M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

AQA : **C4** Edexcel: **C4** OCR: **C4** OCR MEI: **C3**

IMPLICIT DIFFERENTIATION

$$\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \times \frac{dy}{dx}$$

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \times \frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$\frac{d}{dx}(2x^2 + y^2) = \frac{d}{dx}(2x^2) + \frac{d}{dx}(y^2) \;\Rightarrow\; \frac{d}{dx}(2x^2 + y^2) = 4x + 2y\frac{dy}{dx}$$

$$y^2 + 6x = x^2$$

$$\Rightarrow \frac{d}{dx}(y^2) + \frac{d}{dx}(6x) = \frac{d}{dx}(x^2) \;\Rightarrow\; 2y\frac{dy}{dx} + 6 = 2x \;\Rightarrow\; 2y\frac{dy}{dx} = 2x - 6 \;\Rightarrow\; \frac{dy}{dx} = \frac{x-3}{y}$$

$$xy^2 + 5x + 20 = 8xy$$

$$\Rightarrow \frac{d}{dx}(xy^2) + \frac{d}{dx}(5x + 20) = \frac{d}{dx}(8xy) \;\Rightarrow\; y^2 + 2xy\frac{dy}{dx} + 5 = 8y + 8x\frac{dy}{dx}$$

$$\Rightarrow 2xy\frac{dy}{dx} = 8y + 8x\frac{dy}{dx} - y^2 - 5 \;\Rightarrow\; (2xy - 8x)\frac{dy}{dx} = 8y - y^2 - 5 \;\Rightarrow\; \frac{dy}{dx} = \frac{y(8-y)-5}{2x(y-4)}$$

$$\frac{d}{dx}(xy^2) = \left(\frac{d}{dx}(x) \times (y^2)\right) + \left((x) \times \frac{d}{dx}(y^2)\right) \;\Rightarrow\; \frac{d}{dx}(xy^2) = y^2 + 2xy\frac{dy}{dx}$$

$$\frac{d}{dx}(2xy) = \left(\frac{d}{dx}(2x) \times (y)\right) + \left((2x) \times \frac{d}{dx}(y)\right) \Rightarrow\; \frac{d}{dx}(2xy) = 2y + 2x\frac{dy}{dx}$$

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IMPLICIT DIFFERENTIATION

A function involving x and y is said to be **explicit** if y is clearly defined as a function of x alone.

$y = 2x^2 + 3x - 5$ is an example of an explicit function; y is on the LHS, and the RHS is expressed fully in terms of x.

$y^2 + 3xy = 3x^2$ and $x^2 + y^2 = 16$, on the other hand, are **implicit** functions involving x and y.

Sometimes an implicit equation in x and y can be rearranged into an explicit one by separating the variables.

Example (1): Differentiate $xy = 3$ with respect to x.

The equation can be rearranged as $y = \frac{3}{x}$, and from there we can directly work out $\frac{dy}{dx} = \frac{-3}{x^2}$.

Example (2): Differentiate $x^2 - 2y^2 = 7$ with respect to x.

Again we can separate the variables: $x^2 - 2y^2 = 7 \Rightarrow x^2 - 7 = 2y^2$

$$\Rightarrow y^2 = \frac{x^2 - 7}{2} \Rightarrow y = \sqrt{\frac{x^2 - 7}{2}}$$

Having obtained y explicitly in terms of x, we can differentiate using the chain rule.

Let $u = \frac{x^2 - 7}{2} \Rightarrow \frac{dy}{du} = \frac{1}{2\sqrt{u}}$ and $\frac{du}{dx} = x$, $\Rightarrow \frac{dy}{dx} = \frac{x}{2\sqrt{\left(\frac{x^2 - 7}{2}\right)}}$

When an equation in x and y is given implicitly and the variables cannot be separated, we find $\frac{dy}{dx}$ by differentiating each term with respect to x. We must also remember that $\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \times \frac{dy}{dx}$ by the chain rule.

Example (3): Differentiate $2x^2 + y^2$ with respect to x.

Here there is no right-hand side to the expression, and so we cannot separate the variables as in the first two examples, and so the result will be given in terms of x, y and $\frac{dy}{dx}$.

$$\frac{d}{dx}(2x^2 + y^2) = \frac{d}{dx}(2x^2) + \frac{d}{dx}(y^2) \quad \Rightarrow \quad \frac{d}{dx}(2x^2 + y^2) = 4x + 2y\frac{dy}{dx}.$$

Note the use of the chain rule in when differentiating y^2 with respect to x.

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \times \frac{dy}{dx} = 2y\frac{dy}{dx}.$$

Example (4): Differentiate $2xy$ with respect to x.

We need to use the chain rule and the product rule here.

$$\frac{d}{dx}(2xy) = \left(\frac{d}{dx}(2x)\times(y)\right) + \left((2x)\times\frac{d}{dx}(y)\right) \Rightarrow \frac{d}{dx}(2xy) = 2y + 2x\frac{dy}{dx}.$$

Example (5): Differentiate xy^2 with respect to x.

Again, we need to use the chain rule and the product rule.

$$\frac{d}{dx}(xy^2) = \left(\frac{d}{dx}(x)\times(y^2)\right) + \left((x)\times\frac{d}{dx}(y^2)\right) \Rightarrow \frac{d}{dx}(xy^2) = y^2 + 2xy\frac{dy}{dx}.$$

Example (6): Find $\frac{dy}{dx}$ in terms of x and y if $y^2 + 6x = x^2$.

This time, there is a little more algebra involved, where we need to bring the $\frac{dy}{dx}$ term over to the LHS.
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Differentiating each term with respect to x we have

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(6x) = \frac{d}{dx}(x^2).$$

$$\Rightarrow 2y\frac{dy}{dx} + 6 = 2x \quad \Rightarrow 2y\frac{dy}{dx} = 2x - 6 \Rightarrow \frac{dy}{dx} = \frac{x-3}{y}.$$

Example (7): Find the gradient of the tangent to the curve $x^2 - 2y^2 = 7$ (from Example 2) at the point (5, 3) by implicit methods.

We could work out the gradient at $x = 5$ by substituting into the explicit derivative

$$\frac{dy}{dx} = \frac{x}{2\sqrt{\left(\frac{x^2-7}{2}\right)}} \quad \text{(from Ex. 2)} \Rightarrow \quad \frac{dy}{dx} = \frac{5}{2\sqrt{9}} = \frac{5}{6}.$$

The question asks for implicit methods, so rewriting as $x^2 - 2y^2 - 7 = 0$ and differentiating term by term will produce an expression for $\frac{dy}{dx}$ in terms of x and y.

$$\frac{d}{dx}(x^2 - 2y^2 - 7) = \frac{d}{dx}(x^2) - \frac{d}{dx}(2y^2) \quad = 2x - 4y\frac{dy}{dx}.$$

Rearranging $= 2x - 4y\frac{dy}{dx} = 0$ to make $\frac{dy}{dx}$ the subject, we have $4y\frac{dy}{dx} = 2x$
and finally $\frac{dy}{dx} = \frac{x}{2y}$.

Substituting $x = 5$ and $y = 3$ gives a gradient of $\frac{5}{6}$ at that point.

Example (8): Find $\frac{dy}{dx}$ in terms of x and y if $xy^2 + 5x + 20 = 8xy$.

This example is a little trickier, since here we need to use the product rule to differentiate xy^2 and $8xy$.

$$\frac{d}{dx}(xy^2) = \left(\frac{d}{dx}(x) \times (y^2)\right) + \left((x) \times \frac{d}{dx}(y^2)\right) \Rightarrow \frac{d}{dx}(xy^2) = y^2 + 2xy\frac{dy}{dx}.$$

$$\frac{d}{dx}(8xy) = \left(\frac{d}{dx}(8x) \times (y)\right) + \left((8x) \times \frac{d}{dx}(y)\right) \Rightarrow \frac{d}{dx}(8xy) = 8y + 8x\frac{dy}{dx}.$$

Having worked those two products out, we evaluate the whole derivative term by term and rearrange to make $\frac{dy}{dx}$ the subject .

$$\frac{d}{dx}(xy^2) + \frac{d}{dx}(5x + 20) = \frac{d}{dx}(8xy)$$

$$\Rightarrow \quad y^2 + 2xy\frac{dy}{dx} + 5 = 8y + 8x\frac{dy}{dx}$$

$$\Rightarrow \quad 2xy\frac{dy}{dx} = 8y + 8x\frac{dy}{dx} - y^2 - 5$$

$$\Rightarrow \quad (2xy - 8x)\frac{dy}{dx} = 8y - y^2 - 5$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y(8-y) - 5}{2x(y-4)}$$

Examination questions usually include revision of AS-level techniques such as maxima, minima, tangents and normals.

Example (9): (Extension to Example (8)).

Find the equation of i) the tangent to the curve $xy^2 + 5x + 20 = 8xy$ at the point (-4, 0); ii) the normal to the same curve at the point (2,5).

Part i) Substitute $(x, y) = (-4, 0)$ into the expression

$\frac{dy}{dx} = \frac{y(8-y)-5}{2x(y-4)}$ obtained in the previous question to obtain the gradient of the tangent at that point. The gradient at (-4, 0) is $\frac{dy}{dx} = \frac{-5}{(-8)(-4)} = -\frac{5}{32}$.

The equation of the tangent to the curve, in the form $y = mx + c$, is

$$y - 0 = -\frac{5}{32}(x+4) \Rightarrow y = -\frac{5}{32}x - \frac{5}{8}.$$

(In the form $ax + by + c = 0$, the equation is $32y = -5x - 20 \Rightarrow 5x + 32y + 20 = 0$.)

Part ii) Substitute $(x, y) = (2, 5)$ into the expression for $\frac{dy}{dx}$ in i) to obtain the gradient of the tangent at that point.

This gradient is $\frac{5(8-5)-5}{4(5-4)} = \frac{10}{4} = \frac{5}{2}$, so the gradient of the normal at (2, 5) is $-\frac{2}{5}$.

The equation of the normal to the curve, in the form $y = mx + c$, is

$$y - 5 = -\frac{2}{5}(x-2) \Rightarrow y = 5 - \frac{2}{5}x + \frac{4}{5} \Rightarrow y = 5\frac{4}{5} - \frac{2}{5}x.$$

(In the form $ax + by + c = 0$, the equation is $5y - 25 = -2x + 4 \Rightarrow 2x + 5y - 29 = 0$.)