M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

AQA : **C4** Edexcel: **C3** OCR: **C4** OCR MEI: **C4**

RATIONAL EXPRESSIONS

$$\frac{3x^2+12x}{x^2-x-20}=\frac{3x(x+4)}{(x-5)(x+4)}=\frac{3x}{(x-5)}$$

$$\frac{5x^4-15x^3+10x^2}{10x^4-15x^3+5x^2}=\frac{x^2-3x+2}{2x^2-3x+1}=\frac{(x-2)(x-1)}{(2x-1)(x-1)}=\frac{(x-2)}{(2x-1)}$$

$$\frac{2x^2-18}{3-x}=\frac{2(x^2-9)}{3-x}=\frac{2(x+3)(x-3)}{3-x}=\frac{-2(x+3)(3-x)}{3-x}=-2(x+3)$$

$$\frac{3}{x-4}+\frac{1}{x+3}=\frac{3(x+3)}{(x-4)(x+3)}+\frac{1(x-4)}{(x-4)(x+3)}=\frac{4x+5}{(x-4)(x+3)}$$

$$\frac{2}{(x+1)(x+4)}+\frac{3}{(x-4)(x+4)}=\frac{2(x-4)+3(x+1)}{(x+1)(x+4)(x-4)}=\frac{5(x-1)}{(x+1)(x+4)(x-4)}$$

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Rational Expressions.

A **rational expression** is an algebraic fraction of the form $\frac{P(x)}{Q(x)}$

where $P(x)$ and $Q(x)$ are polynomials in x.

It is often possible to simplify a rational expression by factorising $P(x)$ and $Q(x)$ wherever possible, and cancelling any factors appearing in both the numerator and denominator.

Example (1): Simplify $\frac{x^2-16}{x+4}$.

Factorising the numerator gives $\frac{x^2-16}{x+4} = \frac{(x+4)(x-4)}{x+4} = x-4$.

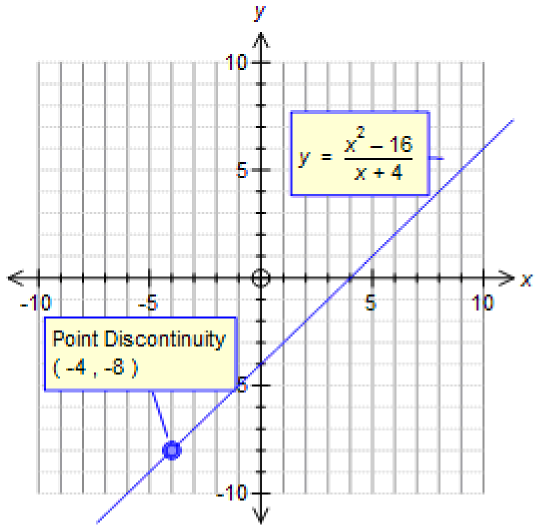
Although we have simplified the expression, the functions $f(x) = \frac{x^2-16}{x+4}$ and $g(x) = x-4$ are not equivalent !

$g(-4) = -8$, but $f(-4)$ is undefined, due to division by zero.

Hence $x = -4$ is in the domain of f but not in the domain of g.

The graph of $y = g(x)$ is the straight line $y = x - 4$.

On the other hand, the graph of $y = f(x)$ (right) is the same line with a gap in it, at (-4, -8) !



These issues are not usually part of exam questions to do with simplifying rational expressions, but they have been included for the sake of awareness.

(Such domain issues are ignored in the rest of the section).

Example (2): Simplify $\dfrac{3x^2 + 12x}{x^2 - x - 20}$.

Factorising both top and bottom of the expression gives

$$\frac{3x^2 + 12x}{x^2 - x - 20} = \frac{3x(x+4)}{(x-5)(x+4)} = \frac{3x}{(x-5)}.$$

This cannot be simplified any further.

Example (3): Simplify $\dfrac{5x^4 - 15x^3 + 10x^2}{10x^4 - 15x^3 + 5x^2}$.

This can be simplified in two stages: first cancel out a factor of $5x^2$, and then factorise.

$$\frac{5x^4 - 15x^3 + 10x^2}{10x^4 - 15x^3 + 5x^2} = \frac{x^2 - 3x + 2}{2x^2 - 3x + 1} = \frac{(x-2)(x-1)}{(2x-1)(x-1)} = \frac{(x-2)}{(2x-1)}$$

Example (4): Simplify $\dfrac{2x^2 - 18}{3 - x}$

This example uses the 'difference of squares' result.

$$\frac{2x^2 - 18}{3 - x} = \frac{2(x^2 - 9)}{3 - x} = \frac{2(x+3)(x-3)}{3 - x} = \frac{-2(x+3)(3-x)}{3 - x} = -2(x+3).$$

Notice the final step of the simplification: we multiplied (x-3) by -1 to obtain (3-x), and brought the minus sign outside the brackets.

Addition and subtraction of algebraic fractions.

Always use the L.C.M. of the denominators when adding and subtracting algebraic fractions. Factorise the denominators when necessary.

Example (5): Express $\frac{3}{x-4}+\frac{1}{x+3}$ as a single fraction.

$$\frac{3}{x-4}+\frac{1}{x+3} = \frac{3(x+3)}{(x-4)(x+3)}+\frac{1(x-4)}{(x-4)(x+3)} = \frac{4x+5}{(x-4)(x+3)}$$

Example (6): Express $\frac{3}{2-x}+\frac{5}{3+x}$ as a single fraction.

$$\frac{3}{2-x}+\frac{5}{3+x} = \frac{3(3+x)}{(2-x)(3+x)}+\frac{5(2-x)}{(2-x)(3+x)} = \frac{9+3x+10-5x}{(2-x)(3+x)} = \frac{19-2x}{(2-x)(3+x)}.$$

Example (7): Express $\frac{2}{x-3}+\frac{3}{2x+1}-\frac{1}{x+4}$ as a single fraction.

$$\frac{2}{x-3}+\frac{3}{2x+1}-\frac{1}{x+4}$$

$$= \frac{2(2x+1)(x+4)}{(x-3)(2x+1)(x+4)} + \frac{3(x-3)(x+4)}{(x-3)(2x+1)(x+4)} - \frac{1(x-3)(2x+1)}{(x-3)(2x+1)(x+4)}.$$

Multiplying and adding out the top line and collecting like terms gives

$$\frac{2(2x^2+9x+4)+3(x^2+x-12)-(2x^2-5x-3)}{(x-3)(2x+1)(x+4)}$$, then

$$\frac{4x^2+18x+8+3x^2+3x-36-2x^2+5x+3}{(x-3)(2x+1)(x+4)}$$

and finally $\frac{5x^2+26x-25}{(x-3)(2x+1)(x+4)}$.

Example (8): Express $\dfrac{4}{x^2+2x-15}-\dfrac{3}{x^2+7x+10}$ as a single fraction.

Here we must factorise the denominators to find the L.C.M. The expression factorises out into

$$\frac{4}{(x-3)(x+5)}-\frac{3}{(x+2)(x+5)}.$$

This gives an L.C.M. of (x+2) (x-3) (x+5). (Note that (x+5) is a common factor of both denominators).

The expression becomes $\dfrac{4(x+2)}{(x+2)(x-3)(x+5)}-\dfrac{3(x-3)}{(x+2)(x-3)(x+5)}$ or

$\dfrac{4x+8-3x+9}{(x+2)(x-3)(x+5)}$ and finally $\dfrac{x+17}{(x+2)(x-3)(x+5)}$.

Example (9): Express $\dfrac{2}{x^2+5x+4}+\dfrac{3}{x^2-16}$ as a single fraction.

Factorising out the denominators we obtain
$\dfrac{2}{(x+1)(x+4)}+\dfrac{3}{(x-4)(x+4)}$, giving an L.C.M. of ($x$+1) ($x$-4) ($x$+4).

(Note that x+4 is a common factor of both denominators).

Multiplying out, we get $\dfrac{2(x-4)+3(x+1)}{(x+1)(x+4)(x-4)}$ and finally $\dfrac{5(x-1)}{(x+1)(x+4)(x-4)}$.

Example (10): Express $\dfrac{2}{x^2+6x+9}-\dfrac{1}{x+3}+\dfrac{3}{x-4}$ as a single fraction.

Since $x^2 + 6x + 9$ factorises out into $(x + 3)^2$, the L.C.M. of the denominators is $(x – 4)(x + 3)^2$.

The required expression is thus $\dfrac{2(x-4)}{(x-4)(x+3)^2}-\dfrac{1(x+3)(x-4)}{(x-4)(x+3)^2}+\dfrac{3(x+3)^2}{(x-4)(x+3)^2}$

$$= \frac{(2x-8)-(x^2-x-12)+(3x^2+18x+27)}{(x-4)(x+3)^2}$$

$$= \frac{2x^2+21x+31}{(x-4)(x+3)^2}.$$