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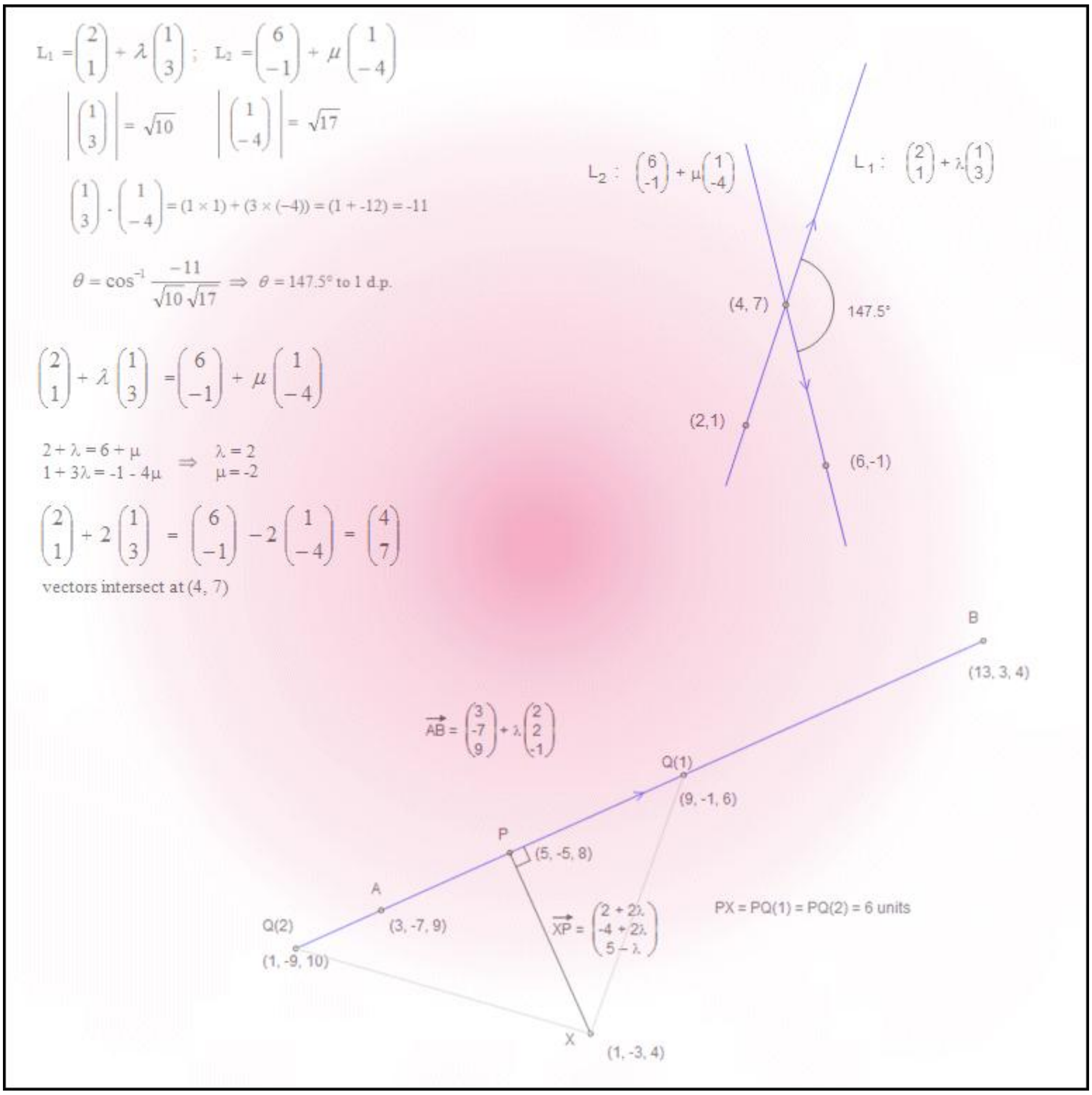
M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

AQA : **C4** Edexcel: **C4** OCR: **C4** OCR MEI: **C4**

VECTORS



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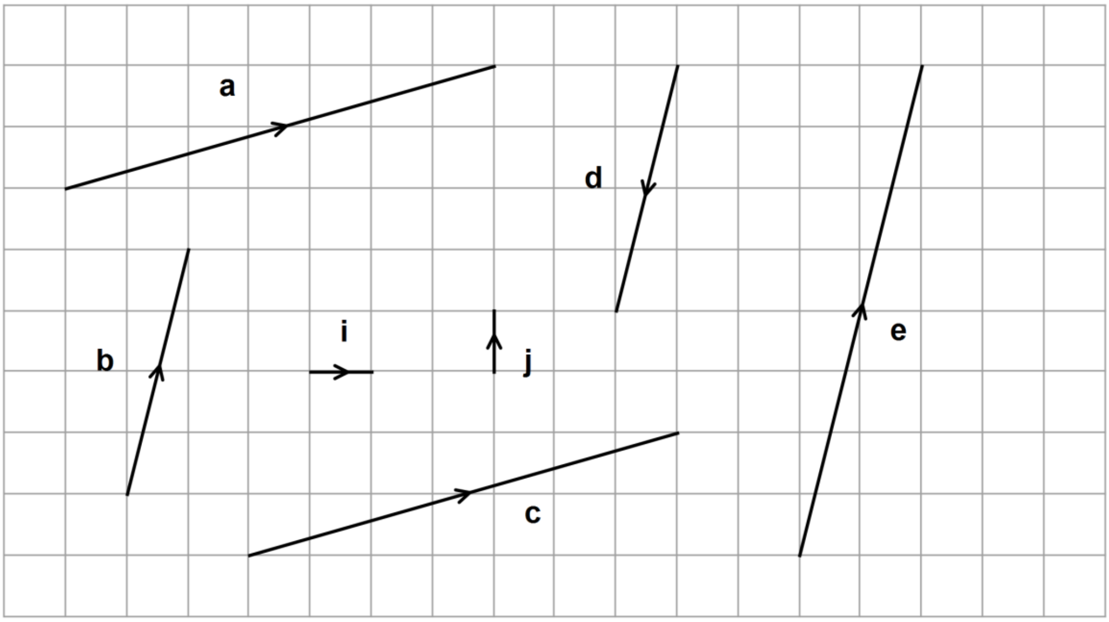
Vectors.

(This version of the document uses column notation over i-j-k notation).

Introductory Revision. (Two dimensions).

Vectors are used in mathematics to illustrate quantities that have size (magnitude) and direction.
Quantities like mass and length have magnitude only, and are called **scalars**.
Velocity and force, on the other hand, have direction as well as size and can be expressed as vectors.

There are various ways of denoting vectors: typed documents use boldface, but written work uses underlining. Thus **a** and <u>a</u> are the same vector.



Example (1): The diagram above shows a collection of vectors in the plane.

Describe the relationships between the following vector pairs :
i) **a** and **c** ; ii) **b** and **d** ; iii) **b** and **e ;** iv) **i** and **j**.

i) Vectors **a** and **c** are equal here; hence **a** = **c**.
Two vectors are equal if they have the same size and the same direction.

The fact that **a** and **c** have different start and end points is irrelevant.

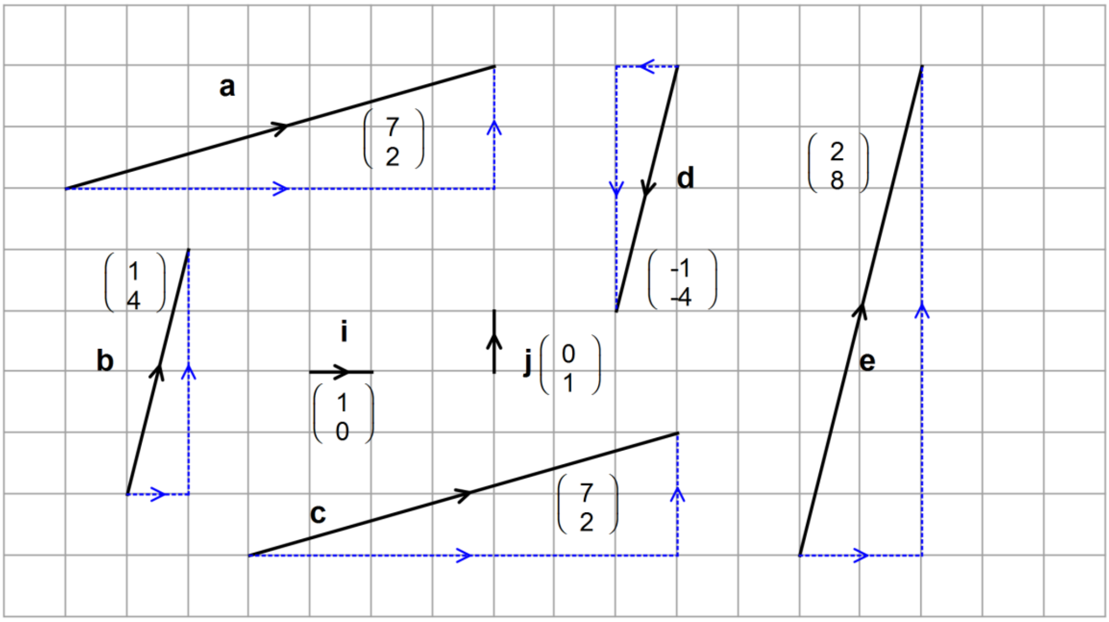
ii) Vectors **b** and **d** have the same size, but opposite directions, therefore **d** = **-b**.
Two vectors are inverses of each other if they have the same size, but opposite directions.

iii) Vector **e** is exactly twice as long as vector **b**, so **e** = 2**b**.
The 2 is what is known as a **scalar** multiplier.
(A scalar multiplier of -1 signifies an inverse vector.)

iv) Vectors **i** and **j** are perpendicular to each other.

Column Vector Notation.

Example (2): Express the six vectors in the last example in column notation.



Vectors can be conveniently expressed in column form, by defining them in terms of their horizontal and vertical components. This is termed **resolving** into components.

A movement in the direction of vector **a** corresponds to 7 units horizontally and 2 units vertically, using normal Cartesian convention, as does that in the direction of vector **c**, given that vectors **a** and **c** are equal .

Hence **a** = **c** = $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$. This is known as column notation.

For vector **b**, the values are 4 horizontally and 1 vertically. Vector **d** is the inverse of vector **b**, so the movement is -4 units horizontally and -1 unit vertically.

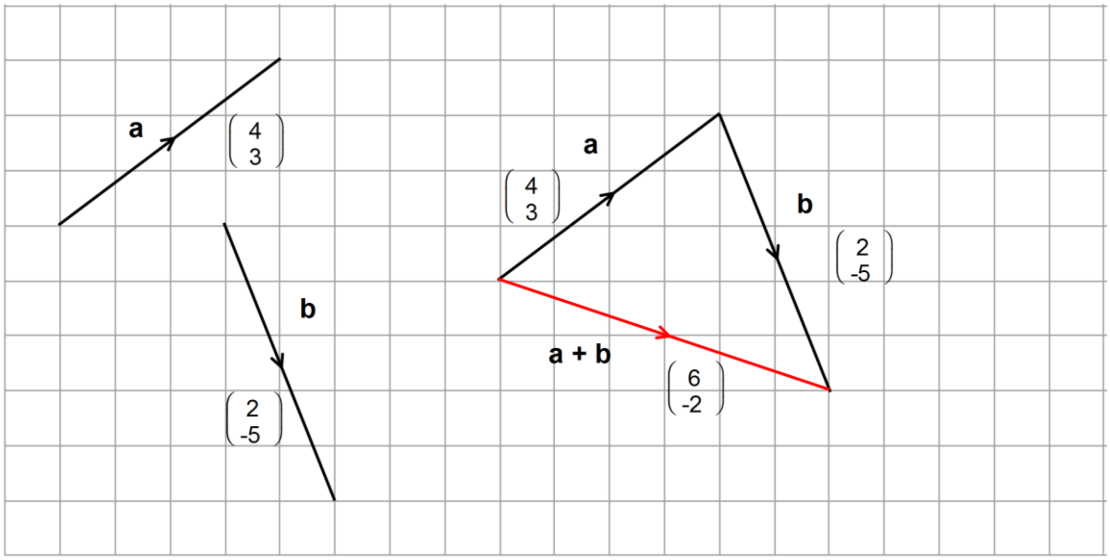
Hence **b** = $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and **d** = - $\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$.

Vector **e** is twice vector **b** , so **e** = $2\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$.

Finally the vector **i** = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the vector **j** = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. These are termed the **standard unit vectors.**

Addition of vectors.

Example (3):



To add two vectors, join them “nose to tail” as in the diagram.

In column notation, **a** = $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, **b** = $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ and **a**+**b** = $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$.

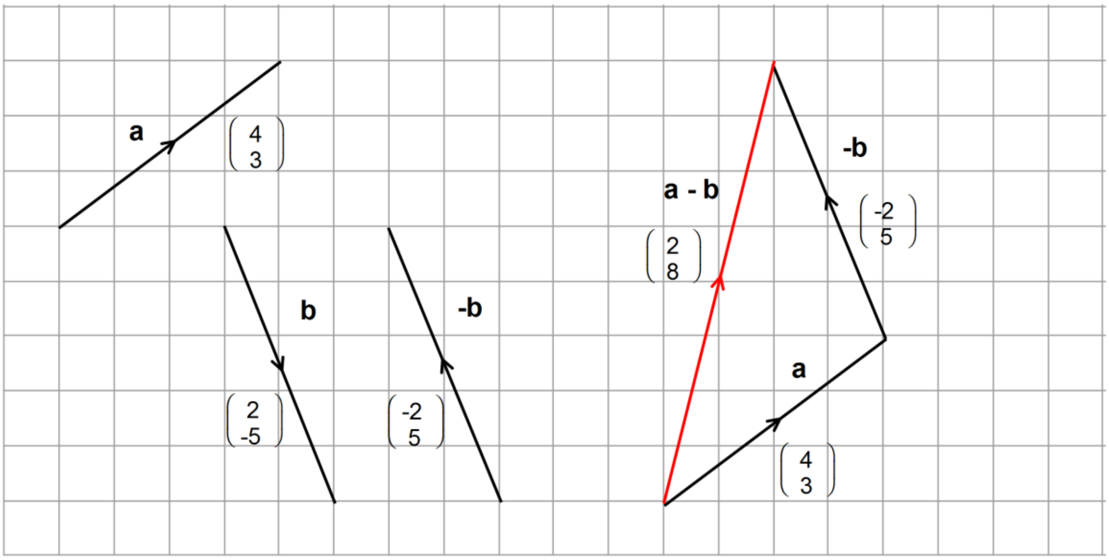
This result could also have been obtained without drawing the diagram.

a+**b** = $\begin{pmatrix} 4+2 \\ 3-5 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$.

Another term for the sum of two vectors is their **resultant**.

Subtraction of vectors.

Example (4):



Subtracting vector **b** from **a** is identical to adding the inverse of vector **b** to **a**.

This time, we join **–b** to **a** "nose to tail".

In column notation, **a** = $\begin{pmatrix}4\\3\end{pmatrix}$, **b** = $\begin{pmatrix}2\\-5\end{pmatrix}$, **-b** = $\begin{pmatrix}-2\\5\end{pmatrix}$ and **a-b =** **a** + **(-b)** = $\begin{pmatrix}2\\8\end{pmatrix}$.

This result could again have been obtained without drawing the diagram.

a-b = $\begin{pmatrix}4-2\\3+5\end{pmatrix}$ = $\begin{pmatrix}2\\8\end{pmatrix}$.

Addition or subtraction of vectors in column form is very easy - just add or subtract the components !

Another special case is **a-a** = $\begin{pmatrix}7-7\\2-2\end{pmatrix}$ = $\begin{pmatrix}0\\0\end{pmatrix}$.

The result here is the **zero vector**, **0**. This is not the same as the number 0, which is a scalar.

Standard Unit Vectors.

In Example (1), we came across two special vectors in the two-dimensional *x*-*y* plane :

i = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$; **j** = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. These are termed the **standard unit vectors** in two dimensions**.**

Vector **i** is parallel to the *x*-axis and vector **j** is parallel to the *y*-axis.

All two-dimensional vectors can also be expressed as combinations of **i** and **j**.
(This is also known as component form.)

Thus the vector **a** = $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ from Examples (3) and (4) can be expressed as 4**i** + 3**j** ,

and vector **b** = $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ as 2**i** – 5**j**.

Example (5): Let vector **r** = 3**i** - **j** and **s** = **i** + 4**j**.

Express the following in column form:

i) **r** + 3**s**; ii) 2**s** - **r**; iii) **r** + **i** - **j**.

i) **r** + 3**s** = $\begin{pmatrix} 3 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3+3 \\ -1+12 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \end{pmatrix}$

ii) 2**s** - **r** = $2\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2-3 \\ 8+1 \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \end{pmatrix}$

iii) **r** + **i** - **j** = $\begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

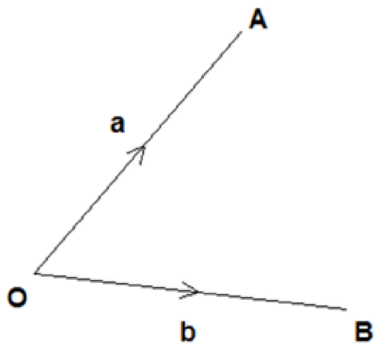
Another way of denoting vectors is by stating their end points and writing an arrow above them.

In the right-hand diagram, vector **a** joins points *O* and *A* and vector **b** joins point *O* and *B*.

Therefore $\overrightarrow{OA}$ = **a** and $\overrightarrow{OB}$ = **b**.

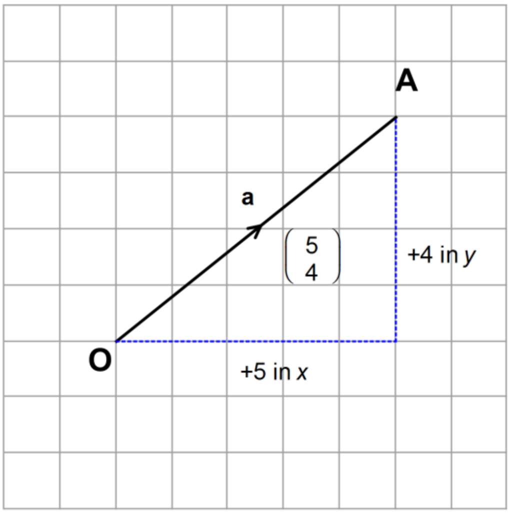
The direction of the arrow is important here;
the vector $\overrightarrow{AO}$ goes in the opposite direction to $\overrightarrow{OA}$ although it has the same magnitude.

Hence $\overrightarrow{AO}$ = - $\overrightarrow{OA}$ = -**a**.



The Magnitude of a Vector.

An important property of a vector is its **magnitude**, and it can be determined very easily by Pythagoras.



Since the vector **a** from Example (1) can be visualised as the hypotenuse of a right-angled triangle with a base of 5 units and a height of 4 units, its magnitude is simply

$\sqrt{5^2 + 4^2} = \sqrt{41}$ units.

In general, the magnitude of any vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is $\sqrt{a^2 + b^2}$.

Example (6): Find the magnitudes of the following vectors:

i) $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$; ii) $\begin{pmatrix} 0.28 \\ 0.96 \end{pmatrix}$

i) The magnitude of the vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is $\sqrt{3^2 + 4^2} = 5$ units.

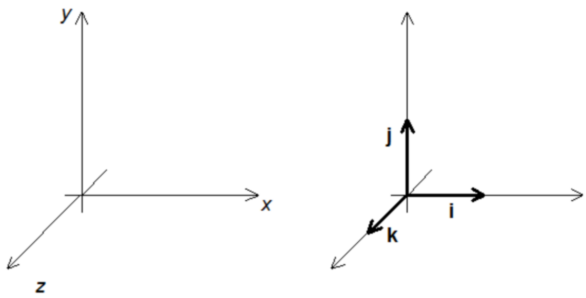
ii) The vector $\begin{pmatrix} 0.28 \\ 0.96 \end{pmatrix}$ has a magnitude of $\sqrt{0.28^2 + 0.96^2} = 1$ unit.

The vector in part (ii) is therefore a unit vector.

Vectors in three dimensions.

The concepts and methods shown up to now can be extended to three dimensions.

The *z*-axis is perpendicular to both the *x*-and *y*-axes, and by convention, the positive direction is 'out of the paper towards the eye'.



The three unit vectors in three dimensions are **i** or $\begin{pmatrix}1\\0\\0\end{pmatrix}$, **j** or $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and **k** or $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

Vector arithmetic is as straightforward in three dimensions as it is in two.

Examples (7): Let **p = 2i + 3j – 2k; q = -i + j – 2k; r = 3i + 4j ; s = 4j – k.**

Give the values of i) 2**p**; ii) –**q**; iii) 3**q** + **r**; iv) **r** – **s**.

In i) 2**p** = 2(2**i** + 3**j** – 2**k**) = 4**i** + 6**j** – 4**k** .

In column notation $2\mathbf{p} = 2\begin{pmatrix}2\\3\\-2\end{pmatrix} = \begin{pmatrix}4\\6\\-4\end{pmatrix}$

In ii) –**q** = -(-**i** + **j** – 2**k**) = **i** - **j** + 2**k**.

In column notation $-\mathbf{q} = -\begin{pmatrix}-1\\1\\-2\end{pmatrix} = \begin{pmatrix}1\\-1\\2\end{pmatrix}$.

In iii) 3**q** + **r** = 3(-**i** + **j** – 2**k**) + (3**i** + 4**j**) = 7**j** – 6**k**.

In column notation $3\mathbf{q} + \mathbf{r} = 3\begin{pmatrix}-1\\1\\-2\end{pmatrix} + \begin{pmatrix}3\\4\\0\end{pmatrix} = \begin{pmatrix}-3\\3\\-6\end{pmatrix} + \begin{pmatrix}3\\4\\0\end{pmatrix} = \begin{pmatrix}0\\7\\-6\end{pmatrix}$.

In iv) **r** – **s** = (3**i** + 4**j**) – (4**j** – **k**) = 3**i** + **k**.

In column notation $\mathbf{r} - \mathbf{s} = \begin{pmatrix}3\\4\\0\end{pmatrix} - \begin{pmatrix}0\\4\\-1\end{pmatrix} = \begin{pmatrix}3\\0\\1\end{pmatrix}$.

The magnitude of a three-dimensional vector $\mathbf{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ or $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is again given by applying Pythagoras' theorem.

$|\mathbf{r}| = \sqrt{a^2 + b^2 + c^2}$.

The working is the same as that with two-dimensional vectors.

Examples (8): Find the magnitudes of the following vectors:

i) $\mathbf{p} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$; ii) $\mathbf{q} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

In i), $|\mathbf{p}|$ is $\sqrt{3^2 + 2^2 + 6^2}$ or 7.
In ii), $|\mathbf{q}|$ is $\sqrt{2^2 + (-2)^2 + 1^2}$ or 3.

Unit vectors.

A unit vector is any vector with a magnitude of 1. Examples include the standard unit vectors **i** and **j** (in two dimensions) and **i**, **j** and **k** (in three dimensions).

A unit vector is denoted by a 'hat' over the symbol, i.e. $\hat{\mathbf{a}}$, and its length, $|\hat{\mathbf{a}}| = 1$.

Examples (9) : Find the unit vectors parallel to i) $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$; ii) $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

i) The length of the vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is $\sqrt{4^2 + 3^2}$ or 5, so the parallel unit vector is

$\frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, or $\begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix}$.

ii) The length of the vector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is $\sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$, so the parallel unit vector is

$\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, or $\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$.

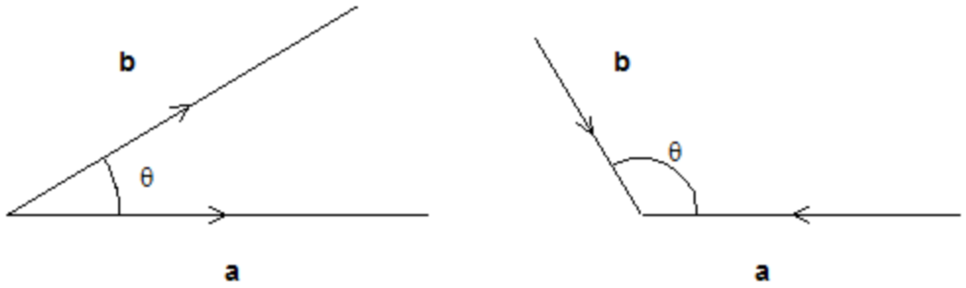
The scalar product (dot product).

Take the vectors **a** = $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and **b** = $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$.

The length of each vector is therefore |**a**| = $\sqrt{x_1^{\ 2} + y_1^{\ 2} + z_1^{\ 2}}$ and |**b**| = $\sqrt{x_2^{\ 2} + y_2^{\ 2} + z_2^{\ 2}}$.

The **scalar product** of **a** and **b**, also known as the **dot product**, is the product of the lengths of **a** and **b**, multiplied by the cosine of the angle between them. As its name suggests, it has magnitude but no direction.

The scalar product is denoted by **a.b** where **a.b** = |**a**||**b**| cos **θ**.



The two vectors must be joined 'tail to tail' or 'nose to nose' to visualise the correct angle between them, as per the above diagram.

Because cos 0° = 1 and cos 90° = 0, we have the following dot product results;

When two vectors are parallel, their dot product is simply the product of their lengths.
In other words, θ = 0° and **a.b** = |**a**|.|**b**|

On the other hand, the dot product of two perpendicular vectors is 0.
This time, θ = 90° and **a.b** = 0

By applying these results to the three unit vectors, it follows that **i.i** = **j.j** = **k.k** = 1; **i.j** = **j.k** = **k.i** = 0.

Using the above rules, we can expand the scalar product of **a** = x_1**i** + y_1**j** + z_1**k** + and **b** = x_2**i** + y_2**j** + z_2**k** as (x_1**i.** x_2**i** + y_1**j.** y_2**j** + z_1**k.** z_2**k**)

= ($x_1 x_2 + y_1 y_2 + z_1 z_2$).

In column vectors, the scalar product is represented as **a.b** = $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} . \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$.

Examples (10): Take the following vectors:

$\mathbf{p} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$; $\mathbf{q} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$; $\mathbf{r} = \begin{pmatrix} -12 \\ 5 \end{pmatrix}$; $\mathbf{s} = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$

i) Find **p.q**, **r.s** and **p.s** . Which two vectors are perpendicular ?
ii) Find the angles between **p** and **q**, also between **r** and **s**.

$\mathbf{p.q} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \end{pmatrix} =$ (3 × (-1)) + (4 × 3) = (-3 + 12) = 9.

(The method is more visually evident using column vectors !)

$\mathbf{r.s} = \begin{pmatrix} -12 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ 6 \end{pmatrix} =$ ((-12)× (-8)) + (5 × 6) = (96 + 30) = 126.

$\mathbf{p.s} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ 6 \end{pmatrix} =$ (3 × (-8)) + (4 × 6) = (-24 + 24) = 0.

Vectors **p** and **s** are perpendicular since their scalar (dot) product is zero.

To find the angle between **p** and **q**, we need to find the lengths of both and their dot product.
The length of **p** is $\sqrt{3^2 + 4^2} = 5$ and the length of **q** is $\sqrt{(-1)^2 + 3^2} = \sqrt{10}$.
The dot product has been worked out as 9, so the angle between them satisfies

$$\cos\theta = \frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}|\,|\mathbf{q}|} \Rightarrow \theta = \cos^{-1}\frac{9}{5\sqrt{10}} \Rightarrow \theta = 55.3° \text{ to 1 d.p.}$$

The process of finding the angle between **r** and **s** is identical:
The length of **r** is $\sqrt{(-12)^2 + 5^2} = 13$ and the length of **s** is $\sqrt{(-8)^2 + 6^2} = 10$.
The dot product has been worked out as 126, so the angle between them satisfies

$$\cos\theta = \frac{\mathbf{r} \cdot \mathbf{s}}{|\mathbf{r}|\,|\mathbf{s}|} \Rightarrow \theta = \cos^{-1}\frac{126}{130} \Rightarrow \theta = 14.3° \text{ to 1 d.p.}$$

The treatment of three-dimensional vectors is the same:

Examples (11): Take the following vectors:

p = $\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$; **q** = $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Find the scalar product and angle between **p** and **q**.

p.q = $\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ = $(3 \times 2) + (2 \times (-2)) + (6 \times 1) = (6 - 4 + 6) = 8$.

Since this question asks for the angle between the vectors, we also need to find the lengths of each .

|**p**| = $\sqrt{3^2 + 2^2 + 6^2} = 7$; |**q**| = $\sqrt{2^2 + (-2)^2 + 1^2} = 3$

The angle can then be worked out using the formula:

$\cos\theta = \dfrac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}|\,|\mathbf{q}|} \Rightarrow \theta = \cos^{-1}\dfrac{8}{21} \Rightarrow \theta = 67.6°$ to 1 d.p.

Other properties of the scalar product.

- As in ordinary multiplication, the scalar product is commutative, namely **p.q = q.p**.
- Also, the scalar product is distributive over addition / subtraction: **a.(b ± c)** = **a.b ± a.c**.
- Multiplication by a scalar λ has the following properties: λ(**a.b**) = **a.**(λ**b**) = (λ**a**).**b**.

Vector equation of a straight line – an introduction.

Example (12) :

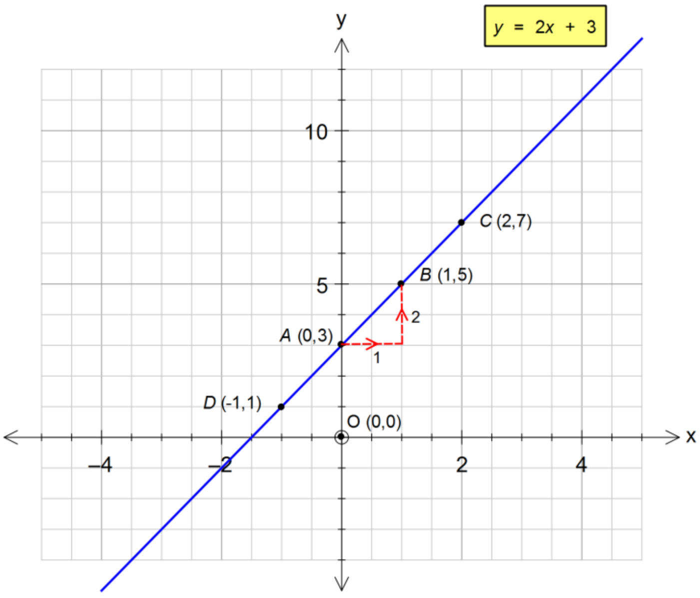
The graph on the right is that of the straight line whose equation of $y = 2x + 3$.

This straight line passes through the points *A* (0, 3), *B* (1,5), *C* (2,7) and *D* (-1,1) .

Using the points *A* and *B* as guides, the gradient is clearly 2 – as *x* increases by 1, *y* increases by 2.

The line's equation is given in **Cartesian form** , but there is another way of describing the line by using vectors.

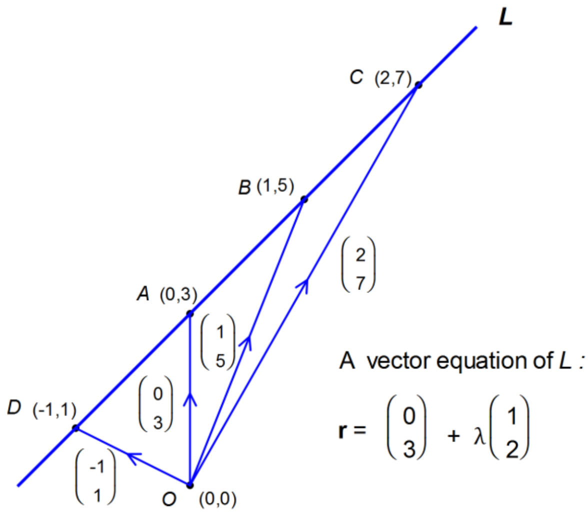
For this we need to state one point on the line using its **position vector** relative to the origin O, and the **direction vector** of the line itself.



The same line is shown on the right, but the axes and grid have been removed, and vectors have been drawn from the origin O to the points *A*, *B*, *C* and *D*.

The position vector of *A* relative to the origin is $\overrightarrow{OA} = 3\mathbf{j}$ or $\begin{pmatrix}0\\3\end{pmatrix}$. The others follow, thus for instance $\overrightarrow{OB} = \mathbf{i} + 5\mathbf{j}$ or $\begin{pmatrix}1\\5\end{pmatrix}$.

To find the direction vector, we choose any two points on the line. Thus if we were to use *A* and *B*, the direction vector is



$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}1\\5\end{pmatrix} - \begin{pmatrix}0\\3\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}.$$ (Note that this column vector translates point *A* to point *B*.)

One vector equation of the line *L* is therefore $\mathbf{r} = \begin{pmatrix}0\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\end{pmatrix}$.

The choice of **r** is arbitrary – it just happens to be the one commonly shown in exams.

The equation of the line is not unique, as will be apparent in the next example.

The letter λ (Greek lambda) is a numerical parameter, and by varying it, we obtain the position vector of a different point on the same line.

Let λ = 0 ; $\begin{pmatrix} 0 \\ 3 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, corresponding to the coordinates of *A* (0, 3).

Let λ = 1 ; $\begin{pmatrix} 0 \\ 3 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$, corresponding to the coordinates of *B* (1, 5).

Let λ = 2 ; $\begin{pmatrix} 0 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$, corresponding to the coordinates of *C* (2, 7).

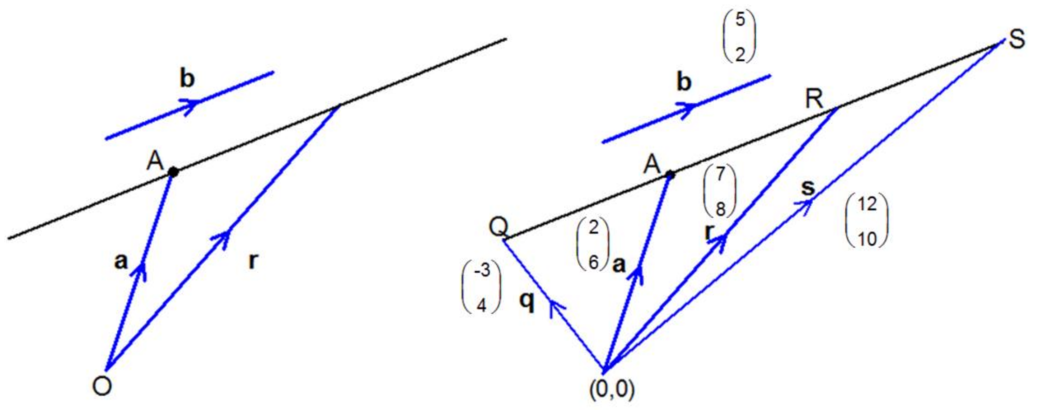
Let λ = -1 ; $\begin{pmatrix} 0 \\ 3 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, corresponding to the coordinates of *D* (-1, 1).

The value of λ does not have to be an integer ; thus, any point on the line can be defined in this way.

Other letters used for the parameter are μ (Greek mu) and ν (Greek nu) , as well as ordinary letters like *s* and *t* .

Vector equation of a line, given one fixed point and a parallel vector.

Example (13):



The above figures show a line passing through a point *A*, and parallel to a vector **b**. The point *A* has a position vector of **a** , and the line as a whole has a vector equation of **r** = **a** + λ**b** **,** where λ is a scalar parameter. Vector **b** is the direction vector of the line.

In the illustration on the right, the position vectors are all measured from the origin for convenience.

The position vector of *A* is **a** = 2**i** + 6**j** or $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$, and the direction vector of the line is 5**i** + 2**j** or $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

The vector equation of the line is therefore **r** = $\begin{pmatrix} 2 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \end{pmatrix}$. The position vector of **r** is 7**i** + 8**j** or

$\begin{pmatrix} 7 \\ 8 \end{pmatrix}$, corresponding to a value of 1 for λ.

Substituting 2 for λ will give the position vector of **s**, i.e. 12**i** + 10**j** or $\begin{pmatrix} 12 \\ 10 \end{pmatrix}$; substituting -1 for λ will similarly give the position vector of **q**, i.e. -3**i** + 4**j** or $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

The equation of the line is not unique; any other point on the line could have been used for *A*, and any multiple of **b** could been used as the direction vector; the vector equation

r = $\begin{pmatrix} -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 10 \\ 4 \end{pmatrix}$ would have been equally valid.

Example (14): Give one vector equation of a line passing through point (2, 1) and parallel to the vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$; also give the coordinates of two other points on the same line.

One vector equation of the line is **r** = $\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

If two lines have the same direction vectors, they are parallel.
This also holds true if the direction vectors of the lines are scalar multiples of each other.

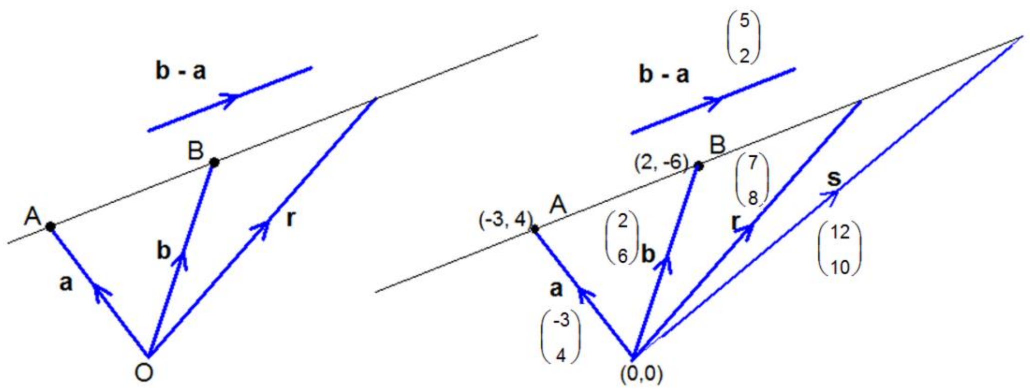
Two other points on the line can be found by substituting certain values for λ; λ = 1 gives point (5,0), and λ = 2 gives (8, -1).

Three-dimensional vectors are treated in exactly the same way:

Example (15): Give one vector equation of a line passing through point (4, -2, -1) and parallel to the vector $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$; also give the coordinates of another point on the same line.

One vector equation of the line is **r** = $\begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$.
Substituting λ = 1 (for example) gives the coordinates of the point (6, -1, -4) on the line.

Vector equation of a line, given two fixed points.



Given the position vectors **a** and **b** of two fixed points *A* and *B* on a line, we can find its vector equation by using the formula **r** = **a** + λ(**b-a**), where λ is a scalar parameter.

The line in the example passes through the points *A* (-3, 4) and *B* (2, -6). Subtracting the position vectors of the points gives the direction vector of the line, **b-a**: (2**i** + 6**j**) – (-3**i** + 4**j**) or 5**i** + 2**j**.

In column notation, **b-a** = $\begin{pmatrix} 2 \\ 6 \end{pmatrix} - \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

Its vector equation is therefore **r** = $\begin{pmatrix} -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

Examples (16): Give one vector equation of i) a line passing through the points *P* (-2, 0) and *Q* (1, 3) and ii) a line passing through the points *P* (2, -3, 1) and *Q* (0, 2, 1).

In i), the position vector of *P*, **p**, is $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ and the direction vector of the line is is given by

q-p: $\begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$.

One vector equation of the line is therefore **r** = $\begin{pmatrix} -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(Note that any multiple of **b-a** could have been used as the direction vector, which is why a factor of 3 had been taken out).

In ii), the position vector of *P*, **p**, is $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and the direction vector of the line is is given by

q-p: $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}$.

One vector equation of the line is therefore $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}$.

The angle between two lines.

To find the angle between two lines, we find the angle between their direction vectors. See the section on Scalar Product and Examples 10 and 11 in this document.

Vector properties of line pairs.

In two-dimensional space, a pair of lines can either intersect or be parallel (or coincident).

Matters are more complicated in three dimensions; a pair of lines can be parallel (or coincident), they can intersect, or they can go in different directions without meeting (**skew lines**).

Two lines are parallel if their direction vectors are multiples of each other (this holds in 2 and 3 dimensions).

Thus, $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -6 \\ 0 \end{pmatrix}$ are parallel, because the direction vector of **s** is –3 times the direction vector of **r**.

Two lines $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{s} = \mathbf{c} + \mu\mathbf{d}$ intersect if unique values of λ and μ can be found such that $\mathbf{a} + \lambda\mathbf{b} = \mathbf{c} + \mu\mathbf{d}$.

Example (17): Two lines have the following vector equations:

$L_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \end{pmatrix}$; $L_2 = \begin{pmatrix} 6 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

Find their point of intersection and the acute angle between them.

The two lines will intersect when λ and μ take values satisfying the equation

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

This can be solved using the elimination method for simultaneous equations.

$2 + \lambda = 6 + \mu$ $\quad A$ (equating **i** – components)
$1 + 3\lambda = -1 - 4\mu$ $\quad B$ (equating **j**– components)

Eliminating λ :

$6 + 3\lambda = 18 + 3\mu$ $\quad 3A$
$1 + 3\lambda = -1 - 4\mu$ $\quad B$
$5 = 19 + 7\mu$ $\quad 3A - B$

This gives $14 + 7\mu = 0 \Rightarrow \mu = -2$.
Substituting into equation A gives $\lambda = 2$.

Thus the point of intersection has position vector $\begin{pmatrix}2\\1\end{pmatrix} + 2\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}6\\-1\end{pmatrix} - 2\begin{pmatrix}1\\-4\end{pmatrix}$, or $\begin{pmatrix}4\\7\end{pmatrix}$.

In other words, the vectors intersect at (4, 7).

To find the angle between the lines, we need to find the length of their direction vectors and their dot product.

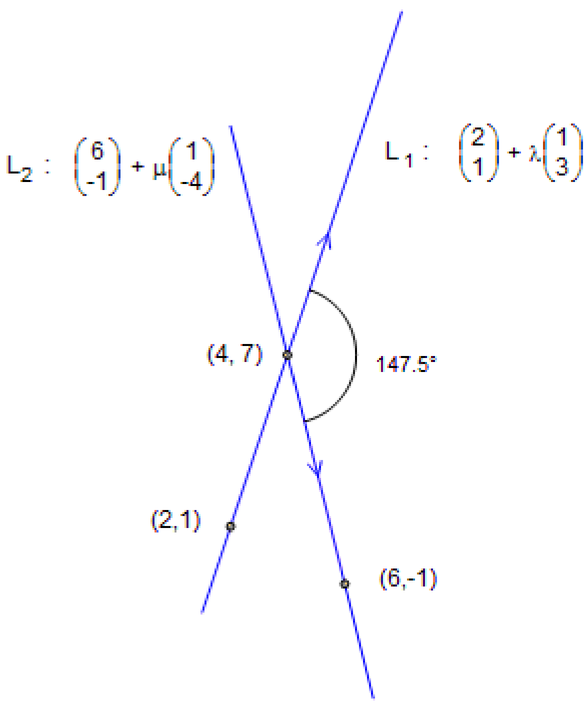
The dot product is $\begin{pmatrix}1\\3\end{pmatrix} \cdot \begin{pmatrix}1\\-4\end{pmatrix}$ = (1 × 1) + (3 × (–4)) = (1 + -12) = -11.

The length of $\begin{pmatrix}1\\3\end{pmatrix}$ is √10 and that of $\begin{pmatrix}1\\-4\end{pmatrix}$ is √17.

The angle between them therefore satisfies $\theta = \cos^{-1}\dfrac{-11}{\sqrt{10}\sqrt{17}} \Rightarrow \theta$ = 147.5° to 1 d.p.

This gives the obtuse angle solution – the acute angle solution can be found by subtracting from 180°. The acute angle between the lines is therefore 32.5°.

(See examples 10 and 11 for greater detail).



Example (16): Two lines have the following vector equations:

$$L_1 = \begin{pmatrix} 5 \\ 7 \\ -8 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} \; ; \quad L_2 = \begin{pmatrix} 3 \\ -9 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} .$$

Find out whether they intersect, and give their point of intersection if they do.

The two lines will intersect when λ and μ take values satisfying the equation

$$\begin{pmatrix} 5 \\ 7 \\ -8 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} .$$

Solving the simultaneous equations we have

$5 + \lambda = 3$ *A* (equating **i** – components)
$7 + 3\lambda = -9 + 2\mu$ *B* (equating **j**– components)
$-8 - 3\lambda = -7 + \mu$ *C* (equating **k**– components)

We immediately see that $\lambda = -2$, so we substitute in equations *B* and *C*:

$1 = -9 + 2\mu$ *B*
$-2 = -7 + \mu$ *C*

Substituting in *B* gives $\mu = 5$, as does substituting in *C*.

The two lines therefore do intersect at the point

$$\begin{pmatrix} 5 \\ 7 \\ -8 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \\ -7 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} .$$

The point of intersection has position vector $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$, i.e. its coordinates are (3, 1, 2).

The next example is almost identical, but there is a slight difference – enough to affect the final result.

Example (17): Two lines have the following vector equations:

$$L_1 = \begin{pmatrix} 5 \\ 7 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} \; ; \quad L_2 = \begin{pmatrix} 3 \\ -9 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

Find out whether they intersect, and give their point of intersection if they do.

The two lines will intersect when λ and μ take values satisfying the equation

$$\begin{pmatrix} 5 \\ 7 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

Solving the simultaneous equations we have

$5 + \lambda = 3$ *A* (equating **i** – components)
$7 + 3\lambda = -9 + 2\mu$ *B* (equating **j**– components)
$-6 - 3\lambda = -7 + \mu$ *C* (equating **k**– components)

$\lambda = -2$, so substitute in equations *B* and *C*:

$1 = -9 + 2\mu$ *B*
$0 = -7 + \mu$ *C*

Substituting in *B* gives $\mu = 5$, but substituting in *C* gives $\mu = 7$.
$\therefore$ The values $\lambda = -2$, $\mu = 5$ only satisfy two equations of the three, as do $\lambda = -2$, $\mu = 7$.

This inconsistency means that the two lines do not meet - they are skew.

Example (18): Two lines have the following vector equations:

$L_1 = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$; $L_2 = \begin{pmatrix} 3 \\ 4 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$.

Do they intersect ? If so, give their point of intersection.

The scalars λ and μ must take values satisfying the equation below for the lines to intersect.

$$\begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} .$$

The simultaneous equations are

$-2 + 2\lambda = 3 - \mu$ *A* (equating **i** – components)
$4 - \lambda = 4 + 3\mu$ *B* (equating **j**– components)
$\lambda = 8 + 5\mu$ *C* (equating **k**– components)

$-2 + 2\lambda = 3 - \mu$ *A*
$4 = 12 + 8\mu$ *B* + *C*

Eliminating λ, we have $12 + 8\mu = 4 \Rightarrow 8 + 8\mu = 0 \Rightarrow \mu = -1$.

Substituting for μ in equation *A* , $-2 + 2\lambda = 4 \Rightarrow \lambda = 3$.
Substituting for μ in equation *B* , $4 - \lambda = 1 \Rightarrow \lambda = 3$.
Substituting for μ in equation *C* , $\lambda = 3$ as well.

The two lines therefore do intersect at the point

$$\begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 8 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} .$$

The point of intersection therefore has position vector $\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$, or coordinates of (4, 1, 3).

Example (19): Two lines have the following vector equations:

$$L_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \; ; \quad L_2 = \begin{pmatrix} 2 \\ -5 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} .$$

Do they intersect ? If so, give their point of intersection.

For intersection, $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

As before,
$3 + \lambda = 2 + \mu$ *A* (equating **i** – components)
$2 - \lambda = -5 + 3\mu$ *B* (equating **j**– components)
$-1 + 2\lambda = -7 + 2\mu$ *C* (equating **k**– components)

$5 = -3 + 4\mu$ *A* + *B*
$-1 + 2\lambda = -7 + 2\mu$ *C*

Eliminating λ, we have $4\mu - 3 = 5 \Rightarrow \mu = 2$.

Substituting for μ in *A*, $3 + \lambda = 4 \Rightarrow \lambda = 1$.
Substituting for μ in *B*, $2 - \lambda = 1 \Rightarrow \lambda = 1$.

But...

Substituting for μ in *C*, $-1 + 2\lambda = -3 \Rightarrow \lambda = -1$.

The simultaneous equations are inconsistent, and so the two lines in question do not have a point of intersection. In other words, they are skew lines.

Example (20): Two lines have vector equations $\mathbf{r_1} = \begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix} + t\begin{pmatrix} -8 \\ 1 \\ -2 \end{pmatrix}$; $\mathbf{r}_2 = \begin{pmatrix} -2 \\ a \\ -2 \end{pmatrix} + s\begin{pmatrix} -9 \\ 2 \\ -5 \end{pmatrix}$.

In the equation for $\mathbf{r}_2$, a is a constant to be determined.

i) Calculate the acute angle between the lines.
ii) Given that these two lines intersect, find a and the point of intersection.
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i) To find the angle between the lines, we need to find the length of their direction vectors and their dot product.

The dot product is $\mathbf{r_1}.\,\mathbf{r}_2 = \begin{pmatrix} -8 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -9 \\ 2 \\ -5 \end{pmatrix}$ = (-8 × –9) + (1 × 2) + (-2 × –5) = (72 + 2 +10) = 84.

The length of $\mathbf{r}_1$ is $\sqrt{(-8)^2 + 1^2 + (-2)^2} = \sqrt{69}$; that of $\mathbf{r}_2$ is $\sqrt{(-9)^2 + 3^2 + (-5)^2} = \sqrt{110}$.

The angle between them therefore satisfies $\theta = \cos^{-1} \dfrac{84}{\sqrt{69}\sqrt{110}} \Rightarrow \theta = 15.4°$ to 1 d.p.

ii) We are told that the two lines intersect, and so we begin by equating **i-** and **k-** components and solving the two simultaneous equations in s and t. Having obtained s and t we then substitute their values into the **j**-component equation, and hence find a value of a to ensure consistency between the equations.

$$\begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix} + t\begin{pmatrix} -8 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ a \\ -2 \end{pmatrix} + s\begin{pmatrix} -9 \\ 2 \\ -5 \end{pmatrix}.$$

$4 - 8t = -2 - 9s$ A (equating **i** – components)
$-6 - 2t = -2 - 5s$ C (equating **k**– components)

From the **i-**component equation A we have $9s - 8t = -6$.
From the **k-**component equation C we have $5s - 2t = 4$.

$9s - 8t = -6$ from A
$20s - 8t = 16$ from $4C$
$-11s = 22$ from $A - 4C$

Hence $s = 2$, and substituting in eqn. C gives $10 - 2t = 4$, and thus $t = 3$.

We need to substitute $s = 2$, and $t = 3$ into the **j**-component equation: $2 + t = a + 2s$, giving $a + 4 = 5$, and finally $a = 1$ for consistency.

The equation of $\mathbf{r}_2$ is $\begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} + s\begin{pmatrix} -9 \\ 2 \\ -5 \end{pmatrix}$.

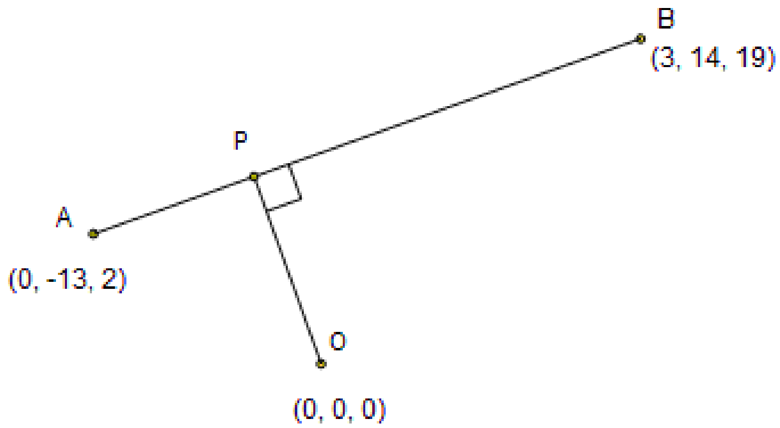
The point of intersection is given by either substituting $s = 2$ or $t = 3$ into the respective line equations:

$$\begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix} + 3\begin{pmatrix} -8 \\ 1 \\ -2 \end{pmatrix} \text{ or } \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} + 2\begin{pmatrix} -9 \\ 2 \\ -5 \end{pmatrix}, \text{ i.e. } \begin{pmatrix} -20 \\ 5 \\ -12 \end{pmatrix}.$$

The distance between a point and a line.

Example (21): The position vectors of two points *A* and *B* are $\mathbf{a} = \begin{pmatrix} 0 \\ -13 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 14 \\ 19 \end{pmatrix}$ respectively, referred to the origin *O*.

i) Find a vector equation for the line *AB*.
ii) Find the position vector of the point *P* on *AB* such that *OP* is perpendicular to *AB*, and hence calculate the perpendicular distance of *P* from the origin, leaving the result as a surd.



i) The direction vector of the line *AB* is given by

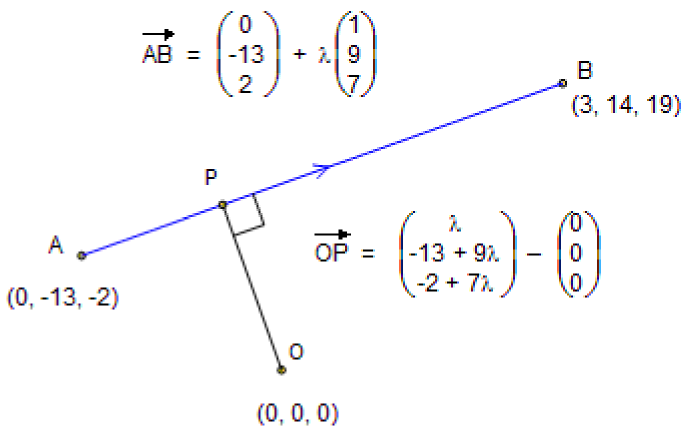
$\mathbf{b}\text{-}\mathbf{a} = \begin{pmatrix} 3 \\ 14 \\ 19 \end{pmatrix} - \begin{pmatrix} 0 \\ -13 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 27 \\ 21 \end{pmatrix}$. We can take out 3 as a factor to give $\begin{pmatrix} 1 \\ 9 \\ 7 \end{pmatrix}$, and that has the advantage of simplifying the rest of the arithmetic.

One vector equation of the line is therefore $\overrightarrow{AB} = \begin{pmatrix} 0 \\ -13 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 9 \\ 7 \end{pmatrix}$.

ii) We are given that $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{AB}$.

Any point on *AB* will have *position* vector $\begin{pmatrix} \lambda \\ -13+9\lambda \\ -2+7\lambda \end{pmatrix}$ where λ is a parameter to be determined.

The direction vector of $\overrightarrow{OP}$ is $\begin{pmatrix} \lambda - 0 \\ -13+9\lambda - 0 \\ -2+7\lambda - 0 \end{pmatrix}$. (Since *O* is the origin, the arithmetic is easy).



Thus $\overrightarrow{OP} \cdot \overrightarrow{AB} = \begin{pmatrix} \lambda \\ -13+9\lambda \\ -2+7\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 9 \\ 7 \end{pmatrix}$

$\Rightarrow \overrightarrow{OP} \cdot \overrightarrow{AB} = (\lambda) + (-117 + 81\lambda) + (-14 + 49\lambda) = -131 + 131\lambda.$

We want point *P* on the line to satisfy $\overrightarrow{OP} \cdot \overrightarrow{AB} = 0$ (perpendicular lines have a zero dot product !)

Hence $\overrightarrow{OP} \cdot \overrightarrow{AB} = 0$ when -131 + 131λ = 0, i.e. when λ = 1.

The direction vector of *OP* can therefore be found by substituting λ = 1 into the vector equation for the line *AB*.

$\vec{O}P = \begin{pmatrix} 0 \\ -13 \\ -2 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 9 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$, and finally, the perpendicular distance of point *P* from the origin is

$\sqrt{1^2 + (-4)^2 + 5^2} = \sqrt{42}$ units.

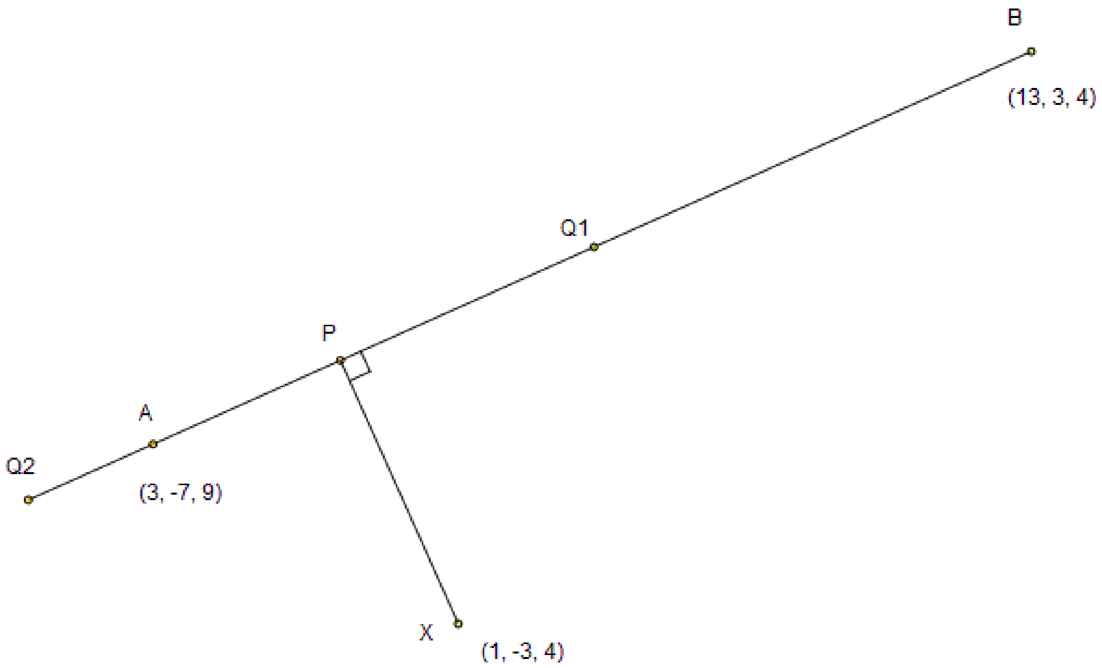
.

Example (22): Points *A* and *B* have coordinates of (3, -7, 9) and (13, 3, 4) respectively.
The point *X* has coordinates (1, -3, 4).

i) Find a vector equation for the line passing through *A* and *B*.

ii) The point *P* lies on *AB* such that its distance from *X* takes the shortest possible value. Find the coordinates of *P*.

iii) The point *Q* also lies on the line passing through *A* and *B* such that the triangle *PXQ* is isosceles. Find the possible coordinates for *Q* and the area of the triangle *PXQ*.



i) The direction vector of the line *AB* is given by

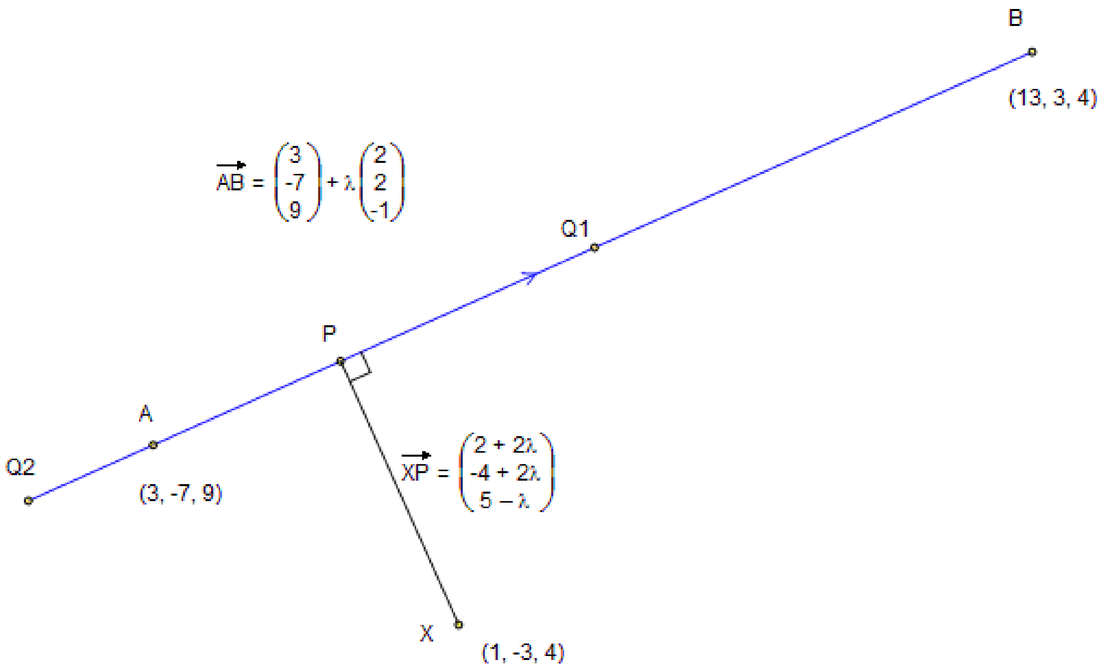
$\overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 13 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -7 \\ 9 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \\ -5 \end{pmatrix}$, or $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ in its simplest form .

One vector equation of the line is therefore $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -7 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$.

ii) The distance *XP* takes its minimum value when *XP* is perpendicular to *AB*.

Any point *P* on *AB* will have position vector $\begin{pmatrix} 3+2\lambda \\ -7+2\lambda \\ 9-\lambda \end{pmatrix}$ where λ is a parameter to be determined.

The direction vector of $\overrightarrow{XP}$ is $\overrightarrow{OP}$ - $\overrightarrow{OX}$ = $\begin{pmatrix} 3+2\lambda \\ -7+2\lambda \\ 9-\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2+2\lambda \\ -4+2\lambda \\ 5-\lambda \end{pmatrix}$.



Thus $\overrightarrow{XP} \cdot \overrightarrow{AB} = \begin{pmatrix} 2+2\lambda \\ -4+2\lambda \\ 5-\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

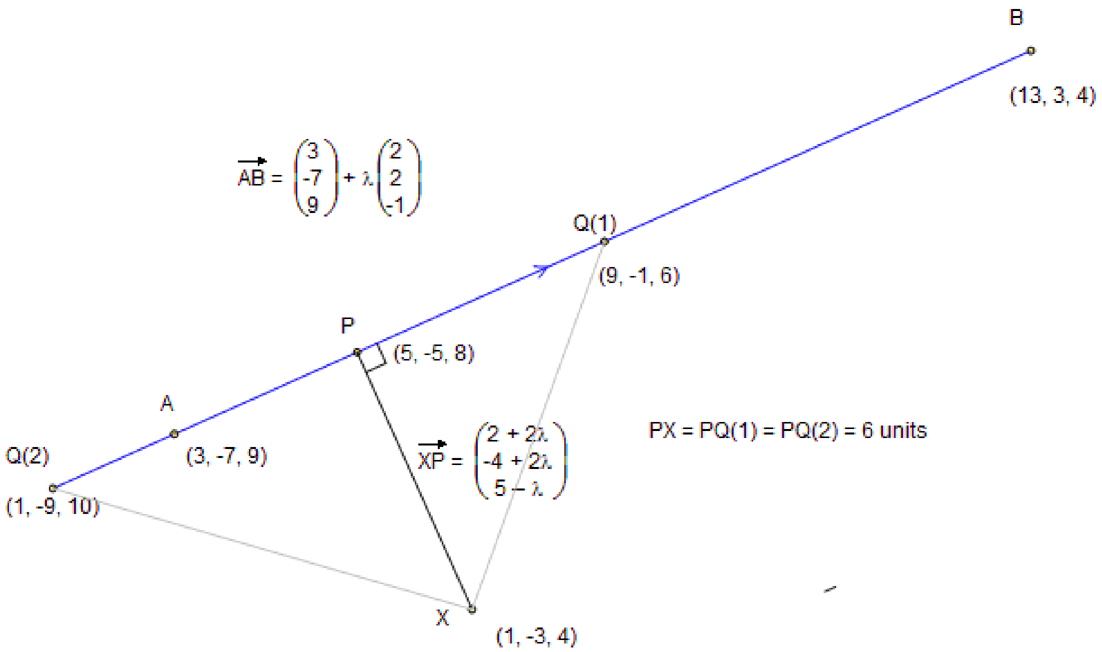
$\Rightarrow$ $\overrightarrow{XP} \cdot \overrightarrow{AB}$ = (4 + 4λ) + (-8 + 4λ) + (-5 + λ) = –9 + 9λ.

Point *P* satisfies $\overrightarrow{XP} \cdot \overrightarrow{AB} = 0$ (zero dot product)

so $\overrightarrow{XP} \cdot \overrightarrow{AB}$ = 0 when -9 + 9λ = 0, i.e. when λ = 1.

Substituting λ = 1 into the equation for line through *AB*,

$$\overrightarrow{XP} = \begin{pmatrix} 3 \\ -7 \\ 9 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} . \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} .$$



iii) We are given that the triangle *PXQ* is isosceles. Therefore *PX* = *PQ*.

The length of *PX* = $\sqrt{4^2 + (-2)^2 + 4^2} = \sqrt{36} = 6$ units.

Since we are given that triangle *XPQ* is isosceles, the distance *PQ* is also 6 units, and the area of the triangle is 18 square units, as angle *XPQ* is a right angle.

The length of the vector *PQ* is therefore some multiple of the length of the direction vector of *AB*.

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.$$

The length of the direction vector of *AB* is $\sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3.$

Since the length *PQ* is twice 3 or 6 units, λ = 2 or -2.

Hence the possible position vectors of *Q* are $\begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \\ 6 \end{pmatrix}$

and $\begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -9 \\ 10 \end{pmatrix}$, corresponding to coordinates of (9, -1, 6) and (1, -9, 10).