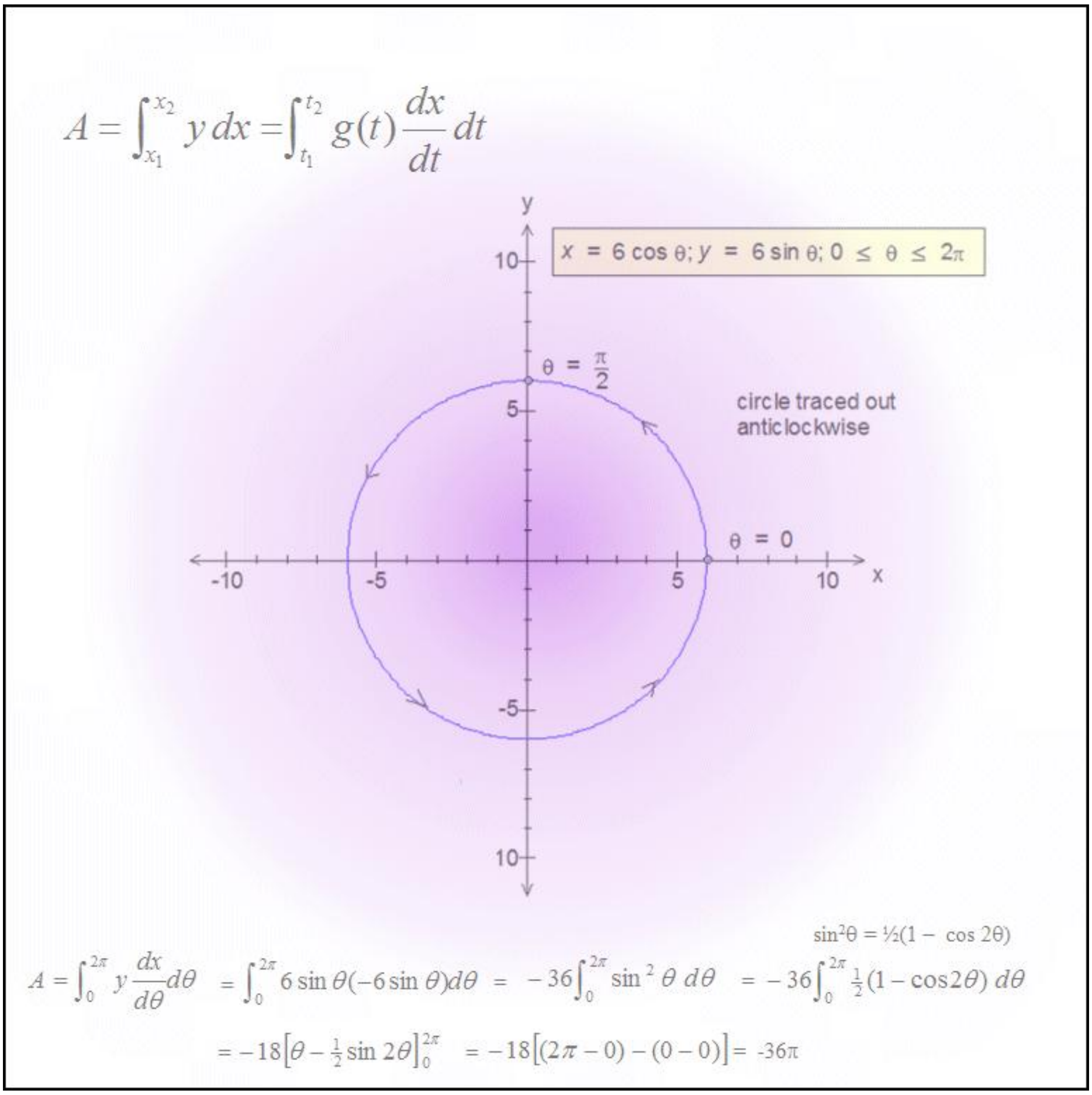
M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

Edexcel: **C4**

PARAMETRIC INTEGRATION



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PARAMETRIC INTEGRATION
(Edexcel only)

A curve can be defined in parametric form - for instance, x could be defined as $f(t)$ and y as $g(t)$, with t as the parameter.

The formula for the area under a curve can be adapted for parametric integrals as follows:

$$A = \int_{x_1}^{x_2} y\,dx = \int_{t_1}^{t_2} g(t)\frac{dx}{dt}\,dt.$$

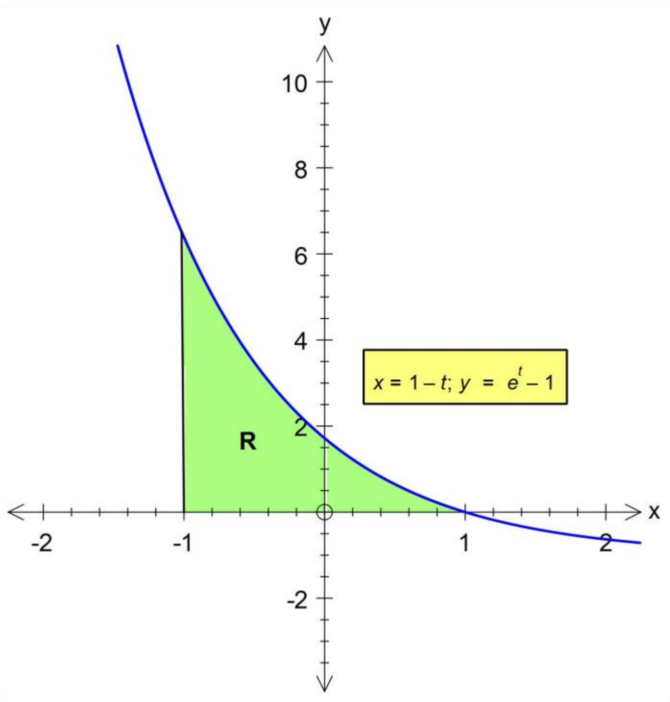
Example (1):

A curve is defined parametrically as $x = 1 - t$, $y = e^t - 1$.

Its graph is shown on the right.

Using parametric methods, find the area of region **R** enclosed by the curve, the x-axis and the line $x = -1$.

Give your answer in an exact form.



Preparatory working :

Change x – limits to t – limits; $x = 1 - t \Rightarrow t = 1 - x$, so when $x = 1, t = 0$; $x = -1, t = 2$.

Find $\frac{dx}{dt}$; here it is simply -1.

The area **R** under the curve is given by

$$\int_{x=-1}^{x=1} y\,dx = \int_{t=2}^{t=0} (e^t - 1)\frac{dx}{dt}\,dt = \int_{t=2}^{t=0} (e^t - 1)(-1)\,dt = \int_{t=0}^{t=2} (e^t - 1)\,dt$$

$$= \left[e^t - t\right]_0^2 = \left[(e^2 - 2) - (1 - 0)\right] = e^2 - 3.$$

Note the reversal of the limits and the multiplication by -1 in the working.

Example (2): The curve shown here has parametric equations

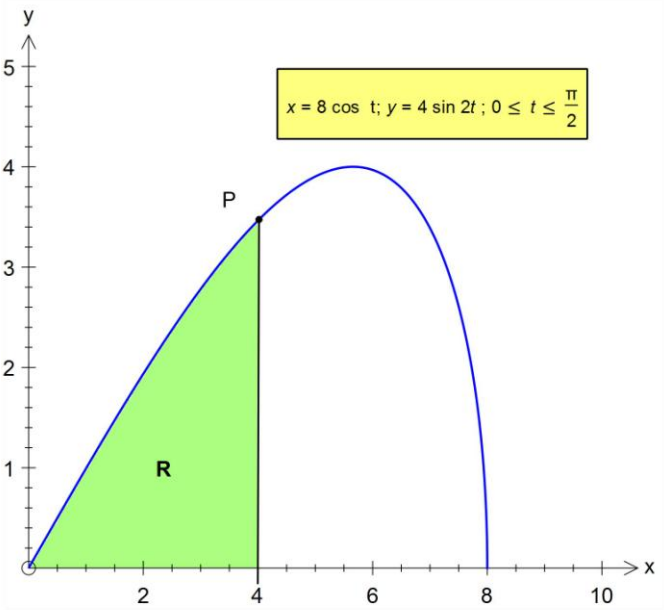
$x = 8 \cos t$, $y = 4 \sin 2t$, where $0 \le t \le \pi/2$.

The point P lies on the curve and its coordinates are $(4, 2\sqrt{3})$.

i) Find the value of t at the point P.

ii) Use parametric integration to find the area of region **R** enclosed by the curve, the x-axis and the line $x = 4$, giving the answer in an exact form.

Hint : $\frac{d}{dt}\left(\sin^3 t\right) = 3\sin^2 t \cos t$.



i) At point P, $8 \cos t = 4 \Rightarrow \cos t = \frac{1}{2} \Rightarrow t = \frac{\pi}{3}$.

ii) $\frac{dx}{dt} = -8\sin t$; also at $x = 0$, $8 \cos t = 0 \Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2}$.

To find the area of region **R** we transform the integrand from $\int_{x=0}^{x=4} y\,dx$ to

$$\int_{t=\pi/2}^{t=\pi/3} 4\sin 2t \frac{dx}{dt}\,dt = \int_{\pi/2}^{\pi/3} 4\sin 2t(-8\sin t)\,dt = \int_{\pi/2}^{\pi/3} 8\sin t\cos t(-8\sin t)\,dt$$

$$= \int_{\pi/2}^{\pi/3} \left(-64\sin^2 t\right)\cos t\,dt = \int_{\pi/3}^{\pi/2} 64\sin^2 t \cos t\,dt.$$

This final integrand is the inverse of the result $\frac{d}{dt}\left(\sin^3 t\right) = 3\sin^2 t\cos t$ stated earlier in the hint, but multiplied by the scale factor of $\frac{64}{3}$.

The area of **R** is therefore $\int_{\pi/3}^{\pi/2} 64\sin^2 t \cos t\,dt = \left[\frac{64}{3}\sin^2 t\right]_{\pi/3}^{\pi/2} = \frac{64}{3}\left(1 - \frac{3\sqrt{3}}{8}\right)$

$$= \frac{64}{3} - \left(\frac{\cancel{64}^{8}}{\cancel{3}} \times \frac{\cancel{3}\sqrt{3}}{\cancel{8}}\right) = \frac{64}{3} - 8\sqrt{3}.$$

For enclosed curves, a negative integral results from tracing the curve in an anticlockwise direction; a positive one results from tracing the curve clockwise.

Example (3): Using parametric integration, find the area of the circle defined by $x=6\cos\theta$, $y=6\sin\theta$, for $0 \le \theta \le 2\pi$.

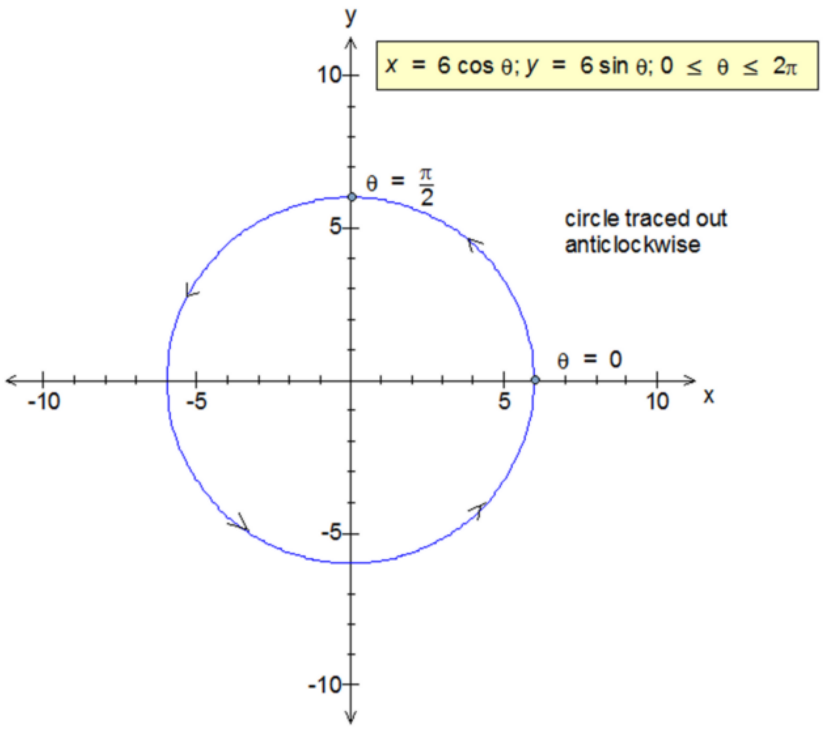
$$A = \int_0^{2\pi} y\frac{dx}{d\theta}d\theta = \int_0^{2\pi} 6\sin\theta(-6\sin\theta)d\theta = -36\int_0^{2\pi}\sin^2\theta\, d\theta$$

We use the trigonometric identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ to evaluate the integral.

$$-36\int_0^{2\pi}\sin^2\theta\, d\theta = -36\int_0^{2\pi}\tfrac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= -18\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} = -18\left[(2\pi - 0) - (0 - 0)\right] = -36\pi.$$

(The resulting integral is negative because the circle has been traced out in an anticlockwise direction.)



Example (4): Using parametric integration, find the area of the ellipse defined by $x=5\cos\theta$, $y=2\sin\theta$, for $0 \le \theta \le 2\pi$.

$$A = \int_0^{2\pi} y\frac{dx}{d\theta}d\theta = \int_0^{2\pi} 2\sin\theta(-5\sin\theta)d\theta = -10\int_0^{2\pi}\sin^2\theta\, d\theta$$

Again using $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, the integrand becomes

$$-10\int_0^{2\pi}\tfrac{1}{2}(1-\cos 2\theta)\, d\theta = -5\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} = -5\left[(2\pi - 0) - (0 - 0)\right] = -10\pi.$$

The last examples lead us to the general result: for any ellipse defined by $x=a\cos\theta$, $y=b\sin\theta$, the area will be πab.