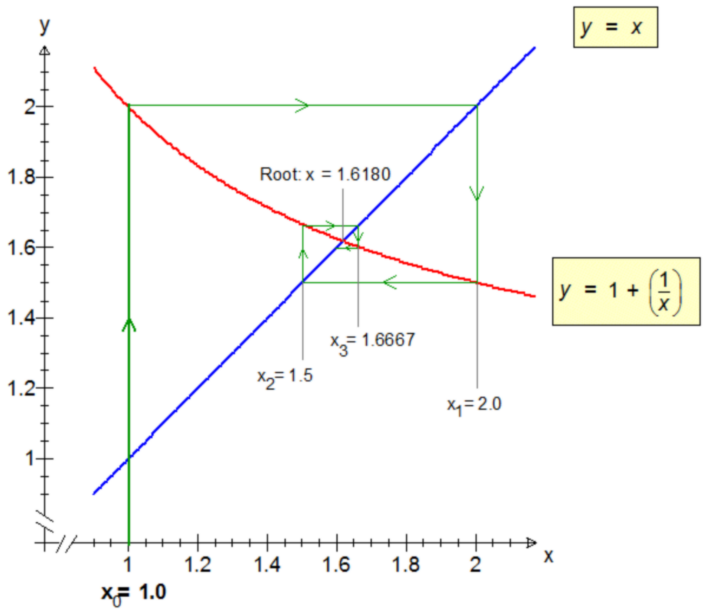
This converges to the positive root when choosing a starting value of 1.

This can be shown diagrammatically: Start from point (1, 0) on the *x*-axis (corresponding to the starting *x*).

Move up to the curve, and then across to the line $y = x$.
This gives the next value of *x*, i.e. 2.

Move down to the curve, and across to the line $y = x$ again.
This gives the next *x*-value , i.e. 1.5.

Repeating this process creates a 'cobweb' diagram as each *x*-value oscillates around the root (1.6180 to 4 dp) in ever-smaller intervals.



When trying to find the negative root, choosing a starting value of 0 or -1 causes the iterative process to 'fall over' – one of several things that can go wrong with a formula iteration.

n	x_n
0	-1
1	0
2	Falls over - division by zero !

Different starting values also lead to wildly varying results:

n	x_n
0	-0.6
1	-0.6667
2	-0.5
3	-1
4	0
5	Falls over !

n	x_n	n	x_n
0	-0.7	6	1.8333
1	-0.4286	7	1.5455
2	-1.3333	8	1.6471
3	0.25	9	1.6071
4	5	10	1.6222
5	1.2	11	1.6164

A starting value of -0.6 causes the iteration to fall over, and a starting value of -0.7 ends up converging to the positive root of the equation.

Using formula (3b):

$$x_{n+1} = \sqrt{(x_n + 1)}$$

(Calculator hint for most makes: Key in 1, = , and then √ (Ans + 1) =   to produce each term in the sequence).

n	x_n	**n**	x_n
0	1	**5**	1.6161
1	1.4142	**6**	1.6174
2	1.5538	**7**	1.6179
3	1.5981	**8**	1.6180
4	1.6118		

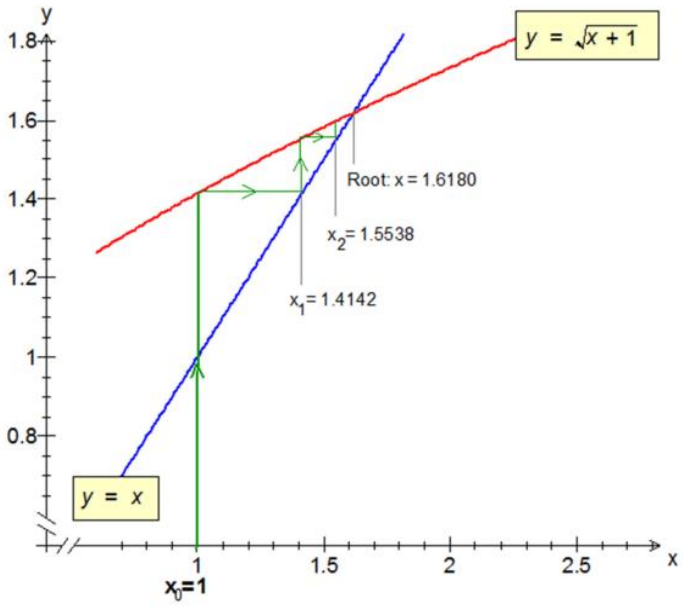
This also converges to the positive root when choosing a starting value of 1.

Start from point (1, 0) on the *x*-axis (corresponding to the starting *x*).

Move up to the curve, and then across to the line $y = x$ to give the next value of *x*, or 1.4142 to 4 d.p.

Move up to the curve, and across to the line $y = x$ again.
This gives the next *x*-value, 1.5538 to 4 d.p.

Repeating this process creates a 'staircase' diagram as each *x*-value approaches the root (1.6180 to 4 dp) more closely. This is a different pattern from the 'cobweb' from formula (3a).



If we tried to find the other root (between 0 and –1) , there would be a problem:

n	x_n
0	-0.5
1	0.7071
2	1.3066
3	1.5187
4	1.5871
5	1.6084

This looks to be converging to the positive root as well.

Using formula (3c): $x_{n+1} = x_n^{\ 2} - 1$

This formula will be found to be unsatisfactory for iteration, but will highlight several of the problems that might be encountered when choosing an inappropriate formula.

We will try and find the positive root of the equation by starting with $x_0 = 1$.

(Calculator hint for most makes: Key in 1, =, and then Ans, x^2 , - , 1 keys., followed by repeated presses of the = key to produce each term in the sequence easily and rapidly).

n	x_n
0	1
1	0
2	-1
3	0
4	-1

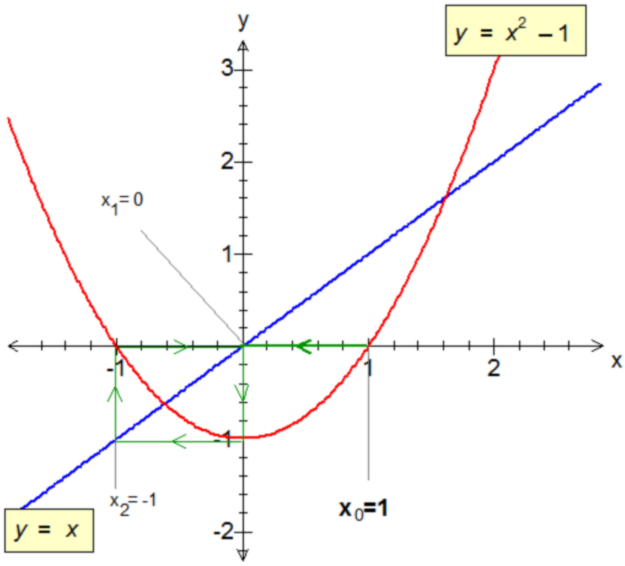
By choosing $x_0 = 1$, the formula appears to end up oscillating between 0 and -1 without giving a root.

Start from point (1, 0) on the *x*-axis, we are already on the curve, so we move across to the line $y = x$ to give the next value of *x*, namely x_1 or 0.

Moving down to the curve, and across to the line $y = x$ again, gives the next value of *x*, x_2 or –1.

Moving up to the curve, and across to the line $y = x$ again, gives the next value of *x*, x_3 or 0.

Moving down to the curve, and across to the line $y = x$ again, gives the next value of *x*, x_4 or –1.



The iteration has become stuck in an infinite loop !

Trying to find the positive root using $x_0 = 2$ is also no help !

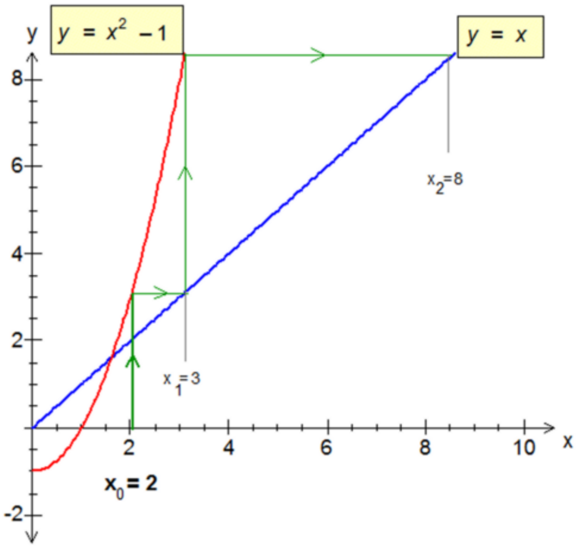
n	x_n
0	2
1	3
2	8
3	63
4	3968

The iteration 'blows up' by diverging further and further from the solution !

We start from point (2, 0) on the *x*-axis, so we move up to the curve and then across to the line $y = x$ to give the next value of *x*, namely x_1 or 3.

Moving up to the curve, and across to the line $y = x$ again, gives the next value of *x*, x_2 or 8.

It can clearly be seen that we have a 'diverging staircase' scenario, where the subsequent values of *x* become indefinitely large.



Finally we try and find the negative root using a value other than 0 or –1, say $x_0 = -0.5$.

We already know that using a starting value of 0 or –1 would lead us into an infinite loop, so we try a different starting value.

n	x_n	**n**	x_n
0	-0.5	**5**	-0.8802
1	-0.75	**6**	-0.2253
2	-0.4375	**7**	-0.9492
3	-0.8086	**8**	-0.0990
4	-0.3462	**9**	-0-9902

Start from point (-0.5, 0) on the *x*-axis (corresponding to x_0).

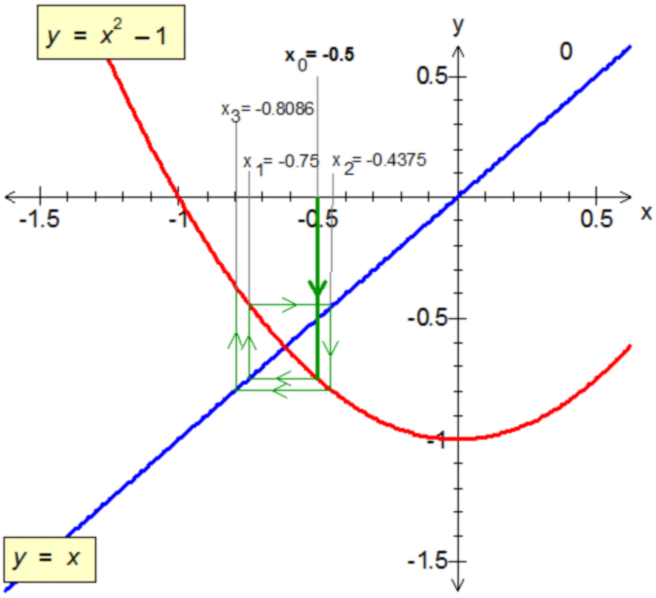
Move down to the curve, and then across to the line $y = x$.
This gives x_1, or –0.75.

Move up to the curve, and across to the line $y = x$ again.
This gives x_2 , i.e. –0.4375.

Move down to the curve, and then across to the line $y = x$.
This gives x_3, or –0.8086.

We can see that, although we have a 'cobweb' diagram here, successive iterations extend *outwards* away from the root, so that the process will not converge.

(In fact, it will end up oscillating between 0
and –1).



Some examination questions give a suggested formula for you.

Example (4): Verify that the formula $x_{n+1} = \frac{x_n^{\ 2} + 1}{2x_n - 1}$ is a rearrangement of $x^2 - x - 1 = 0$.

Substitute $x_0 = 2$ and -1, and hence find the roots of the equation to 6 decimal places. What do you notice about the convergence ?

Removing the subscripts and rearranging the iteration formula we have $x = \frac{x^2 + 1}{2x - 1}$

$\Rightarrow x(2x-1) = x^2 + 1 \Rightarrow 2x^2 - x = x^2 + 1 \Rightarrow 2x^2 - x - x^2 - 1 = 0 \Rightarrow \mathbf{x^2 - x - 1 = 0}$.

Key in 2= and then (Ans x^2 + 1) ÷ ((2 × Ans) – 1) = = =

n	x_n
0	2
1	1.666667
2	1.619048
3	1.618034
4	1.618034

n	x_n
0	-1
1	-0.666667
2	-0.619048
3	-0.618034
4	-0.618034

This rearrangement converges far more rapidly than the other two, giving 6-decimal place values of 1.618034 and –0.618034.

Example (5): Find the roots of $e^x - 5x - 4 = 0$ by choosing suitable iteration formulae, to 4 decimal places. Use 3 and –1 as the starting values.

Rearrange as $e^x = 5x + 4$, and by taking logs of both sides, we have $x = \ln(5x + 4)$
and hence $x_{n+1} = \ln(5x_n + 4)$

n	x_n	*n*	x_n
0	3	**5**	2.9244
1	2.9444	**6**	2.9244
2	2.9297	**7**	2.9243
3	2.9258	**8**	2.9243
4	2.9247	**9**	2.9243

This rearrangement falls over at once when starting with x = -1, as the first step would attempt to find ln (–1).

We can also rearrange the original equation as

$5x = e^x - 4 \Rightarrow x = \frac{e^x - 4}{5}$ and thus $x_{n+1} = \frac{e^{x_n} - 4}{5}$.

The first root was found with a starting value of 3, so try finding the root using a start value of -1:

n	x_n	*n*	x_n
0	-1	**5**	-0.7008
1	-0.7264	**6**	-0.7008
2	-0.7033		
3	-0.7010		
4	-0.7008		

∴ the roots of $e^x - 5x - 4 = 0$ are 2.9243 and –0.7008 to 4 decimal places.

Example (6) : i) Show that the equation $9x^2 - \tan^{-1} x = 0$ where x is in radians and non-zero, produces the iteration formula $x_{n+1} = \frac{1}{3}\sqrt{\tan^{-1}(x_n)}$.

ii) Use a starting value of $x = 0.1$ to solve the equation $9x^2 - \tan^{-1} x = 0$ correct to 3 decimal places.

i) We rearrange the original equation as $9x^2 = \tan^{-1} x$, and taking square roots of both sides,

$$3x = \sqrt{\tan^{-1}(x)} \Rightarrow x = \frac{1}{3}\sqrt{\tan^{-1}(x)}$$

The final iteration formula is $x_{n+1} = \frac{1}{3}\sqrt{\tan^{-1}(x_n)}$.

The iteration results are :

n	x_n	***n***	x_n
0	0.1	**5**	0.1103
1	0.1052	**6**	0.1105
2	0.1079	**7**	0.1106
3	0.1093	**8**	0.1106
4	0.1100	**9**	0.1106

The non-zero solution of $9x^2 - \tan^{-1} x = 0$ is 0.111 to 3 decimal places.

Example (7): Two game shows have a similar format where the prize money increases on a daily basis and remains unclaimed until the champions are beaten by the challengers, who then win the prize money.

On *Smartheads* , the prize money starts at £1000 on day 1 and increases by £1000 per day.
In other words, it forms an arithmetic series with starting value £1000 and common difference £1000.

$\therefore$ The prize money sequence (in £) goes 1000, 2000, 3000 ..... and the general term is 1000*t.*

On *Brainboxes*, the prize money starts at £2000 on day 1 and increases by 20% per day.
This time, it forms a geometric series with starting value £2000 and common ratio 1.2.

$\therefore$ The prize money sequence (in £) goes 2000, 2400, 2880 ..... and the general term is $2000(1.2^{t-1})$

After 3 days, the payout on *Smartheads* is higher, but the payout on *Brainboxes* regains the lead several days later.

i) Show that the time in days, *t*, when the prize money payouts on *Smartheads* and *Brainboxes* are equal again, can be represented by the iterative formula $t_{n+1} = \frac{\ln(0.6t_n)}{\ln 1.2}$.

ii) Find the value of *t* using $t_0 = 5$, and hence state on which day the payout on *Brainboxes* exceeds that on *Smartheads* again.

i) The payouts on both shows are equal when $2000(1.2)^{t-1} = 1000t \Rightarrow 2(1.2)^{t-1} = t$

$\Rightarrow \frac{2(1.2)^t}{1.2} = t$ (multiplying top and bottom by 1.2 to get rid of the $t - 1$ index)

$\Rightarrow \frac{(1.2)^t}{0.6} = t \Rightarrow (1.2)^t = 0.6t \Rightarrow t\ln 1.2 = \ln(0.6t) \Rightarrow t = \frac{\ln(0.6t)}{\ln 1.2}$, and hence the required iterative formula is $t_{n+1} = \frac{\ln(0.6t_n)}{\ln 1.2}$.

Iteration results :

n	t_n	***n***	t_n
0	5	**5**	8.96
1	6.03	**6**	9.23
2	7.05	**7**	9.39
3	7.91	**8**	9.48
4	8.54	**Limit**	9.61

The prize payout in *Brainboxes* equals that in *Smartheads* on 'Day 9.61', so therefore it exceeds it by Day 10.

(Check: *Smartheads* payout for Day 10: £10,000;
Brainboxes payout for Day 10: £(2000×1.2^9) = £ 10,320 to nearest £).

Checking the suitability of an iteration formula.

Sometimes, an iterative formula might be unsatisfactory because a) convergence is too slow, b) the iteration becomes 'stuck' in an infinite loop, c) the iteration 'blows up' or d) the iteration 'falls over'.

The full study of iterative formulae requires difficult analysis beyond the scope of A-level, and therefore an examination question would include a 'ready-tweaked' formula, as in the next examples.

Example (8): Use iterative formulae to find the two positive roots of $f(x) = x^3 - 4x^2 + 6 = 0$.

(One of them has already been found to 2 decimal places using decimal searching, in Example (2)).

Find both to 4 decimal places, and use $x_0 = 2$ in each case.

One arrangement is

$x^3 - 4x^2 + 6 = 0 \Rightarrow x - 4 + \dfrac{6}{x^2} = 0 \Rightarrow x = 4 - \dfrac{6}{x^2}$, to give an iterative formula of

$$x_{n+1} = 4 - \frac{6}{x_n^{\ 2}}.$$

n	x_n	*n*	x_n	*n*	x_n
0	2	**5**	3.5004	**10**	3.5141
1	2.5	**6**	3.5103		
2	3.04	**7**	3.5131		
3	3.3508	**8**	3.5138		
4	3.4656	**9**	3.5141		

This iteration formula has converged to the root between 3 and 4, and its value is 3.5141 to 4 decimal places.

Another arrangement is

$x^3 - 4x^2 + 6 = 0 \Rightarrow 4x^2 = x^3 + 6 \Rightarrow 2x = \sqrt{x^3 + 6} \Rightarrow x = \frac{1}{2}\sqrt{x^3 + 6}$

to give an iterative formula of $x_{n+1} = \frac{1}{2}\sqrt{x_n^{\ 3} + 6}$.

n	x_n	*n*	x_n	*n*	x_n
0	2	**5**	1.6206	**21**	1.5720
1	1.8708	**...**	....		
2	1.7712	**10**	1.5756		
3	1.6997	**...**	....		
4	1.6516....	**20**	1.5720		

This iteration formula converges to the root between 1 and 2, and its value is 1.5720 to 4 decimal places, but the convergence is a bit slow.

Also, neither formula is of use when trying to find the negative root.

The next two examples show how a 'tweaked' formula can be used for better convergence.

Example(9): i) Show that the iterative formula $x_{n+1} = \frac{1}{10}\left(4 - \frac{6}{{x_n}^2} + 9x_n\right)$ can be obtained by rearranging the equation $x^3 - 4x^2 + 6 = 0$.

ii) Hence, find the negative root of $x^3 - 4x^2 + 6 = 0$, using a starting value of -1.

i) We begin by writing the formula as $x = \frac{1}{10}\left(4 - \frac{6}{x^2} + 9x\right) \Rightarrow 10x = 4 - \frac{6}{x^2} + 9x$

$$\Rightarrow x = 4 - \frac{6}{x^2} \Rightarrow x^3 = 4x^2 - 6 \Rightarrow x^3 - 4x^2 + 6 = 0$$

ii) The results of the formula iteration are as follows:

n	x_n
0	-1
1	-1.1
2	-1.0859
3	-1.0861
4	-1.0861

The negative root of $x^3 - 4x^2 + 6 = 0$ = -1.0861 to 4 decimal places.

Example(10): i) Show that the iterative formula $x_{n+1} = \frac{1}{2}\left(\frac{6}{{x_n}^2} + 3x_n - 4\right)$ can be obtained by rearranging the equation $x^3 - 4x^2 + 6 = 0$.

ii) Hence, find the root of $x^3 - 4x^2 + 6 = 0$ lying between 1 and 2, using a starting value of 2. How does this compare to the result from Example (8) ?

In Example (8), we used a different iterative formula and obtained a convergence to 1.5720, but the process was rather slow, taking 20 iterations to achieve accuracy to 4 decimal places.

i) We begin with $x = \frac{1}{2}\left(\frac{6}{x^2} + 3x - 4\right) \Rightarrow 2x = \frac{6}{x^2} + 3x - 4$

$$\Rightarrow -x = \frac{6}{x^2} - 4 \Rightarrow x = 4 - \frac{6}{x^2} \Rightarrow x^3 = 4x^2 - 6 \Rightarrow x^3 - 4x^2 + 6 = 0$$

ii) The results of the formula iteration are:

n	x_n
0	2
1	1.75
2	1.6046
3	1.5721
4	1.5720
5	1.5720

This iterative formula converges to the root of 1.5720 much more rapidly than the iterative formula of $x_{n+1} = \frac{1}{2}\sqrt{{x_n}^3 + 6}$ from Example (8), which took over 20 iterations.

Example (11): (Omnibus exam-style question)

i) A function is defined as $f(x) = x - \sin x - 1$. (**Note:** x is measured in radians !)
Show that the equation $f(x) = 0$ has a solution in the range $1.9 < x < 2.0$.

ii) Use a decimal search method to find the solution of $f(x) = 0$ to two decimal places.

iii) Form an iteration formula from the equation $f(x) = 0$.

iv) Taking the result from part ii) as the starting value x_0 for the iteration, use this formula to find the values of x_1, x_2 and x_3.

v) Hence find the solution of the equation $f(x) = 0$ to four decimal places.

i) Substituting $x = 1.9$ into the equation gives $1.9 - \sin(1.9^c) - 1 = -0.046$.
Substituting $x = 2.0$ into the equation gives $2.0 - \sin(2.0^c) - 1 = 0.091$.

Given that $f(x) = x - \sin x - 1$ is a continuous function, the change of sign in the value of the function between $x = 1.9$ and $x = 2.0$ shows that there is a root in the interval, probably closer to 1.9.

ii) We calculate $f(1.91)$, $f(1.92)$ .... until the sign of $f(x)$ changes.

x	1.91	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99
$f(x)$	-0.033	-0.020	-0.006	0.007	no need	no need	no need	no need	no need

The equation $f(x) = 0$ changes sign between $x = 1.93$ and $x = 1.94$.

The midpoint of those two values is 1.935, and $f(1.935) = 0.001$. The change of sign occurs between $x = 1.93$ and $x = 1.935$. The correct value of the root to 2 decimal places is therefore **1.93**.

iii) The expression $x - \sin x - 1$ can be rewritten as $x = \sin x + 1$, from which the iteration formula of $x_{n+1} = \sin x_n + 1$ can be derived.

iv) Using the iteration formula of $x_{n+1} = \sin x_n + 1$, and a starting value of $x_0 = 1.93$,
the next three iterations give:

$x_1 = 1.9362$ $x_2 = 1.9340$ $x_3 = 1.9348$.

v) Further iterations give:

$x_4 = 1.9345$ $x_5 = 1.9346$ $x_6 = 1.9346$.

The iteration converges to the root of $f(x) = x - \sin x - 1 = 0$, or 1.9346 to four decimal places.

The Bisection (Midpoint) method. (Skip this if not on your syllabus)

This method is less hit-miss than formula iteration, but again is fairly slow, though slightly faster than the decimal search. This time we start with two values on each side of the root and keep halving the desired interval by 'chopping out' the irrelevant half.

Example (12) : The equation $f(x) = x^3 + 4x - 12 = 0$ has a root between 1 and 2. Use the bisection method to find its value correct to 1 decimal place.

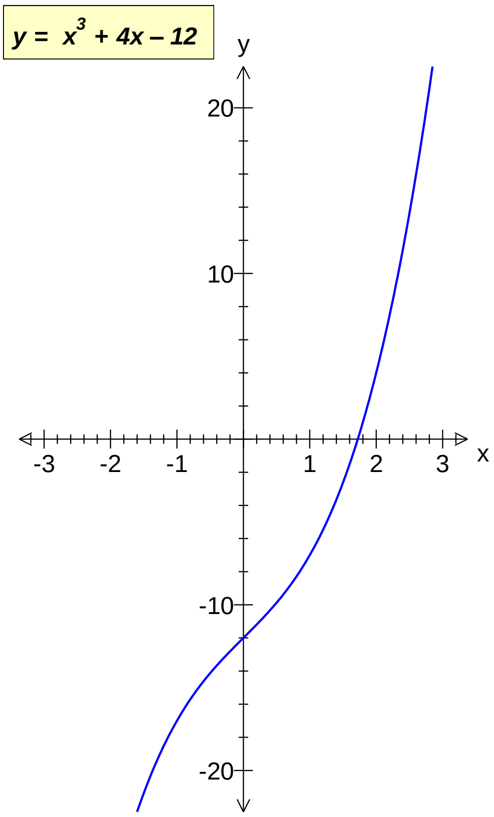
We begin by working out $f(x)$ for the end values of x. Here $f(1) = -7$ and $f(2) = 4$, so $f(x)$ changes sign from negative to positive between those two values.

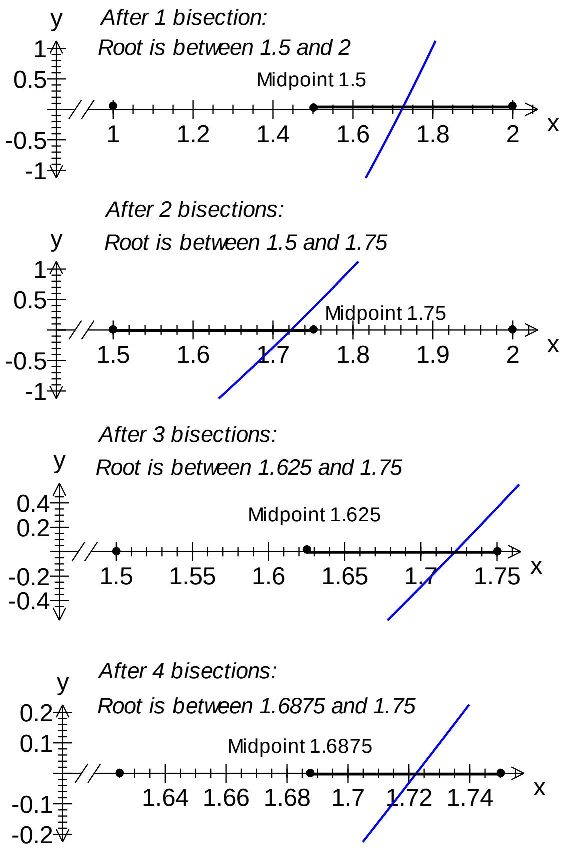
We therefore proceed to halve the interval by finding out $f(1.5)$, that being the mean of the two end values. The value of $f(1.5) = -2.625$, and so the function changes sign between 1.5 and 2, and we can discard the interval $1 < x < 1.5$, and keep the interval $1.5 < x < 2$.

$\therefore$ After one bisection, we have narrowed down the possible range of values for the root – it is now in the interval $1.5 < x < 2$.

We can continue the process by taking the midpoint of this reduced interval, namely 1.75, and then finding out $f(1.75)$. This works out at 0.359, so $f(x)$ changes sign in the interval $1.5 < x < 1.75$. We therefore chop out the irrelevant interval $1.75 < x < 2$.

$\therefore$ After two bisections, the root has been found to lie in the interval $1.5 < x < 1.75$.





We then continue the process until there is no uncertainty about the first decimal place.
(Third bisection) The midpoint of the interval $1.5 < x < 1.75$ is 1.625. Now $f(1.625) = -1.201$, and so $f(x)$ changes sign in the interval $1.625 < x < 1.75$.

(Fourth bisection) The midpoint of the interval 1.625 < x < 1.75 is 1.6875. Now f(1.6875) = -0444, and so $f(x)$ changes sign in the interval 1.6875 < x < 1.75.
Both end values in the interval are 1.7 to one decimal place, so we have obtained the required accuracy after four bisections.

$\therefore$ The root of $f(x) = x^3 + 4x - 12 = 0$ = 1.7 to one decimal place.

The working can be seen more clearly when the results are tabulated:

Interval	f(lower bound)	f(upper bound)	Midpoint of interval	f(midpoint)	Sign change in interval
1 < x < 2	-7	4	1.5	-2.625	1.5 < x < 2
1.5 < x < 2	-2.625	4	1.75	0.359	1.5 < x < 1.75
1.5 < x < 1.75	-2.625	0.359	1.625	-1.209	1.625 < x < 1.75
1.625 < x < 1.75	-1.209	0.359	1.6875	-0.444	1.6875 < x < 1.75

Example (12a) : Continue the previous example to find the root of the equation $f(x) = x^3 + 4x - 12 = 0$ correct to 2 decimal places.

We left off the last example with the root in the range 1.6875 < x < 1.75, thus obtaining a value of 1.7 to 1 decimal place.

We continue taking the midpoints as before until the estimate is correct to 2 decimal places. (Previous workings in italics).

Interval	f (lower bound)	f (upper bound)	Midpoint of interval	f (midpoint)	Sign change in interval
1 < x < 2	*-7*	*4*	*1.5*	*-2.625*	*1.5 < x < 2*
1.5 < x < 2	*-2.625*	*4*	*1.75*	*0.359*	*1.5 < x < 1.75*
1.5 < x < 1.75	*-2.625*	*0.359*	*1.625*	*-1.209*	*1.625 < x < 1.75*
1.625 < x < 1.75	*-1.209*	*0.359*	*1.6875*	*-0.444*	*1.6875 < x < 1.75*
1.6875 < x < 1.75	-0.444	0.359	1.7188	-0.048	1.7188 < x < 1.75
1.7188 < x < 1.75	-0.048	0.359	1.7344	0.155	1.7188 < x < 1.7344
1.7188 < x < 1.7344	-0.048	0.155	1.7266	0.053	1.7188 < x < 1.7266
1.7188 < x < 1.7266	-0.048	0.053	1.7227	0.003	1.7188 < x < 1.7227

The root lies in the interval 1.7188 < x < 1.7227, and since each end value = 1.72 to 2 decimal places, the solution of $f(x) = x^3 + 4x - 12 = 0$ is x = 1.72 to 2 d.p.

The bisection method is a slight improvement on the decimal search, but is still slow, and eventually the fractions become tedious to input in full (hence the approximations to 4 decimal places).

The Newton-Raphson method. (Skip this if not on your syllabus)

This is another method of finding roots of an equation.
Take the equation $f(x) = 0$ with an unknown root α, in other words $f(\alpha) = 0$.

If x_0 is a reasonable approximation to α, then an improved one is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} .$$

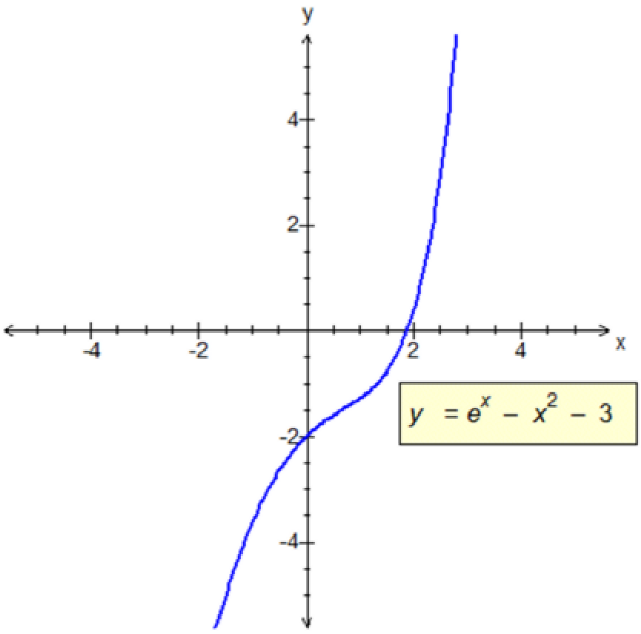
The method usually converges quickly to the required root, but it might fail in exceptional cases, such as when $f'(x)$ is close to zero or if there are discontinuities in the curve.

(On the occasions when the method fails, a better first approximation will usually give convergence.)

Example (13): Find the root of $e^x - x^2 - 3 = 0$, using a starting value, x_0, of 2. Use three iterations of the Newton-Raphson method and give the result to 4 decimal places.

This equation has a root fairly close to 2, as the graph shows.

We need to differentiate the function to be able to find the root.



$f(x) = e^x - x^2 - 3$
$f'(x) = e^x - 2x$

Substituting for $x_0 = 2$, we have

$$x_1 = 2 - \frac{f(2)}{f'(2)} \Rightarrow x_1 = 2 - \frac{e^2 - 7}{e^2 - 4} \Rightarrow x_1 = 2 - \frac{0.389056}{3.389056} \Rightarrow x_1 = 1.885202.$$

The diagram illustrates the method geometrically. By continuing the tangent to the curve until it meets the *x*-axis, we will have an improved estimate of the actual root – here it is $x_1 = 1.8852$ to 4 d.p.

If we draw another tangent to the curve, this time at $(x_1, f(x_1))$, then that tangent will meet the *x*-axis even closer to the root.

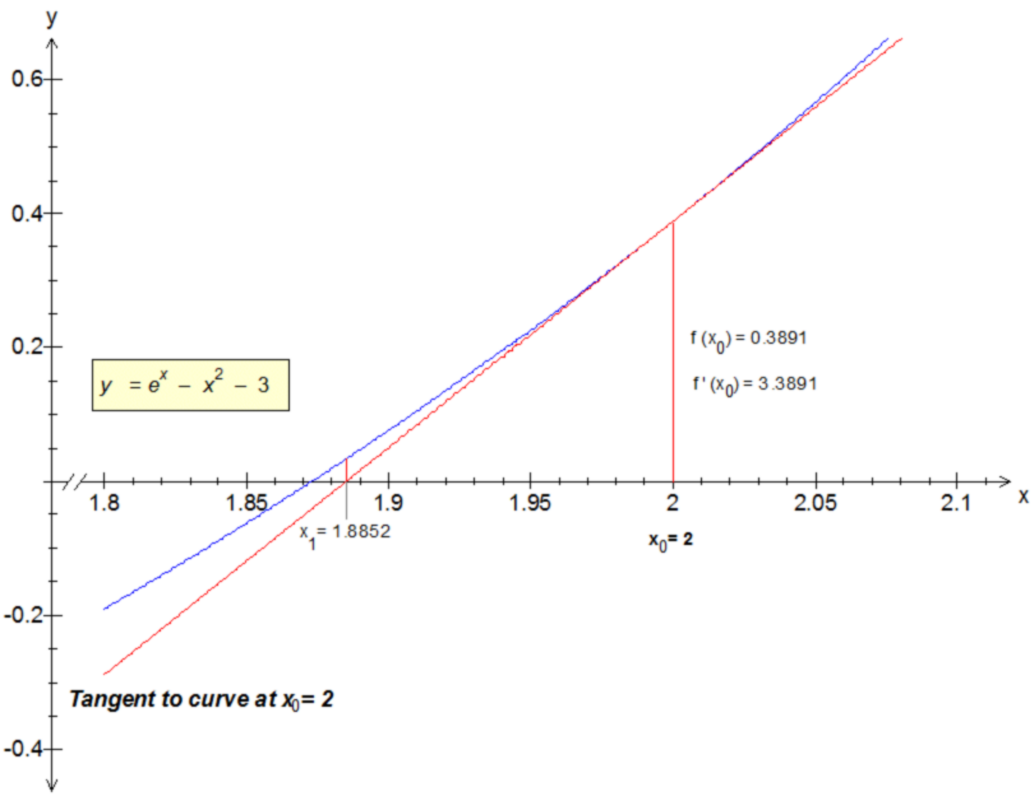
Repeated iterations give

$$x_2 = 1.885202 - \frac{f(1.885202)}{f'(1.885202)} \Rightarrow x_2 = 1.885202 - \frac{0.033699}{2.817282} \Rightarrow x_2 = 1.873241.$$

$$x_3 = 1.873241 - \frac{f(1.873241)}{f'(1.873241)} \Rightarrow x_3 = 1.873241 - \frac{0.000326}{2.762876} \Rightarrow x_3 = 1.873123.$$

The root of the equation is 1.8731 to 4 decimal places.

(Since $f(1.87305) = -0.0002$ and $f(1.873123) = 0.000001$, there is no danger of the result creeping down to 1.8730).



Example (14): Use the Newton-Raphson method with two iterations to find to 4 decimal places the three roots of the equation $x^3 - 4x^2 + 6 = 0$, used in earlier examples.

Use starting values of x = -1.1, 1.6 and 3.5.

Differentiating $f(x) = x^3 - 4x^2 + 6$, we have $f'(x) = 3x^2 - 8x$.

$\boldsymbol{x_0 = -1.1}$

$$x_1 = -1.1 - \frac{f(-1.1)}{f'(-1.1)} \Rightarrow x_1 = -1.1 - \frac{-0.171}{12.43} \Rightarrow x_1 = -1.086243\,.$$

$$x_2 = -1.086243 - \frac{f(-1.086243)}{f'(-1.086243)} \Rightarrow x_2 = -1.086243 - \frac{-0.00138}{12.22971} \Rightarrow x_2 = -1.086130\,.$$

$\boldsymbol{x_0 = 1.6}$

$$x_1 = 1.6 - \frac{f(1.6)}{f'(1.6)} \Rightarrow x_1 = 1.6 - \frac{-0.144}{-5.12} \Rightarrow x_1 = 1.571875\,.$$

$$x_2 = 1.571875 - \frac{f(1.571875)}{f'(1.571875)} \Rightarrow x_2 = 1.571875 - \frac{0.000611}{-5.16263} \Rightarrow x_2 = 1.571993\,.$$

$\boldsymbol{x_0 = 3.5}$

$$x_1 = 3.5 - \frac{f(3.5)}{f'(3.5)} \Rightarrow x_1 = 3.5 - \frac{-0.125}{8.75} \Rightarrow x_1 = 3.514286\,.$$

$$x_2 = 3.514286 - \frac{f(3.514286)}{f'(3.514286)} \Rightarrow x_2 = 3.514286 - \frac{0.001329}{8.936327} \Rightarrow x_2 = 3.514137\,.$$

$\therefore$ The roots of $x^3 - 4x^2 + 6 = 0$ are -1.0862, 1.5720 and 3.5141 to 4 decimal places.

In actual fact, the values are also correct to six places !

Example (15): The equation of $f(x) = x^3 - x^2 - 8$ has a root between 2 and 3.

i) Use three iterations of the Newton-Raphson method and a starting value, x_0, of 2.5. Give the result to 4 decimal places.

ii) Use a suitable lower bounding value of x to show that the result in part i) is indeed correct to 4 decimal places

i) Differentiating $f(x) = x^3 - x^2 - 8$, we have $f'(x) = 3x^2 - 2x$.

$$x_1 = 2.5 - \frac{f(2.5)}{f'(2.5)} \Rightarrow x_1 = 2.5 - \frac{1.375}{13.75} \Rightarrow x_1 = 2.4.$$

$$x_2 = 2.4 - \frac{f(2.4)}{f'(2.4)} \Rightarrow x_2 = 2.4 - \frac{0.064}{12.48} \Rightarrow x_2 = 2.394872.$$

$$x_3 = 2.394872 - \frac{f(2.394872)}{f'(2.394872)} \Rightarrow x_3 = 2.394872 - \frac{0.000163}{12.416489} \Rightarrow x_3 = 2.394859.$$

The root of $f(x) = x^3 - x^2 - 8 = 0$ is thus 2.3949 to 4 decimal places.

ii) Since $f(2.39485) = -0.0001$ and $f(2.394872) = 0.000163$, there is a change of sign in $f(x)$ between $x = 2.39485$ and $x = 2.394872$.

There is therefore no danger of the result 'converging down' to a value that rounds down to 2.3948.