M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

Edexcel: **C4** OCR: **C3** OCR MEI: **C3**

RELATED RATES OF CHANGE

$$\frac{dr}{dt} = 0.5,\quad A = \pi r^2,\quad \frac{dA}{dr} = 2\pi r \quad \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

When $r = 4$, $\frac{dA}{dt} = 8\pi \times 0.5 = 4\pi = 12.6$ (3 s.f.)

$$V = \sqrt{(h^6 + h - 2)} \Rightarrow \frac{dV}{dh} = \frac{6h^5 + 1}{2\sqrt{(h^6 + h - 2)}} \qquad \frac{dV}{dt} = 10$$

When $h = 2$, $\frac{dV}{dh} = \frac{6(32)+1}{2\sqrt{(64+2-2)}} \Rightarrow \frac{dV}{dh} = \frac{193}{16}$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} \Rightarrow \frac{dh}{dt} = \frac{16}{193} \times 10 \text{ or } 0.829$$

$$\frac{dy}{dt} = 0.1 \qquad V = \pi \int_0^p 9 + 6e^{\frac{y}{2}} + e^y\,dy \qquad \frac{dV}{dy} = \pi\left(9 + 6e^{\frac{y}{2}} + e^y\right)$$

when $y = p = 4$, $\frac{dV}{dy} = \pi\left(9 + 6e^2 + e^4\right) = 339$

$$\frac{dV}{dt} = \frac{dV}{dy} \times \frac{dy}{dt} \Rightarrow \frac{dV}{dt} = 339 \times 0.1 = 34 \text{ to 2 s.f.}$$

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Application of the Chain Rule to Related Rates of Change.

The chain rule can also be used to establish results for related rates of change.

Example (1): Crude oil leaking out of a tanker produces a circular slick with radius r km and area A km^2. The radius of the slick increases with time at a rate given by $\frac{dr}{dt} = 0.5$ km/h.

Find the related rate of change of area of the slick, $\frac{dA}{dt}$, and also the rate of increase of the area of the slick at a time when the radius is 4 km.

The area of a circle is given by $A = \pi r^2$. The change in area with respect to the change in radius is therefore $\frac{dA}{dr} = 2\pi r$.

By the chain rule, $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$. Here it is $2\pi r \times 0.5$ or πr.

$\therefore$ At the time when the radius of the slick is 4km, the rate of increase in the slick's area is 4π km^2/h or about 12.6 km^2/h.

Example (2): During a tunnel-building project, it is found that the volume V of the excavated earth is related to the height h of the resulting pile by the formula

$$V = \sqrt{\left(h^6 + h - 2\right)}$$

i) Find the value of $\frac{dV}{dh}$ at the instant when the pile of earth is 2 m high.
ii) The volume of the pile of excavated earth is increasing at a constant rate of 10 m^3 per hour. Find the rate (per hour) at which the height of the pile is increasing at the instant the pile is 2 m high. Give the final result to the nearest centimetre.

i) By the chain rule, $u = h^6 + h - 2$ and $V = \sqrt{u}$.

Therefore $\frac{du}{dh} = 6h^5 + 1$ and $\frac{dV}{du} = \frac{1}{2\sqrt{u}}$. (or $\frac{dV}{du} = \frac{1}{2}u^{-\frac{1}{2}}$).

Hence $\frac{dV}{dh} = \frac{dV}{du} \times \frac{du}{dh} \Rightarrow \frac{dV}{dh} = \frac{6h^5 + 1}{2\sqrt{(h^6 + h - 2)}}$.

Substituting $h = 2$ into the above derivative gives

$$\frac{dV}{dh} = \frac{6(32) + 1}{2\sqrt{(64 + 2 - 2)}} \Rightarrow \frac{dV}{dh} = \frac{193}{16}.$$

ii) From the given data, we deduce that the change (per hour) in volume of the earth, or $\frac{dV}{dt} = 10$.

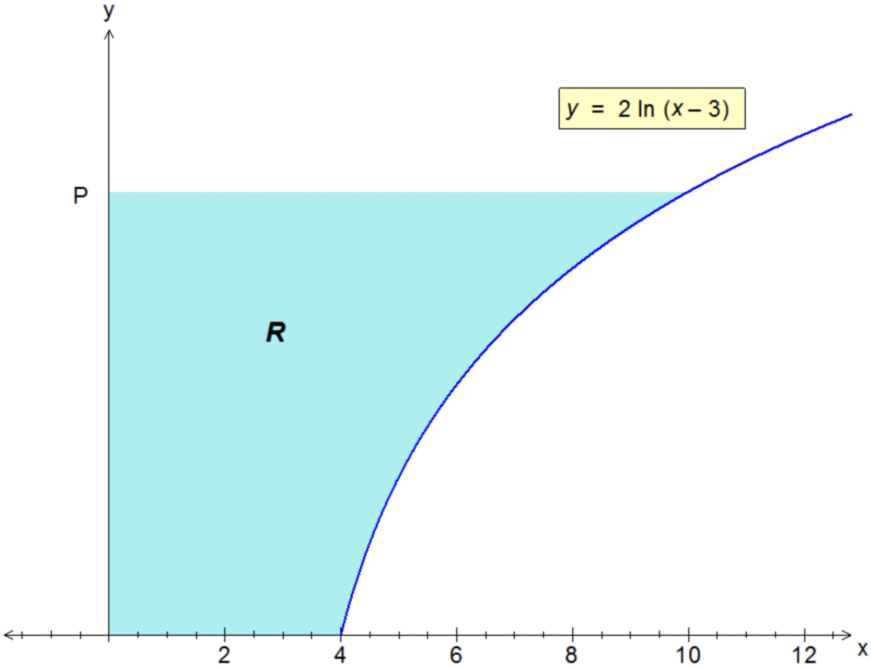
We want to find the rate of change of height when the pile is 2 m high, or $\frac{dh}{dt}$, when $h = 2$.

Now $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ by the chain rule, so at $h = 2$, from (i), $\frac{dh}{dt} = \frac{16}{193} \times 10$ or 0.829.

Note that $\frac{dh}{dV} = \frac{1}{dV/dh}$.

$\therefore$ At the instant when the pile of excavated earth is 2 m high, the height is increasing at a rate of 83 cm per hour.

Example (3): (This question assumes knowledge of volumes of revolution).



The diagram above shows the curve with equation $y = 2 \ln (x - 3)$. The point P has coordinates $(0, p)$, and the region R (shaded) is bounded by the curve, the axes, and the line $y = p$.

The units on the axes are in centimetres, and the region R is rotated completely about the ***y*-axis** to form a solid.

i) Show that the volume, V cm^3, of the solid formed is given by

$$V = \pi\left(e^{p} + 12e^{\frac{p}{2}} + 9p - 13\right).$$

ii) It is given that the point P is moving in the positive direction along the y-axis at a constant rate of 0.1 cm per minute. Find the rate at which the volume of the solid is increasing at the instant when p = 4, giving your answer correct to 2 significant figures.

i) Because we are dealing with a rotation about the *y*-axis, the first step is to rewrite $y = 2\ln(x-3)$ with *x* as the subject.

Taking exponents of both sides, $e^y = e^{2\ln(x-3)} \Rightarrow e^y = e^{\ln(x+3)^2} \Rightarrow e^y = (x-3)^2$.

(Remember, for any function $f(x)$, $e^{\ln(f(x))} = f(x)$ and also $\ln e^{(f(x))} = f(x)$ because e^x and $\ln x$ are inverses of each other !)

We then take square roots: $(x-3) = e^{\frac{y}{2}} \Rightarrow x = 3 + e^{\frac{y}{2}}$.

Having made *x* the subject, we can now use the volume formula :

$$V = \pi\int_0^p x^2\,dy \Rightarrow V = \pi\int_0^p \left(3 + e^{\frac{y}{2}}\right)^2 dy.$$

Expanding, $V = \pi\int_0^p 9 + 6e^{\frac{y}{2}} + e^y\,dy$, and integrating, $V = \pi\left[9y + 12e^{\frac{y}{2}} + e^y\right]_0^p$.

Putting in the limits of 0 and *p*, we then have $V = \pi\left[\left(9p + 12e^{\frac{p}{2}} + e^p\right) - (0 + 12 + 1)\right]$,

and finally $V = \pi\left[\left(e^p + 12e^{\frac{p}{2}} + 9p - 13\right)\right]$.

ii) We need to find the rate of change of volume with respect to time (here in minutes), i.e. $\frac{dV}{dt}$.

We are however given that point *P* is moving in the positive direction along the *y*-axis at a constant rate of 0.1 cm per minute, therefore $\frac{dy}{dt} = 0.1$.

To find $\frac{dV}{dy}$, we differentiate the expression $V = \pi\int_0^p 9 + 6e^{\frac{y}{2}} + e^y\,dy$, which is merely a case of removing the integral sign and limits (but keeping the factor of π !).

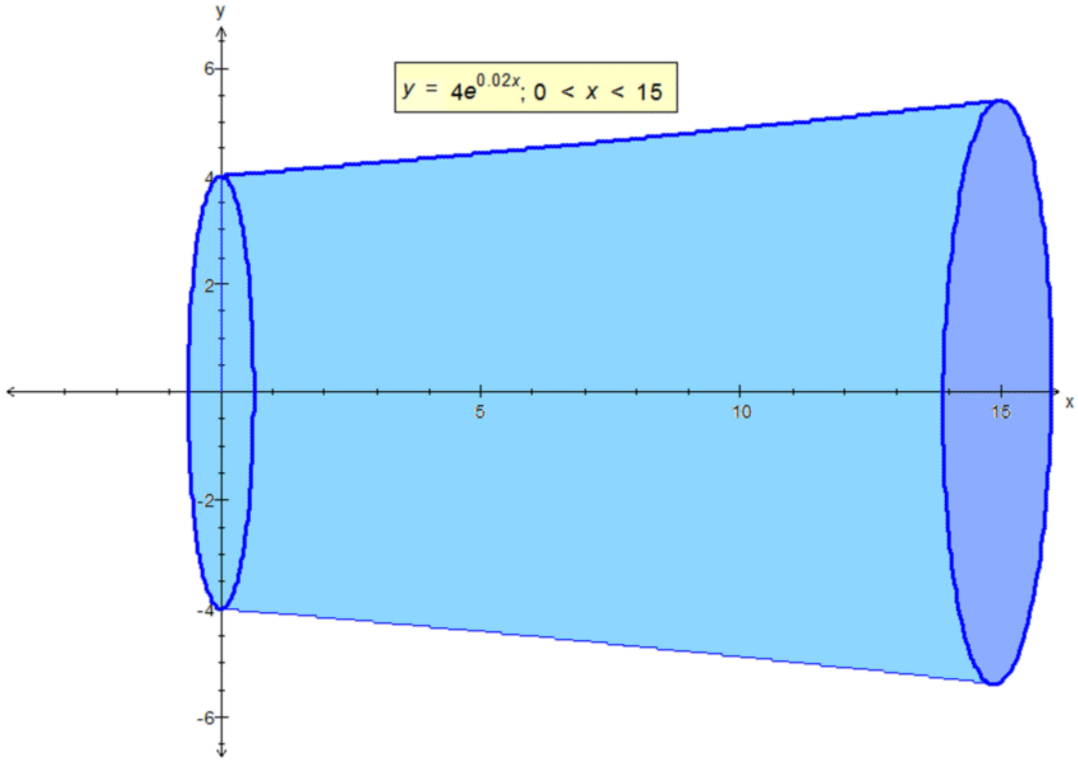
Hence $\frac{dV}{dy} = \pi\left(9 + 6e^{\frac{y}{2}} + e^y\right)$, and when $y = p = 4$, $\frac{dV}{dy} = \pi\left(9 + 6e^2 + e^4\right)$ or 339.

We then use the chain rule to find $\frac{dV}{dt} = \frac{dV}{dy} \times \frac{dy}{dt} \Rightarrow \frac{dV}{dt} = 339 \times 0.1 = 34$ to 2 s.f.

$\therefore$ the volume of the solid is increasing at the rate of 34 cm^3 per minute at the instant when $y = p = 4$.

Example (4): (Again, knowledge of volumes of revolution is required.)
A drinking glass is 16 cm tall, inclusive of a solid base 1cm thick (i.e. the effective height is 15cm.).

Its interior curved surface is generated by the equation $y = 4e^{0.02x}$, followed by a revolution about the *x*-axis. (Diagram below excludes solid base.)



i) Show that the inside diameter of the glass is 8 cm at the base, and 10.8 cm at the top.

ii) Find the volume of revolution of the resulting solid, i.e. the capacity of the glass.

iii) The glass is filled up to its maximum effective height of 15cm, and a small quantity removed to bring the volume down to exactly one litre. Calculate the volume of liquid removed to the nearest cm^3.

iv) Find the rate at which the volume of the liquid in the glass decreases for each 1cm decrease in the height level from the maximum.

v) The glass is then marked to indicate a liquid level of exactly one litre. Using the results of parts iii) and iv), work out how far the mark should be placed below the rim of the glass.

i) At $x = 0$, $y = 4$, so the inside diameter of the top of the glass is twice that, i.e. 8 cm.
At $x = 15$, $y = 4e^{0.3} = 5.4$, so the inside radius of the top of the glass is 5.4 cm, and hence the diameter is twice that, or 10.8 cm.

ii) The volume of revolution of the resulting solid is $V = \pi \int_0^{15} y^2 \, dx$ or $V = \pi \int_0^{15} 16e^{0.04x} \, dx$.

This integrates to $\pi \left[\frac{16e^{0.04x}}{0.04} \right]_0^{15} = 400\pi \left[e^{0.04x} \right]_0^{15} = 400\pi \left(e^{0.6} - 1 \right)$.

The volume of revolution is $400\pi \left(e^{0.6} - 1 \right)$ cm^3.

iii) The volume in part ii) is 1033 ml to the nearest ml, so 33 ml of liquid need removing to bring the volume down to one litre.

iv) We need to find the rate at which the volume changes with respect to the height of the liquid, i.e. we need to find $\frac{dV}{dx}$.

Now $V = \pi \int 16e^{0.04x} \, dx$, so by removing the integral sign, $\frac{dV}{dx} = 16\pi \left(e^{0.04x} \right)$.

v) When the glass is full, i.e. when $x = 15$, the value of $\frac{dV}{dx}$ is $16\pi \left(e^{0.6} \right)$ or 92.

This result means, that when starting from a full glass, the volume of liquid decreases at the rate of 92ml for each cm below the rim, or 9.2ml for each mm.

We need to remove 33ml of liquid from the full glass to make an exact litre, and therefore the resulting loss in height of the liquid level is $\frac{33}{9.2}$ mm or 3.6 mm.

$\therefore$ the 1-litre mark should be placed between 3 and 4 mm from the rim of the glass.