# M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

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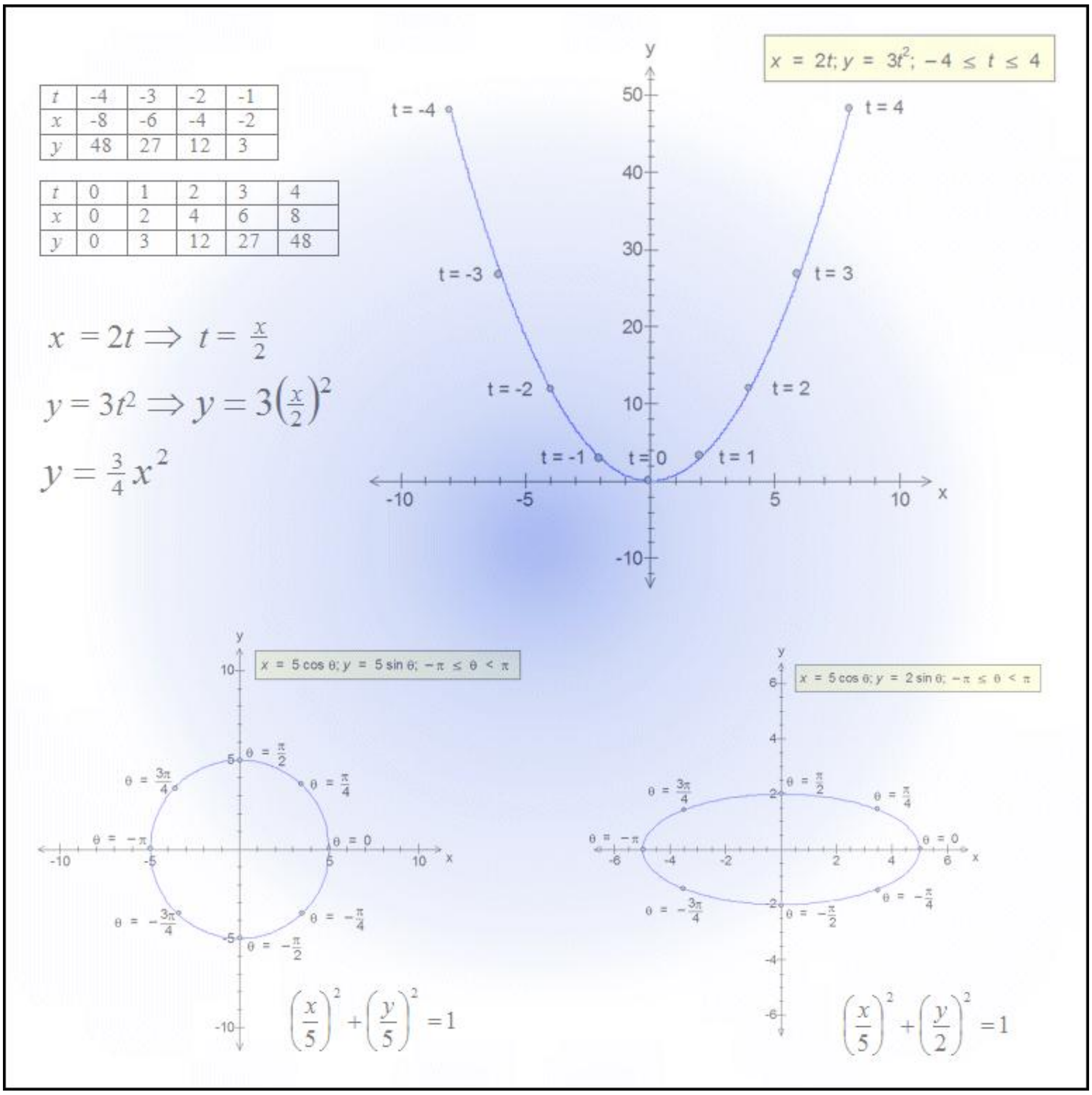
# CARTESIAN AND PARAMETRIC FORM

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**Cartesian Form.**

A curve in Cartesian form is defined in terms of the two variables *x* and *y* only.

The quadratic equation $y = x^2 + 7x + 10$ is an example where *y* is given **explicitly** in terms of *x*.
By contrast, the equation $x^2 - 2xy + y^2 = 16$ is stated **implicitly**.

**Parametric Form.**

A curve is said to be in **parametric form** if the variables *x* and *y* are expressed in terms of a third variable. For example, *x* could be defined as $f(t)$ and *y* as $g(t)$. The variable *t* is the parameter in this case.

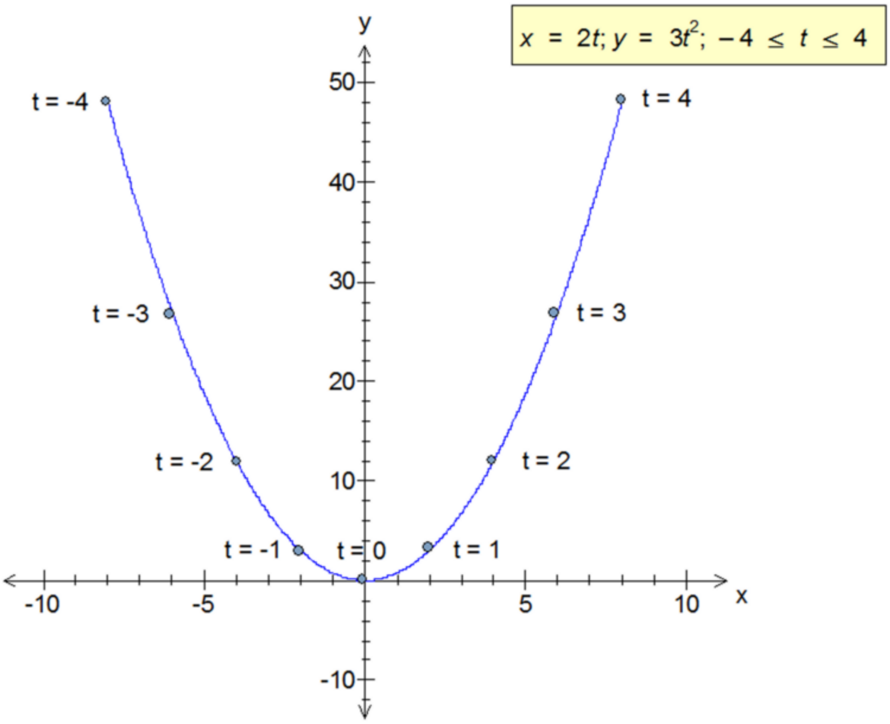
When plotting a parametric curve, it is necessary to work out *x* and *y* whilst varying the values for the parameter.

**Example (1):** Plot the curve $x = 2t$ , $y = 3t^2$ for $-4 \le t \le 4$, and give the Cartesian equation of the curve.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| *t* | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
| *x* | -8 | -6 | -4 | -2 | 0 | 2 | 4 | 6 | 8 |
| *y* | 48 | 27 | 12 | 3 | 0 | 3 | 12 | 27 | 48 |

The Cartesian equation for the curve can be obtained by eliminating *t*, the parameter.

If $x = 2t$, then $t = \frac{x}{2}$ . Substituting into $y = 3t^2$ gives the Cartesian equation for the curve, which is a parabola with equation $y = \frac{3}{4}x^2$.



**Example (2):** A curve is defined by the parametric equations $x = t^2 - 8t + 6$ , $y = t - 4$, for $t > 4$.

Express the equation of the curve in Cartesian form with $y$ as the subject.

We complete the square with $x = t^2 - 8t + 6 \Rightarrow x = (t - 4)^2 - 10 \Rightarrow x = y^2 - 10$ (eliminating $t$)

$\Rightarrow y^2 = x + 10 \Rightarrow y = \sqrt{x+10}$ .

**Example (3):** A curve is defined parametrically as $x = 4t + 3t^2$ , $y = 4t^2 + 3t^3$ .

Find its Cartesian equation in the form, $ax^3 + bxy + cy^2$ where $a$, $b$ and $c$ are integer constants.

We spot $\frac{y}{x} = \frac{4t^2 + 3t^3}{4t + 3t^2} \Rightarrow \frac{y}{x} = t$ , and substituting for $t$ in the (easier) equation for $x$

we have $x = \frac{4y}{x} + \frac{3y^2}{x^2}$ , and after multiplying by $x^2$ on both sides, $x^3 = 4xy + 3y^2$

and finally $x^3 - 4xy - 3y^2 = 0$ .

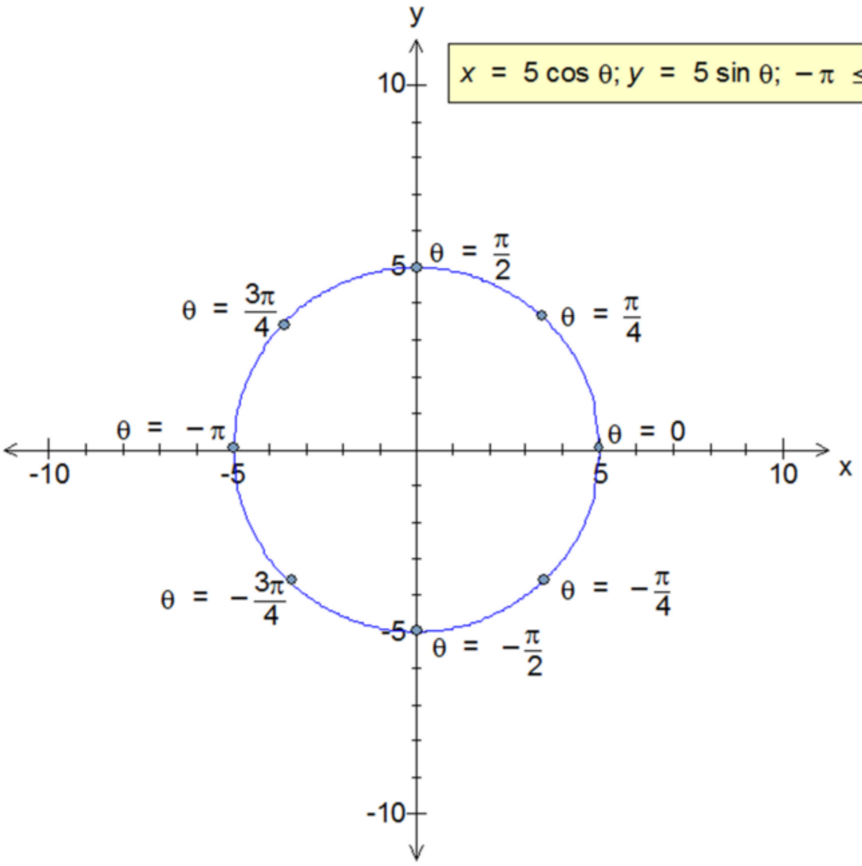
When the parameter is θ, trigonometric identities may be required.

**Example (4):** Plot the curve $x = 5 \cos \theta$ , $y = 5 \sin \theta$ for $-\pi \le \theta < \pi$, and give the Cartesian equation of the curve.

(The approximation $5/\sqrt{2} = 3.54$ is used here.)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $t$ | $-\pi$ | $-3\pi/4$ | $-\pi/2$ | $-\pi/4$ | 0 | $\pi/4$ | $\pi/2$ | $3\pi/4$ | $\pi$ |
| $x$ | -5 | -3.54 | 0 | 3.54 | 5 | 3.54 | 0 | -3.54 | -5 |
| $y$ | 0 | -3.54 | -5 | -3.54 | 0 | 3.54 | 5 | 3.54 | 0 |

Starting at $-\pi$, the values of θ are measured anticlockwise from the negative $x$ –axis, here at (-5, 0).




The curve is a circle – to find the Cartesian equation we use the identity $\cos^2\theta + \sin^2\theta = 1$.

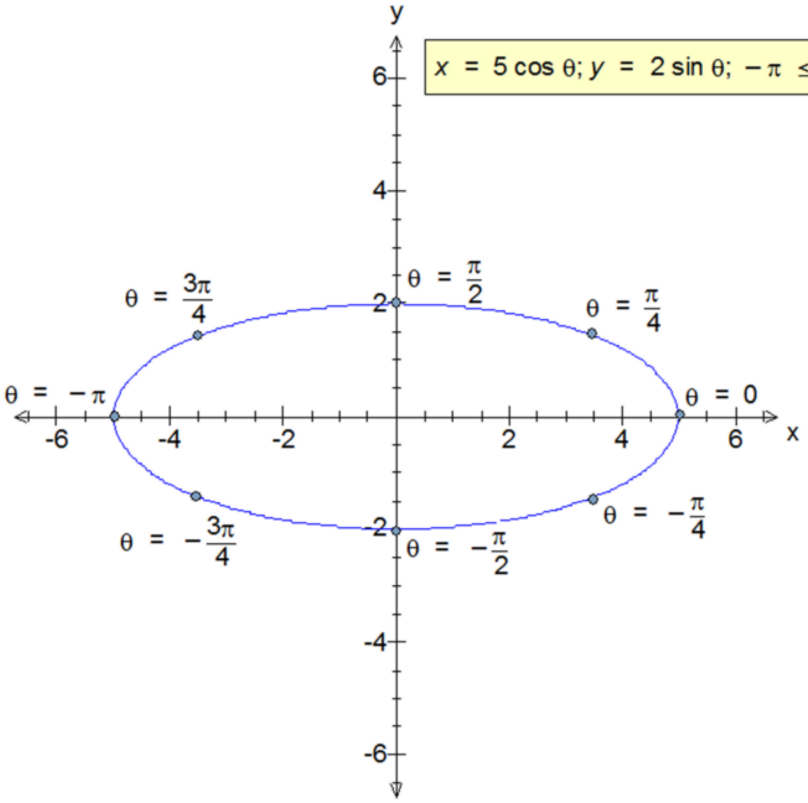
The Cartesian equation is therefore $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$ or $x^2 + y^2 = 25$.

The general Cartesian equation of the circle $x = r \cos \theta$ , $y = r \sin \theta$ is thus $x^2 + y^2 = r^2$.

**Example (5):** Plot the curve $x = 5 \cos \theta$ , $y = 2 \sin \theta$ for $-\pi \le \theta < \pi$, and give the Cartesian equation of the curve.

(The approximations $2/\sqrt{2} = 1.41$ and $5/\sqrt{2} = 3.54$ are used here.)

| $t$ | $-\pi$ | $-3\pi/4$ | $-\pi/2$ | $-\pi/4$ | 0 | $\pi/4$ | $\pi/2$ | $3\pi/4$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | -5 | -3.54 | 0 | 3.54 | 5 | 3.54 | 0 | -3.54 | -5 |
| $y$ | 0 | -1.41 | -2 | -1.41 | 0 | 1.41 | 2 | 1.41 | 0 |




This time, the curve is an ellipse, and again we use the identity $\cos^2\theta + \sin^2\theta = 1$ to find the Cartesian equation of the curve.

Here it is $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ .

The general Cartesian equation of the ellipse $x = a \cos \theta$ , $y = b \sin \theta$ is thus $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

**Example (6):** The parametric equation of a curve is $x = \cos\theta$, $y = 1 + 2\cos 2\theta$, $0 \le \theta \le \pi$ .

Show that it coincides with part of the curve $y = (2x + 1)(2x - 1)$, and state its domain and range.

Using the double angle identity , we have $\cos 2\theta = 2\cos^2\theta - 1$, hence $y = 1 + 2(2\cos^2\theta - 1)$.

Substituting $x = \cos\theta$ gives $y = 1 + 2(2x^2 - 1) \Rightarrow y = 4x^2 - 1$, factorising to $y = (2x + 1)(2x - 1)$..

Because $\cos\theta$ can only take values between -1 and 1 inclusive, the domain is $-1 \le x \le 1$.
The corresponding range is $-1 \le y \le 3$, as shown by the graph below.