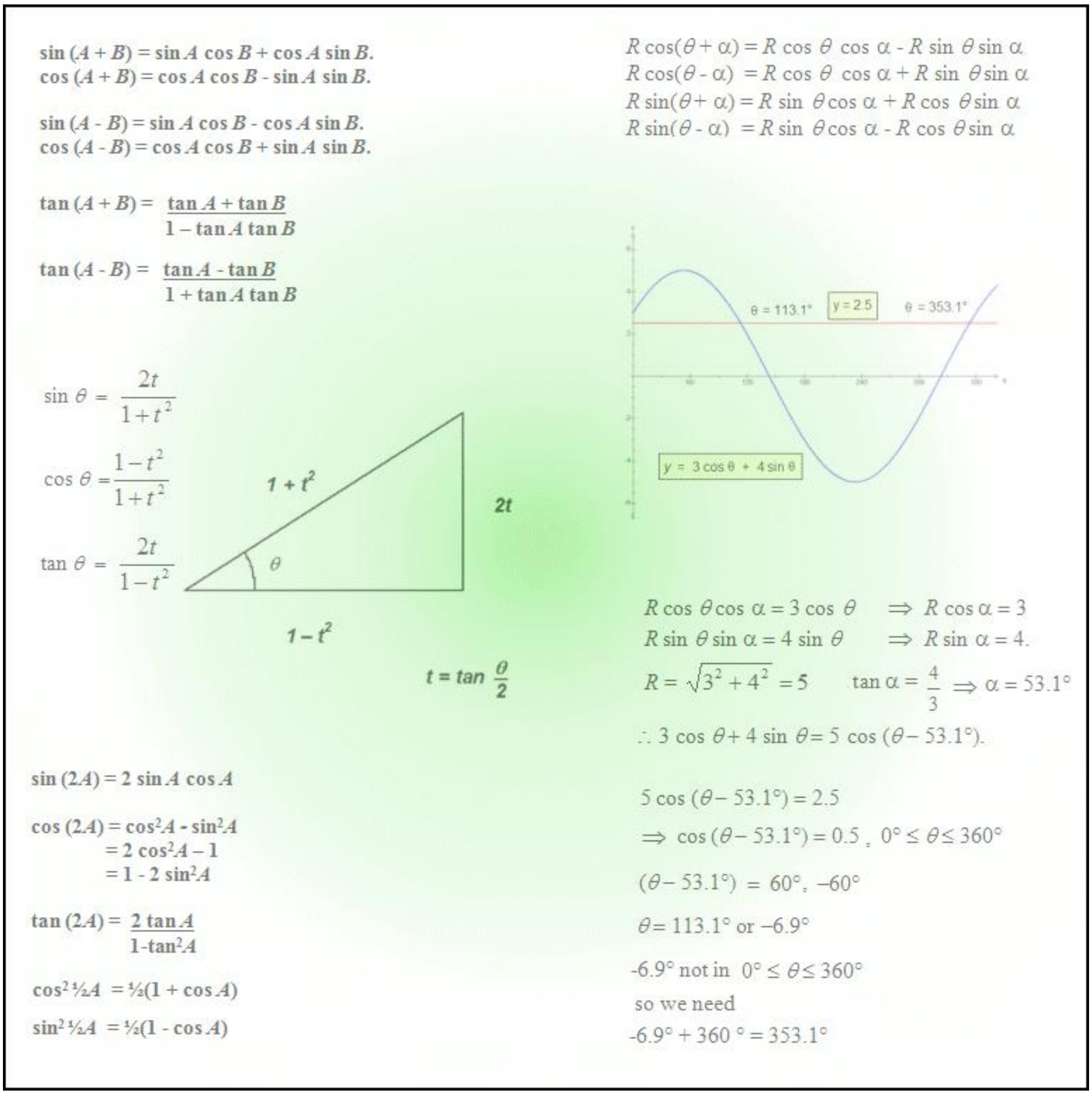
M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

AQA : **C4** Edexcel: **C3** OCR: **C3** OCR MEI: **C4**

FURTHER TRIGONOMETRIC IDENTITIES AND EQUATIONS



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Further Trigonometric identities.

The Pythagorean Identities (revision).

For all angles θ:

$\mathbf{\cos^2 \theta + \sin^2 \theta = 1}$ (This is the Pythagorean identity)

$$\mathbf{\tan \theta = \frac{\sin \theta}{\cos \theta}}$$

From the definitions of the reciprocal functions of

$\frac{1}{\sin x}$ = cosec x (the cosecant of x), sometimes abbreviated to csc x

$\frac{1}{\cos x}$ = sec x (the secant of x)

$\frac{1}{\tan x}$ = cot x (the cotangent of x)

it is possible to obtain several new identities.

Because the cotangent is the reciprocal of the tangent, we have

$$\mathbf{\cot \theta = \frac{\cos \theta}{\sin \theta}}$$

By dividing the Pythagorean identity $\cos^2 \theta + \sin^2 \theta = 1$ throughout by $\cos^2 \theta$, we have
$\mathbf{1 + \tan^2 \theta = \sec^2 \theta}$.

A similar division of the identity $\cos^2 \theta + \sin^2 \theta = 1$ throughout by $\sin^2 \theta$ gives
$\mathbf{\cot^2 \theta + 1 = cosec^2 \theta}$.

Compound Angle Identities.

The following identities hold true for all angles *A* and *B*.

sin (*A* + *B*) = sin *A* cos *B* + cos *A* sin *B*.
cos (*A* + *B*) = cos *A* cos *B* - sin *A* sin *B*.

sin (*A* - *B*) = sin *A* cos *B* - cos *A* sin *B*.
cos (*A* - *B*) = cos *A* cos *B* + sin *A* sin *B*.

The formula for tan (*A* +*B*) can be worked out as

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B} = \frac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

This formidable expression simplifies to

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

The expression for tan (*A* -*B*) can be obtained similarly.

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

In each case the results for (*A*-*B*) can be derived from those for (*A*+*B*) by reversing the '+' and '-' signs – a useful revision hint !

Double and Half Angles.

By taking the compound angle formulae and replacing B with A, we obtain the double angle identities.

$\sin(A + A) = \sin A \cos A + \cos A \sin A = 2 \sin A \cos A.$

$\cos(A + A) = \cos A \cos A - \sin A \sin A = \cos^2 A - \sin^2 A.$

$$\tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2 \tan A}{1 - \tan^2 A}$$

The formula for $\cos 2A$ can be written in two other useful variant forms by using $\cos^2 A + \sin^2 A = 1$.

$\cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1.$
$\cos^2 A - \sin^2 A = (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A .$

The double angle formulae are thus:

$\mathbf{\sin(2A) = 2 \sin A \cos A}$

$$\begin{aligned}\mathbf{\cos(2A)} &= \mathbf{\cos^2 A - \sin^2 A}\\ &= \mathbf{2\cos^2 A - 1}\\ &= \mathbf{1 - 2\sin^2 A}\end{aligned}$$

$$\mathbf{\tan(2A) = \frac{2 \tan A}{1 - \tan^2 A}}$$

By replacing A with $\frac{1}{2}A$ in the last two formulae for $\cos 2A$, we have the half angle identities:

$\cos A = 2\cos^2 \frac{1}{2}A - 1$
$\Rightarrow\ 2\cos^2 \frac{1}{2}A = 1 + \cos A$
$\Rightarrow\ \mathbf{\cos^2 \frac{1}{2}A = \frac{1}{2}(1 + \cos A)}$

$\cos A = 1 - 2\sin^2 \frac{1}{2}A$
$\Rightarrow\ -2\sin^2 \frac{1}{2}A = \cos A - 1$
$\Rightarrow\ 2\sin^2 \frac{1}{2}A = 1 - \cos A$
$\Rightarrow\ \mathbf{\sin^2 \frac{1}{2}A = \frac{1}{2}(1 - \cos A)}$

The above identities are useful in further calculus, especially in integration.

Using and Proving Trigonometric Identities.

The above identities can be manipulated to give rise to new ones. Using and proving identities is a skill which improves with practice.

Example (1): Find the values of sin 75° and cos 75°, leaving the results as surds. From the results, also find the values of tan 75° and tan 15°. (Note that tan 15° = cot 75°)

We treat 75° as 45° + 30° and apply the identity
sin (45° + 30°) = sin 45° cos 30° + cos 45° sin 30°.

Since sin 45° = cos 45° = $\frac{1}{\sqrt{2}}$, the expression for sin 75° becomes $\frac{1}{\sqrt{2}}$(cos 30° + sin 30°)

or $\frac{1}{\sqrt{2}}\left(\frac{1+\sqrt{3}}{2}\right)$ or $\frac{\sqrt{3}+1}{2\sqrt{2}}$.

To find cos 75°, we substitute into the identity
cos (45° + 30°) = cos 45° cos 30° - sin 45° sin 30°.

This expression simplifies into $\frac{1}{\sqrt{2}}$(cos 30° - sin 30°)

or $\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}-1}{2}\right)$ or $\frac{\sqrt{3}-1}{2\sqrt{2}}$.

To find tan 75°, we could either divide the result for cos 75° into the one for sin 75°, or we could substitute into the compound angle tangent formula. The question asks for the former method.

Dividing sin 75° by cos 75° we have

tan 75° = $\frac{\sqrt{3}+1}{2\sqrt{2}} \div \frac{\sqrt{3}-1}{2\sqrt{2}}$

= $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ = $\frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$ (rationalising the denominator)

= $\frac{4+2\sqrt{3}}{2}$

= $2+\sqrt{3}.$

To find tan 15°, we can divide the result for sin 75° into cos 75°, or find the reciprocal of tan 75°.

= $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ = $\frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$ (rationalising the denominator)

= $\frac{4-2\sqrt{3}}{2}$

= $2-\sqrt{3}.$

Example (2): Prove that $\dfrac{\cos 2A}{\cos A - \sin A} = \cos A + \sin A.$

(Copyright Letts Educational, *Revise AS and A2 Mathematics* (2004) ISBN 1-8431-477-3 , Example 'Proving Identities' p.33)

The first step is to start with the expression on one side of the identity, and then to replace parts of it with an equivalent form. The LHS is more complex, and so we can rewrite it using the double angle result for cos 2*A*.

The LHS thus becomes $\dfrac{\cos^2 A - \sin^2 A}{\cos A - \sin A}$, but then the top line can be recognised as the difference of two squares, and so it can be rewritten $\dfrac{(\cos A + \sin A)(\cos A - \sin A)}{(\cos A - \sin A)}$.

After taking out cos *A* – sin *A* as a factor, the expression becomes cos *A* + sin *A*, thus LHS = RHS.

Example (3): Prove that $\dfrac{\csc A}{\csc A - \sin A} = \sec^2 A$

(Copyright Letts Educational, *Revise AS and A2 Mathematics* (2004) ISBN 1-8431-477-3 , Example 'Proving Identities', p.33)

(The software on the computer uses 'csc' for 'cosec').

Since cosec $A = \frac{1}{\sin A}$, we can multiply both top and bottom of the LHS by sin *A* to give

$\dfrac{1}{1 - \sin^2 A}$, which is equivalent to $\dfrac{1}{\cos^2 A}$ (using $\cos^2 A + \sin^2 A = 1$), and finally to $\sec^2 A$

as on the RHS (using sec $A = \frac{1}{\cos A}$).

Example (4): Prove that $\sin 2\theta = \dfrac{2\tan\theta}{1 + \tan^2\theta}$.

(Copyright Letts Educational, *Revise AS and A2 Mathematics* (2004) ISBN 1-8431-477-3 , Trigonometric Equations, Progress Check Ex. 1, p.34)

Rewrite the RHS as $\dfrac{2\tan\theta}{\sec^2\theta}$ (using $1 + \tan^2\theta = \sec^2\theta$)

$= 2\tan\theta\cos^2\theta$ (using sec $\theta = \frac{1}{\cos\theta}$)

$= \dfrac{2\sin\theta\cos^2\theta}{\cos\theta}$ (using tan $\theta = \frac{\sin\theta}{\cos\theta}$)

$= 2\sin\theta\cos\theta$, which is the same as $\sin 2\theta \Rightarrow$ LHS = RHS.

Example (5): Use the result of the previous example to express cos 2*A* in terms of tan *A*.

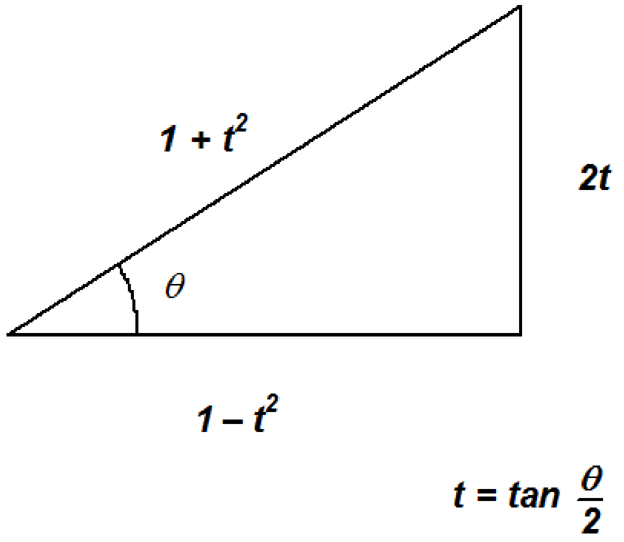
Since $\tan 2A = \dfrac{\sin 2A}{\cos 2A}$, it follows that $\cos 2A = \dfrac{\sin 2A}{\tan 2A} \Rightarrow \cos 2A = \sin 2A \times \dfrac{1}{\tan 2A}$.

$\Rightarrow \cos 2A = \dfrac{2\tan A}{1+\tan^2 A} \times \dfrac{1-\tan^2 A}{2\tan A}$ (previous example plus double angle formula)

$\Rightarrow \cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

These examples are also often written as half-angle identities involving tan ½*θ*.

If *t* = tan ½*θ*, then the other ratios for θ can be obtained in terms of t, as per the diagram.



The standard double-angle formula is

$\tan\theta = \dfrac{2t}{1-t^2}$

Using Pythagoras, the length of the hypotenuse is

$\sqrt{(2t)^2 + (1-t^2)^2} = \sqrt{4t^2 + 1 - 2t^2 + t^4} = \sqrt{1 + 2t^2 + t^4} = 1 + t^2$.

From this result, it follows that

$\sin\theta = \dfrac{2t}{1+t^2}$ and $\cos\theta = \dfrac{1-t^2}{1+t^2}$.

Example (6): Show that $\sin P + \sin Q = 2\sin \frac{1}{2}(P+Q) \cos \frac{1}{2}(P-Q)$. Hint: use $P = A + B$ and $Q = A - B$

Take the two compound angle sine identities and add them together:

$\sin (A + B) + \sin (A - B) = (\sin A \cos B + \cos A \sin B) + (\sin A \cos B - \cos A \sin B) = 2 \sin A \cos B$.

Substituting $P = A + B$ and $Q = A – B$, we have $A = \frac{1}{2}(P+Q)$ and $B = \frac{1}{2}(P-Q)$.
The summed expression above therefore becomes
$\mathbf{\sin P + \sin Q = 2 \sin \frac{1}{2}(P + Q) \cos \frac{1}{2}(P - Q)}$.

Had we subtracted the second sine identity from the first, we would have had
$\sin (A + B) - \sin (A - B) = (\sin A \cos B + \cos A \sin B) - (\sin A \cos B - \cos A \sin B) = 2 \cos A \sin B$, or
$\mathbf{\sin P - \sin Q = 2 \cos \frac{1}{2}(P + Q) \sin \frac{1}{2}(P - Q)}$.

The cosine identities are treated similarly.

$\cos (A + B) + \cos (A - B) = (\cos A \cos B - \sin A \sin B) + (\cos A \cos B + \sin A \sin B) = 2 \cos A \cos B$.
$\mathbf{\cos P + \cos Q = 2\cos \frac{1}{2}(P + Q) \cos \frac{1}{2}(P - Q)}$.

$\cos (A + B) - \cos (A - B) = (\cos A \cos B - \sin A \sin B) - (\cos A \cos B + \sin A \sin B) = -2 \sin A \sin B$.
(watch the minus sign here !)
$\mathbf{\cos P - \cos Q = -2\sin \frac{1}{2}(P + Q) \sin \frac{1}{2}(P - Q)}$.

Example (7): Express $\cos 3x$ in terms of $\cos x$.

$\cos 3x = \cos (2x + x) = \cos 2x \cos x – \sin 2x \sin x$

$= (2 \cos^2 x - 1)(\cos x) – (2 \sin x \cos x)(\sin x)$ (Use double angle formulae)
$= (2 \cos^3 x – \cos x) – (2 \sin^2 x \cos x)$
$= (2 \cos^3 x – \cos x) – (2 (1 - \cos^2 x))(\cos x)$ (Use $\cos^2 x + \sin^2 x = 1$)
$= (2 \cos^3 x – \cos x) – (2 \cos x - 2 \cos^3 x)$
$\mathbf{= 4 \cos^3 x – 3 \cos x}$.

Note: The corresponding working for $\sin 3x$ is similar.

$\sin 3x = \sin (2x + x) = \sin 2x \cos x + \cos 2x \sin x$

$= (2 \sin x \cos x)(\cos x) + (1 – 2 \sin^2 x)(\sin x)$ (Use double angle formulae)
$= (2 \sin x \cos^2 x) + (\sin x - 2 \sin^3 x)$
$= (2 (\sin x)(1 - \sin^2 x)) + (\sin x - 2 \sin^3 x)$ (Use $\cos^2 x + \sin^2 x = 1$)
$= (2 \sin x – 2 \sin^3 x) + (\sin x - 2 \sin^3 x)$
$\mathbf{= 3 \sin x - 4 \sin^3 x}$.

Example (8): Prove that $\sin\theta\,\tan\theta + \cos\theta = \sec\theta$.
(Copyright OUP, *Understanding Pure Mathematics*, Sadler & Thorning, ISBN 9780199142590, Exercise 4D, Q.3a)

$\sin\theta\,\tan\theta + \cos\theta = \sec\theta$

$$\Rightarrow \quad \sin\theta\left(\frac{\sin\theta}{\cos\theta}\right) + \cos\theta = \sec\theta \quad \Rightarrow \quad \left(\frac{\sin^2\theta}{\cos\theta}\right) + \cos\theta = \sec\theta$$

Multiplying both sides by $\cos\theta$ we have the standard identity $\sin^2\theta + \cos^2\theta = 1$, because $\sec\theta$ and $\cos\theta$ are reciprocals of each other).

Example (9): Prove that $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} = 2\csc\theta$.

(The software on the computer uses 'csc' for 'cosec').

(Copyright OUP, *Understanding Pure Mathematics*, Sadler & Thorning, ISBN 9780199142590, Exercise 4D, Q.8a)

$$\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\csc\theta \Rightarrow \frac{\sin^2\theta + (1+\cos\theta)^2}{(1+\cos\theta)(\sin\theta)} = 2\csc\theta$$

$$\Rightarrow \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{(1+\cos\theta)(\sin\theta)} = 2\csc\theta \Rightarrow \frac{2+2\cos\theta}{(1+\cos\theta)(\sin\theta)} = 2\csc\theta$$

$$\Rightarrow \frac{2(1+\cos\theta)}{(1+\cos\theta)(\sin\theta)} = 2\csc\theta \quad \Rightarrow \frac{2}{\sin\theta} = 2\csc\theta$$

$\therefore$ LHS = RHS because $\operatorname{cosec}\theta$ and $\sin\theta$ are reciprocals of each other.

Example (10): Prove that $\dfrac{\csc\theta}{\cot\theta + \tan\theta} = \cos\theta$.

(Copyright OUP, *Understanding Pure Mathematics*, Sadler & Thorning, ISBN 9780199142590, Exercise 4D, Q.8b)

We begin with $\cot\theta + \tan\theta = \dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta} = \dfrac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} = \dfrac{1}{\sin\theta\cos\theta}$.

Therefore $\dfrac{1}{\cot\theta + \tan\theta} = \sin\theta\cos\theta$.

Hence $\dfrac{\csc\theta}{\cot\theta + \tan\theta} = \csc\theta \times \dfrac{1}{\cot\theta + \tan\theta} = (\csc\theta)(\sin\theta\cos\theta)$

and finally $\dfrac{\csc\theta}{\cot\theta + \tan\theta} = \cos\theta$ (because $\operatorname{cosec}\theta$ and $\sin\theta$ are reciprocals of each other).

Example (11): Prove that $\cos^4\theta - \sin^4\theta + 1 = 2\cos^2\theta$.

(Copyright OUP, *Understanding Pure Mathematics*, Sadler & Thorning, ISBN 9780199142590, Exercise 4D, Q.8d)

$\cos^4\theta - \sin^4\theta + 1 = 2\cos^2\theta \Rightarrow \cos^4\theta - \sin^4\theta = 2\cos^2\theta - 1$

$\Rightarrow (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) = 2\cos^2\theta - 1 \Rightarrow (1)(\cos^2\theta - \sin^2\theta) = 2\cos^2\theta - 1$

$\Rightarrow \cos 2\theta = \cos 2\theta$ (Both LHS and RHS are expressions for $\cos 2\theta$).

Example (12): Prove that $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$.

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Since $\sec^2\theta = 1 + \tan^2\theta$, we have $\sec^4\theta = (\sec^2\theta)^2 = (1 + \tan^2\theta)^2 = 1 + 2\tan^2\theta + \tan^4\theta$.

$\therefore \sec^4\theta - \sec^2\theta = (1 + 2\tan^2\theta + \tan^4\theta) - (1 + \tan^2\theta) = \tan^4\theta + \tan^2\theta \;\therefore$ LHS = RHS.

Example (13): Prove the identity: $\tan(\theta + 60°)\tan(\theta - 60°) = \dfrac{\tan^2\theta - 3}{1 - 3\tan^2\theta}$.

(Copyright OCR, GCE Mathematics Paper 4723, Jun. 2007., Q.9, part (i).

We begin with $\tan(\theta + 60°) = \dfrac{\tan\theta + \tan 60°}{1 - \tan\theta\tan 60°} \Rightarrow \tan(\theta - 60°) = \dfrac{\tan\theta - \tan 60°}{1 + \tan\theta\tan 60°}$.

Multiplying, we have $\tan(\theta + 60°)\tan(\theta - 60°) = \dfrac{\tan\theta + \tan 60°}{1 - \tan\theta\tan 60°} \times \dfrac{\tan\theta - \tan 60°}{1 + \tan\theta\tan 60°}$

$\Rightarrow \tan(\theta + 60°)\tan(\theta - 60°) = \dfrac{\tan^2\theta - \tan^2 60°}{1 - \tan^2\theta\tan^2 60°}$ (using difference of squares)

Because $\tan 60° = \sqrt{3}$, we can substitute 3 for $\tan^2 60°$; thus

$$\tan(\theta + 60°)\tan(\theta - 60°) = \frac{\tan^2\theta - 3}{1 - 3\tan^2\theta}.$$

Solving more trigonometric equations.

Identities may be used to re-write an equation in a form that is easier to solve.

Example (14) : Solve the equation $2\sin^2 x + \cos x - 2 = 0$ for $-\pi \le x \le \pi$.

We must first use $\sin^2 x = 1 - \cos^2 x$ to change the equation into a quadratic in $\cos x$.
Thus $2\sin^2 x + \cos x - 2 = 0 \Rightarrow 2(1-\cos^2 x) + \cos x - 2 = 0 \Rightarrow 2 - 2\cos^2 x + \cos x - 2 = 0$
$\Rightarrow \cos x - 2\cos^2 x = 0$

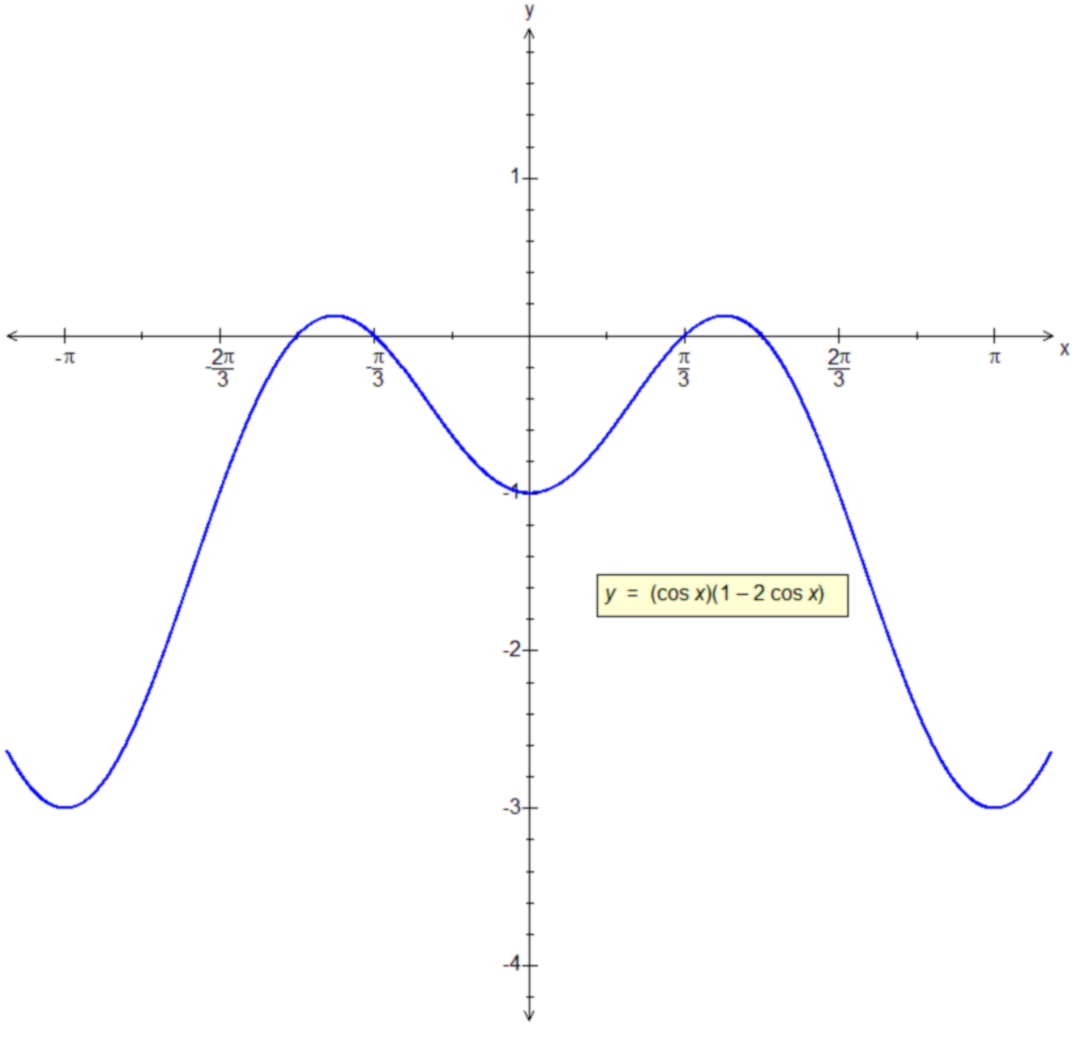
This factorises at once into $(\cos x)(1 - 2\cos x) = 0$.

$\cos x = 0$ when $x = \pi/2$ or $x = -\pi/2$.

$2\cos x = 1$, or $\cos x = 0.5$, when $x = \pi/3$ or $x = -\pi/3$.

The solutions to the equation are therefore $x = \pm\pi/2$ and $\pm\pi/3$ (illustrated below).
It is important not to simply cancel out $\cos x$ from the equation, because there are solutions where $\cos x = 0$, i.e. where $x = \pm\pi/2$.

(Graph for illustration only – students are not expected to sketch it !)



Example (15): Solve the equation $\cos 2x + \sin x + 1 = 0$ for $-\pi \le x \le \pi$.

This equation can be turned into a quadratic in sin x by using the identity $\cos 2x = 1 - 2\sin^2 x$.

This gives $1 - 2\sin^2 x + \sin x + 1 = 0$, or $-2\sin^2 x + \sin x + 2 = 0$.

Using the general formula, we substitute $x = \sin x$, $a = -2$, $b = 1$ and $c = 2$.

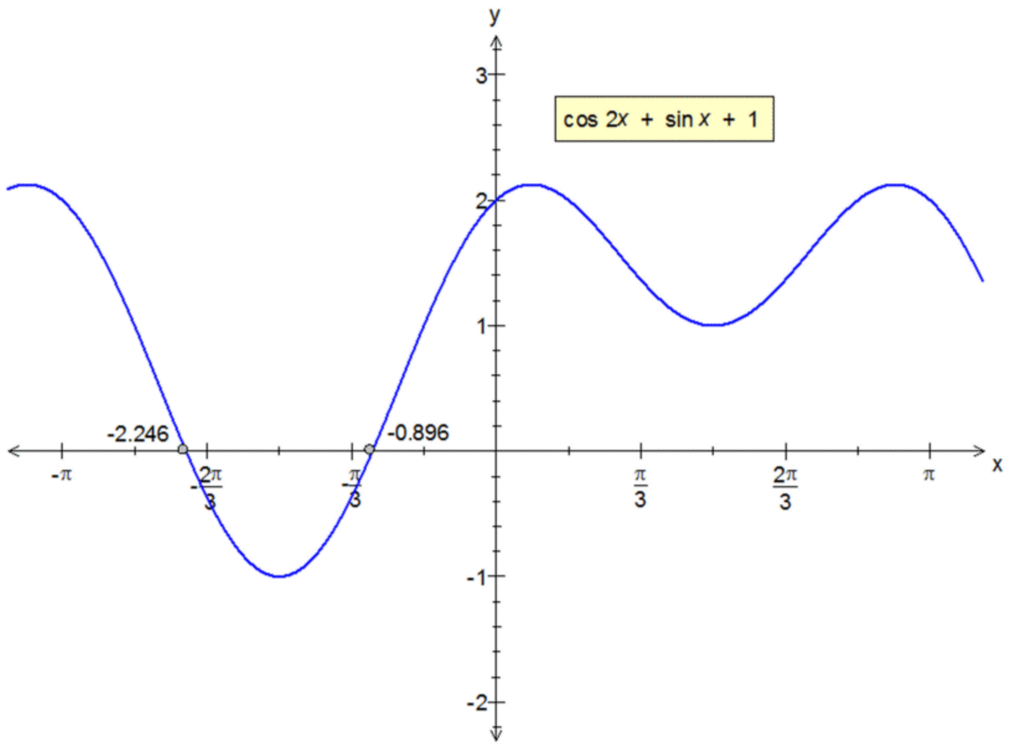
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \sin x = \frac{-1 \pm \sqrt{17}}{-4} .$$

The two solutions are $\sin x = 1.281$ or $\sin x = -0.781$ to 3 d.p.

The positive value can be rejected at once, since no angle has a sine greater than 1.

The principal value of the other result is -0.896^c and it falls within the range $-\pi \le x \le \pi$.

Using the fact that $\sin(\pi - x) = \sin x$, another possible solution is 4.037^c , but it falls outside the range.
Since the sine function has a period of 2π, the other solution in the range is $(4.037 - 2\pi)$ or -2.246^c.



Example (16): Solve the equation sec θ = 4 cosec θ for 0 < θ < 2π, giving results in radians to 3 decimal places.

sec θ = 4 cosec θ $\Rightarrow$ $\frac{1}{\cos\theta} = \frac{4}{\sin\theta}$ $\Rightarrow$ $\frac{\sin\theta}{\cos\theta} = 4$ $\Rightarrow$ $\tan\theta = 4$.

The principal solution is θ = 1.326^c, but because the tangent function has a period of π, another solution is $(1.326 + \pi)^c$ or 4.467^c.

Example (17): Given that sin 3θ = 3 sin θ - 4 $\sin^3\theta$ (see Example 6), use this result to solve, for 0° < β < 90°, the equation 3 sin 6β cosec 2β = 4.

(Copyright OCR, GCE Mathematics Paper 4723, Jan. 2006., Q.9 part iii).

Firstly we substitute θ=2β to obtain 3 sin 3θ cosec θ = 4, for 0° < θ < 180°

3 sin 3θ cosec θ = 4 $\Rightarrow$ $\frac{3\sin 3\theta}{\sin\theta} = 4$ (cosec is reciprocal of sin)

$\Rightarrow$ $\frac{3(3\sin\theta - 4\sin^3\theta)}{\sin\theta} = 4$ $\Rightarrow$ 3(3-4 $\sin^2\theta$) = 4 $\Rightarrow$ 9 –12 $\sin^2\theta$ = 4 $\Rightarrow$ 5 –12 $\sin^2\theta$ = 0

$\Rightarrow$ $\sin^2\theta = \frac{5}{12}$ $\Rightarrow$ $\theta = \sin^{-1}\left(\sqrt{\frac{5}{12}}\right)$, or 40.2°, 139.8°. (There is no need to consider the negative value since sin θ is positive for 0° < θ < 180°)

Finally, we must remember that θ=2β , so we must divide by 2 to give β = 20.1°, 69.9°.

Example (18): Use the identity $\tan(\theta + 60°)\tan(\theta - 60°) = \frac{\tan^2\theta - 3}{1 - 3\tan^2\theta}$ from Example (13), to solve,

for 0 < θ < 180°, the equation tan(θ + 60°) tan(θ–60°) = 4$\sec^2\theta$ – 3.

(Copyright OCR, GCE Mathematics Paper 4723, Jun. 2007., Q.9, part (ii).

$\frac{\tan^2\theta - 3}{1 - 3\tan^2\theta} = 4\sec^2\theta - 3$ $\Rightarrow$ $\frac{\tan^2\theta - 3}{1 - 3\tan^2\theta} = 4(\tan^2\theta + 1) - 3$ (using $\sec^2\theta = 1 + \tan^2\theta$)

$\Rightarrow$ $\frac{\tan^2\theta - 3}{1 - 3\tan^2\theta} = 4\tan^2\theta + 1$ $\Rightarrow$ $\tan^2\theta - 3 = (4\tan^2\theta + 1)(1 - 3\tan^2\theta)$

$\Rightarrow \tan^2\theta - 3 = 4\tan^2\theta + 1 - 12\tan^4\theta - 3\tan^2\theta$
$\Rightarrow 12\tan^4\theta + 3\tan^2\theta - 4\tan^2\theta - 1 + \tan^2\theta - 3 = 0$ (bringing RHS over to left)
$\Rightarrow 12\tan^4\theta - 4 = 0$ $\Rightarrow$ $3\tan^4\theta - 1 = 0$

$\Rightarrow$ $\tan^4\theta = \frac{1}{3}$ $\Rightarrow$ $\theta = \tan^{-1}\left(\sqrt[4]{\frac{1}{3}}\right)$ = $\tan^{-1}$ (±0.7598).

∴ θ = 37.2°, -37.2°. The negative value is outside the range, but must be included since the tangent function repeats every 180°, and (-37.2 + 180)° or 142.8° is also within the range.

∴ Solutions are θ = 37.2° , 142.8°.

Solution of trigonometric equations of the form $a \cos \theta + b \sin \theta = c$. (Harmonic Form)

Equations of the form above can be rewritten in the form $R \cos (\theta \pm \alpha)$ or $R \sin (\theta \pm \alpha)$ for suitable choices of angles θ and α.

The four compound angle identities are :
$R \cos(\theta + \alpha) = R \cos \theta \cos \alpha - R \sin \theta \sin \alpha$
$R \cos(\theta - \alpha) = R \cos \theta \cos \alpha + R \sin \theta \sin \alpha$
$R \sin(\theta + \alpha) = R \sin \theta \cos \alpha + R \cos \theta \sin \alpha$
$R \sin(\theta - \alpha) = R \sin \theta \cos \alpha - R \cos \theta \sin \alpha$

Any expression of the form $a \cos \theta \pm b \sin \theta$ can be rewritten to match the above forms.

Example (19): Express $3 \cos \theta + 4 \sin \theta$ in the form $R \cos (\theta \pm \alpha)$. Give α in degrees, and from there solve $3 \cos \theta + 4 \sin \theta = 2.5$ for $0° \le \theta \le 360°$.

$R \cos(\theta - \alpha) = 3 \cos \theta + 4 \sin \theta$ is the closest match :
$R \cos \theta \cos \alpha + R \sin \theta \sin \alpha = 3 \cos \theta + 4 \sin \theta$

Equating sine and cosine terms we have:
$R \cos \theta \cos \alpha = 3 \cos \theta \quad \Rightarrow \quad R \cos \alpha = 3.$
$R \sin \theta \sin \alpha = 4 \sin \theta \quad \Rightarrow \quad R \sin \alpha = 4.$

Squaring the right-hand sides gives $R^2 \cos^2 \alpha = 9$ and $R^2 \sin^2 \alpha = 16 \quad \Rightarrow \quad R^2 (\cos^2 \alpha + \sin^2 \alpha) = 25$.
The bracketed expression is simply 1, therefore $R = \sqrt{3^2 + 4^2} = 5$.

When any expression of the form $a \cos \theta \pm b \sin \theta$ is rewritten in the form $R \cos (\theta \pm \alpha)$ or $R \sin (\theta \pm \alpha)$, then $R = \sqrt{a^2 + b^2}$.

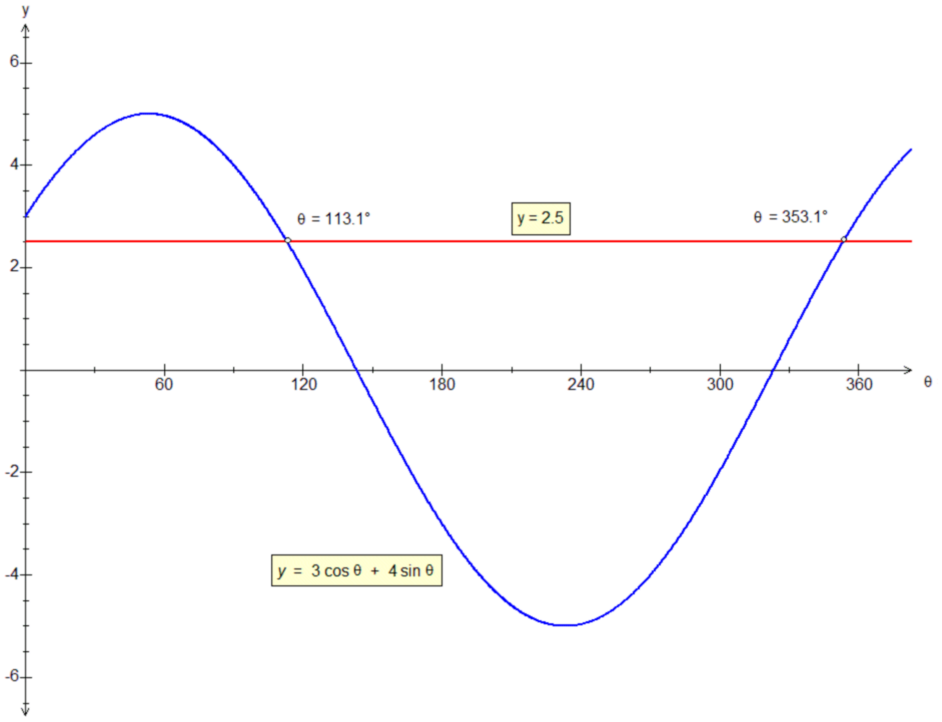
To find α, we see that $\frac{R \sin \alpha}{R \cos \alpha} = \frac{4}{3}$, or that $\tan \alpha = \frac{4}{3}$, and thus $\alpha = 53.1°$.
$\therefore\ 3 \cos \theta + 4 \sin \theta = 5 \cos (\theta - 53.1°)$.

To solve
$5 \cos (\theta - 53.1°) = 2.5$
for $0° \le \theta \le 360°$,
first divide by 5 to give
$\cos (\theta - 53.1°) = 0.5$
for $0° \le \theta \le 360°$.

Two values of
$(\theta - 53.1°)$ satisfying
the condition are
$60°$ and $-60°$;

$\theta = 113.1°$ or $-6.9°$.

The second value is
out of the required
range for θ, but by
adding $360°$ to it we
obtain $\theta = 353.1°$,
which is within the
range.



Example (20): Express sin θ - 2 cos θ in the form R sin ($\theta \pm \alpha$). Give α in radians.

$R \sin(\theta - \alpha) = \sin\theta - 2\cos\theta$ is the closest match :
$R \sin\theta\cos\alpha - R\cos\theta\sin\alpha = \sin\theta - 2\cos\theta$

Equating sine and cosine terms we have:

$R\sin\theta\cos\alpha = \sin\theta \quad \Rightarrow \quad R\cos\alpha = 1.$
$R\cos\theta\sin\alpha = 2\cos\theta \quad \Rightarrow \quad R\sin\alpha = 2.$

Hence $R = \sqrt{1^2 + 2^2} = \sqrt{5}$ and tan α = 2 , and thus $\alpha = 1.107^c$.

$\therefore \sin\theta - 2\cos\theta = \sqrt{5}\sin(\theta - 1.107^c)$.

Example (21): Express 5 sin θ + 12 cos θ in the form R sin ($\theta \pm \alpha$). Give α in radians.

$R \sin(\theta + \alpha) = 5\sin\theta + 12\cos\theta$ is the closest match.
$R\sin\theta\cos\alpha + R\cos\theta\sin\alpha = 5\sin\theta + 12\cos\theta$

$\Rightarrow R\cos\alpha = 5;\ R\sin\alpha = 12 \ \Rightarrow\ R = \sqrt{5^2 + 12^2} = 13$ and $\tan\alpha = \frac{12}{5} \Rightarrow \alpha = 1.176^c$.
$\Rightarrow 5\sin\theta + 12\cos\theta = 13\sin(\theta + 1.176^c)$.

Example (22): The water temperature (°C) of a swimming pool is modelled on the equation
$W = 27 - 2\sin(15t) - 2.2\cos(15t))$, where t is the time in hours after midnight.

(The multiple of 15 enters into the questions because the period of 360 degrees needs modifying into a period of 24 hours: 24 × 15 = 360.)

i) Express the model equation in the form $W = 27 - R\sin(15t + \alpha)°$, where α is positive.
ii) Show that the maximum temperature of the water cannot exceed 30°C according to the model.
iii) Find the time of day when the water reaches this maximum temperature.
iv) The management decide that the water temperature should not fall below 26°C during the opening hours of between 0730 and 2200. Does the given model satisfy the requirements ?

i) We must rearrange the brackets in the formula first:
$W = 27 - 2\sin(15t) - 2.2\cos(15t)) \Rightarrow W = 27 - (2\sin(15t) + 2.2\cos(15t))$

Then continuing in the usual way we have
$R\sin(15t + \alpha) = 2\sin(15t) + 2.2\cos(15t)$.
$R\sin(15t)\cos\alpha + R\cos(15t)\sin\alpha = 2\sin(15t) + 2.2\cos(15t)$.

Equating sine and cosine terms we have:

$R\sin(15t)\cos\alpha = 2\sin(15t) \quad \to \quad R\cos\alpha = 2.$
$R\cos(15t)\sin\alpha = 2.2\cos(15t) \quad \to \quad R\sin\alpha = 2.2.$

Hence $R = \sqrt{(-2)^2 + (2.2)^2} = \sqrt{8.84}$ or 2.973, and $\tan\alpha = \dfrac{2.2}{2} = 1.1$, hence $\alpha = 47.7°$.

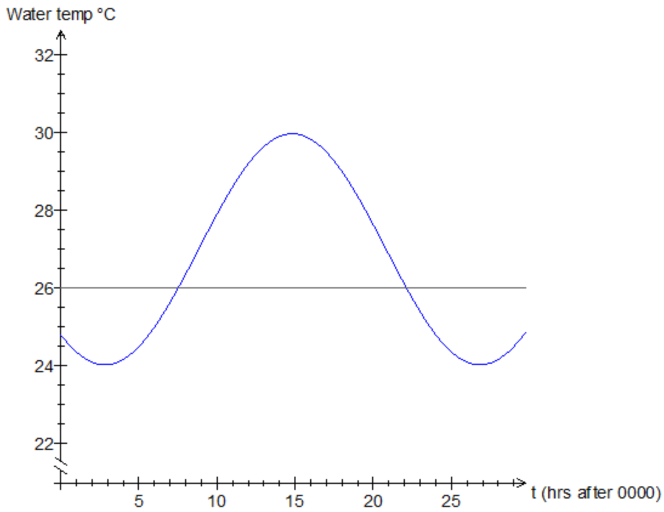
$\therefore$ the model equation is given by $W = 27 - 2.973\sin(15t + 47.7)°$.

ii) Since the sine function cannot values outside the range -1 to 1, the temperature of the water cannot exceed (27 + 2.973)°C, or 29.97°C according to this model.

iii) This maximum temperature is reached when $\sin(15t + 47.7)° = -1$ (remember, we're subtracting !), i.e. when $15t + 47.7 = 270° \Rightarrow 15t = 222.3 \Rightarrow t = 14.82$ hours after midnight, or at about 1449.

iv) (A sketch graph is useful here.)

The temperature of the water is at a maximum at about 1500, and so it must take a minimum at about 0300, because this is a sine function with a 24-hr period. By sketching, it can be seen that the water temperature exceeds 26° from about 0800 to 2200.



We must solve the equation $27 - 2.973\sin(15t + 47.7)° = 26 \Rightarrow -2.973\sin(15t + 47.7)° = -1$
$\Rightarrow 2.973\sin(15t + 47.7)° = 1 \Rightarrow \sin(15t + 47.7)° = 0.3363.$

$\Rightarrow (15t + 47.7)° = 19.654°, 160.346° \Rightarrow 15t = -28.07, \ 112.62$

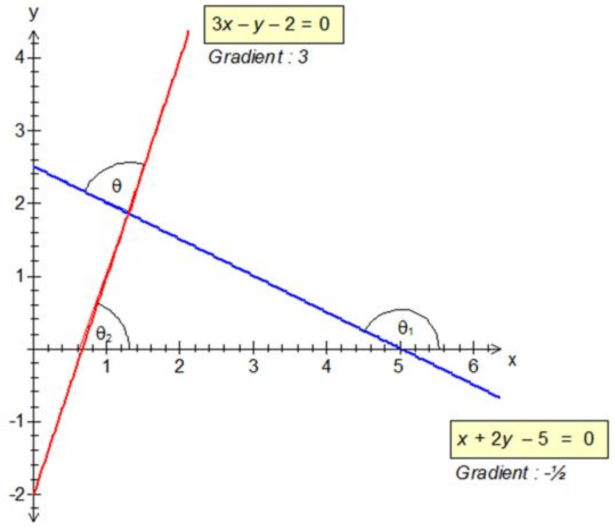
The answers in degrees must finally be converted into hours after midnight by first adding 360° to the negative value , giving 331.93, and then dividing both results by 15.

Hence the water temperature = 26°C when $t = 22.13$ hours after midnight, or 2208, or when $t = 7.51$ hours after midnight, or 0731.

By looking at the graph, the water temperature is above 26° between 0731 and 2208, which practically coincide with the opening hours !

The angle between two straight lines.

Example (23): Two straight lines have the equations $x + 2y - 5 = 0$ and $3x - y - 2 = 0$ respectively. What is the acute angle between them ?



Each of the lines makes a particular angle with the x-axis, measured anticlockwise.

Thus θ_1 is the angle between the line $x + 2y - 5 = 0$ and the x – axis, while θ_2 is the angle between the line $3x - y - 2 = 0$ and the x – axis.

The gradient of the line $x + 2y - 5 = 0$ is also the tangent of θ_1 , and its value is -½. Similarly the gradient of the line $3x - y - 2 = 0$ is the tangent of θ_2 , namely 3.

The acute angle between the lines is θ, and, by the triangle laws, $\theta = \theta_1 - \theta_2$.

Taking tangents we have $\tan\theta = \tan(\theta_1 - \theta_2)$

or $\tan\theta = \dfrac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$

Substituting $\tan\theta_1 = -\frac{1}{2}$ and $\tan\theta_2 = 3$, we can calculate the acute angle θ between the two lines using

$$\tan\theta = \frac{(-\frac{1}{2}) - 3}{1 + (-\frac{1}{2})(3)} = \frac{-3\frac{1}{2}}{-\frac{1}{2}} = 7 \Rightarrow \theta = 81.9° \text{ or } 1.429^c.$$

Also note that the constant terms in the equations of the lines do not affect the result.
Thus the acute angle between the lines $x + 2y - 6 = 0$ and $3x - y + 5 = 0$ would also be 81.9° or 1.429^c .

The general formula for the angle θ between two straight lines (not in scope of syllabus), given their gradients m_1 and m_2 , is $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$.

(It is not important which gradient is chosen as m_1; if $\tan\theta$ comes out negative, then disregarding the sign will still give the acute angle result).