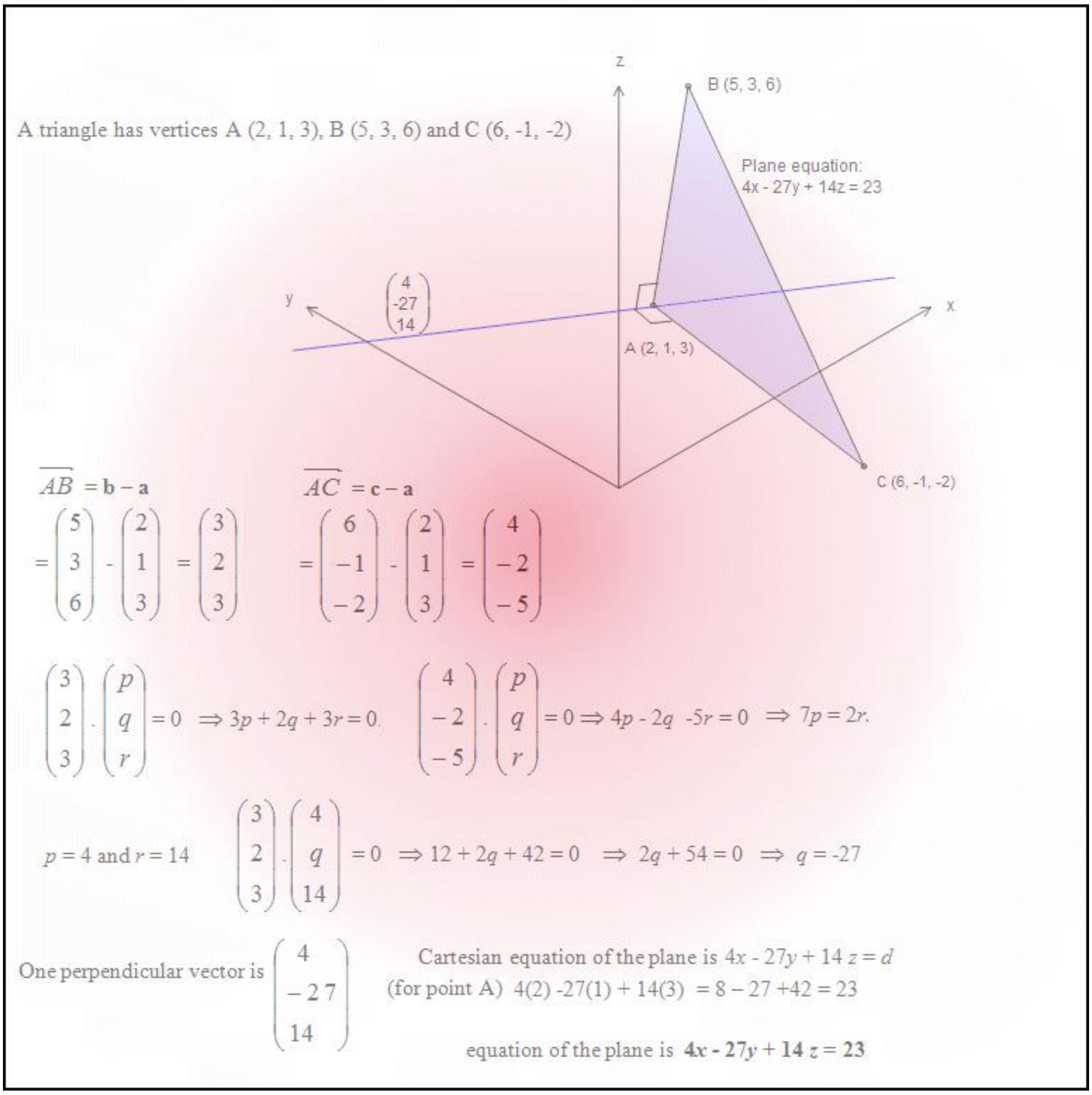
M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level - MEI

OCR MEI: **C4**

FURTHER VECTORS (MEI)



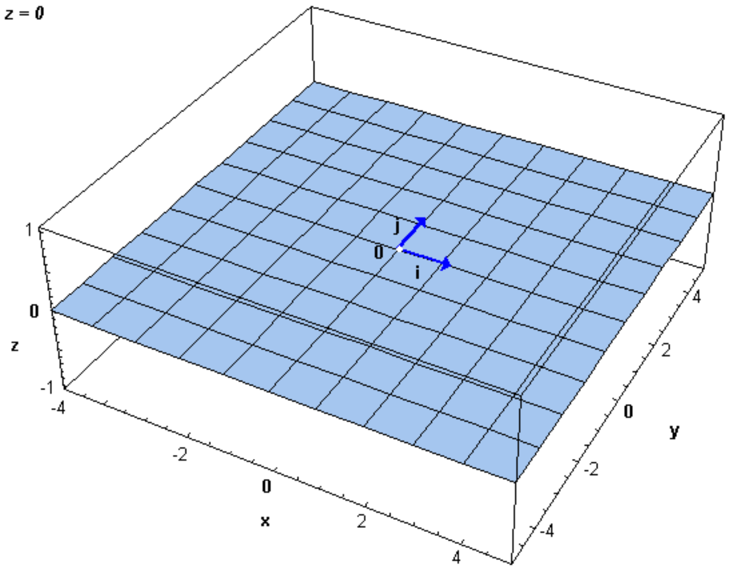
Version : 2.1 Date: 25-09-2007

Further 3-D Vector Problems.

Parametric Vector Equation of a Plane.

In two dimensions, it was possible to obtain a vector equation of a line given the position vector, **a** , of one point and the direction vector, **b**, of a line joining two other points. In other words, a line could be described in the form **r** = **a** + λ**b** otherwise known as the **parametric vector equation** of a line.

A **plane** is a two-dimensional portion of three-dimensional space, and it too can be defined in terms of vectors.



The diagram on the right represents the *x*-*y* plane in three dimensions, in other words the set of all points (*x*, *y*, *z*) where *z* = 0.

It can be seen that any point in the plane can be defined in terms of the position vector of one point (here the origin) and a combination of two direction vectors, here **i** and **j** for convenience.

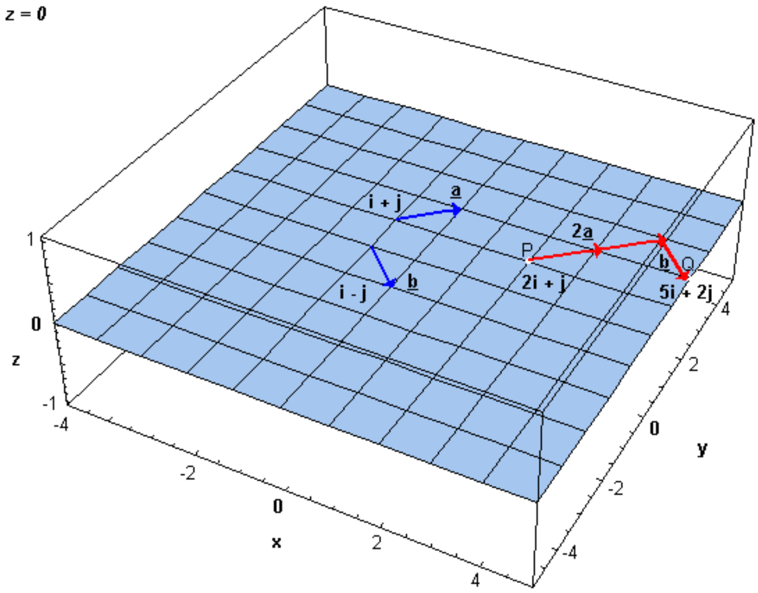
∴ Any point in the *x*-*y* plane can be defined as
0 + λ**i** + μ**j**, where the **k-**component is zero**.**

For example the point (1, 5, 0) = **0 +** $\begin{pmatrix}1\\5\\0\end{pmatrix}$.

Remember that **i** = $\begin{pmatrix}1\\0\\0\end{pmatrix}$, **j** = $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and **k** = $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

The previous example was deliberately chosen for simplicity. However, *any* two vectors in the plane can be chosen to define the plane, provided they are not parallel.

Again, z = 0, but this time we have chosen point P with position vector **p** = 2**i** + **j** as the starting point, and direction vectors **a** = **i** + **j** and **b** = **i** – **j**.



In column notation,

$$\mathbf{p} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

The example shows how the point Q with position vector

$$\mathbf{q} = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \text{ can be expressed as}$$

$\mathbf{q} = \mathbf{p} + \lambda\mathbf{a} + \mu\mathbf{b}$ where λ = 2 and μ = 1.

The equation above is the **parametric vector equation** of the plane, analogous to $\mathbf{q} = \mathbf{p} + \lambda\mathbf{a}$, the parametric vector equation of a line.

Example (1): The vector $\mathbf{r} = \begin{pmatrix} 1 \\ 8 \\ 0 \end{pmatrix}$ is of the form $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Find the values of λ and μ.

$2 + \lambda + \mu = 1$ A (equating **i** – components)
$1 + \lambda - \mu = 8$ B (equating **j** – components)
$1 + 2\mu = -7$ $A - B$

This gives μ = -4, and substituting in A gives λ = 3.

$$\therefore \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 4\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

The previous example was deliberately set to be ‘easy’ – the next one calls for a little more work.

Example (2): Three points A, B and C have position vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\begin{pmatrix}3\\4\\5\end{pmatrix}$ and $\begin{pmatrix}-4\\3\\-3\end{pmatrix}$ respectively.

i) Give the parametric vector equation of the plane containing the three points.

ii) Point P has position vector $\begin{pmatrix}-7\\8\\-1\end{pmatrix}$. Is it coplanar with points A, B and C ?

iii) Point Q has position vector $\begin{pmatrix}8\\10\\-4\end{pmatrix}$. Is it coplanar with points A, B and C ?

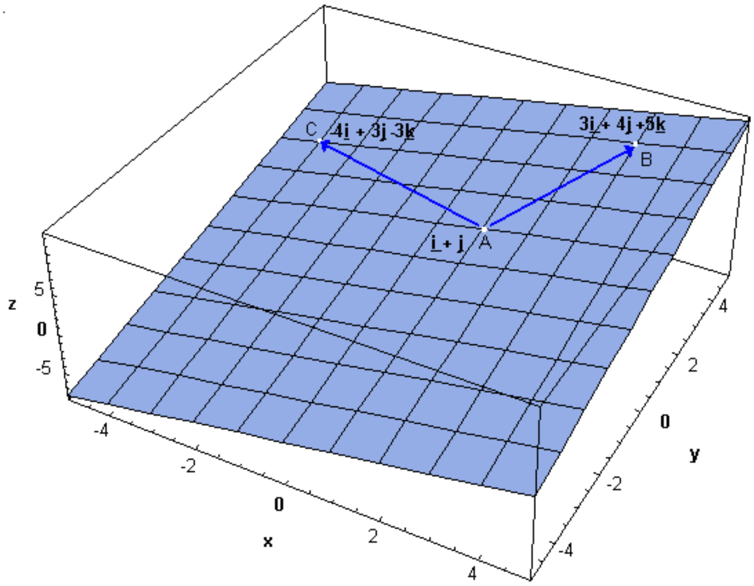
i) Firstly, we find the direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$\overrightarrow{AB}$ = **b** – **a**

$$= \begin{pmatrix}3\\4\\5\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\3\\5\end{pmatrix}.$$

$\overrightarrow{AC}$ = **c** – **a**

$$= \begin{pmatrix}-4\\3\\-3\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}-5\\2\\-3\end{pmatrix}.$$



∴ One vector equation of the plane is

$$\begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\5\end{pmatrix} + \mu\begin{pmatrix}-5\\2\\-3\end{pmatrix}.$$

ii) Having established a vector equation for the plane, we then need to prove that values of λ and μ exist such that $\begin{pmatrix}-7\\8\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\5\end{pmatrix} + \mu\begin{pmatrix}-5\\2\\-3\end{pmatrix}$.

Equating **i**-, **j**- and **k**-components we have

$1 + 2\lambda - 5\mu = -7$ — *A* (equating **i** – components)
$1 + 3\lambda + 2\mu = 8$ — *B* (equating **j** – components)
$0 + 5\lambda - 3\mu = -1$ — *C* (equating **k** – components)
$(5 + 15\lambda + 10\mu) - (15\lambda - 9\mu) = (40+3) \rightarrow$ — *5B – 3C*
$5 + 19\mu = 43$

This gives $\mu = 2$, and substituting into equation *A* gives $1 + 2\lambda - 10 = -7 \Rightarrow 2\lambda = 2 \Rightarrow \lambda = 1$.

Checking equation *B* gives 1 + 3 + 4 = 8, and likewise checking equation *C* gives 5 – 6 = -1.
All the equations are consistent, and so point P is coplanar with A, B and C.

iii) This time we check for values of λ and μ satisfying

$$\begin{pmatrix} 8 \\ 10 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 2 \\ -3 \end{pmatrix}.$$

$1 + 2\lambda - 5\mu = 8$	*A* (equating **i** – components)
$1 + 3\lambda + 2\mu = 10$	*B* (equating **j** – components)
$0 + 5\lambda - 3\mu = -4$	*C* (equating **k** – components)
$(5 + 15\lambda + 10\mu) - (15\lambda - 9\mu) = (50+12) \rightarrow$	*5B – 3C*
$5 + 19\mu = 62$	

This gives μ = 3, and substituting into equation *A* gives $1 + 2\lambda - 15 = 8 \Rightarrow 2\lambda = 22 \Rightarrow \lambda = 11$.

Substituting λ = 11 into equation *B* gives 1 - 33 + 6 = 10 which is inconsistent.
∴ point Q is not coplanar with A, B, C and P.

Example (3): Three points A, B and C have position vectors

$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$ respectively.

Give the parametric vector equation of the plane containing the three points.
Does anything go wrong here ?

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

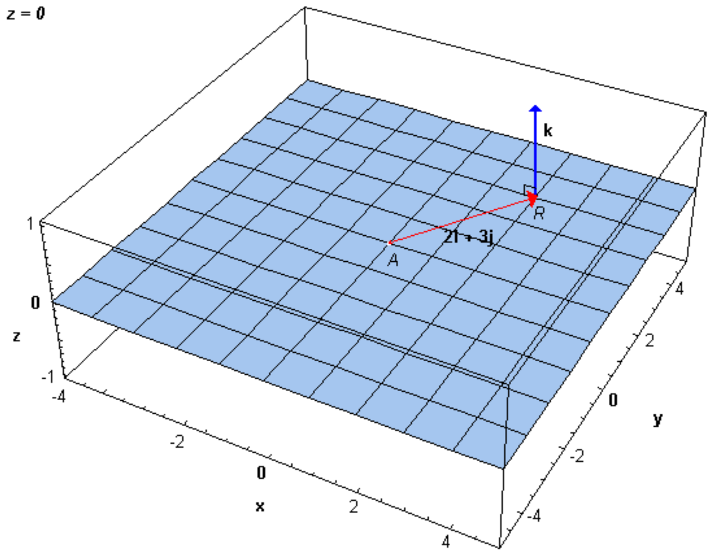
$$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}.$$

One vector equation of the plane is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$.

We have hit a problem here, because $\overrightarrow{AC}$ is a scalar multiple of $\overrightarrow{AB}$, in fact $\overrightarrow{AC} = 3\overrightarrow{AB}$, meaning that the two direction vectors are parallel.

The resulting parametric vector equation can be simplified to $\begin{pmatrix}1\\1\\0\end{pmatrix} + \nu\begin{pmatrix}1\\-1\\1\end{pmatrix}$, which is the equation of a straight line and not a plane.

Scalar product form.



Another way of defining a plane is to select a vector on the plane and another vector perpendicular to it.

Here we have defined the illustrated *x*-*y* plane by a random vector $\overrightarrow{AR}$ within it and a perpendicular vector to it.

Since we are dealing with the *x*-*y* plane, a suitable perpendicular vector is **k**, parallel to the *z*-axis.

The perpendicular vector is labelled **n** (normal) by convention.

$\overrightarrow{AR}$. **n** = 0 and so (**r** - **a**).**n** = 0
and thus **r.n** = **a.n** by dot product rules.

If we know **a** and **n** we can work out **a.n**.

Since A is the origin, its position vector **a** is **0**.

The position vector of R, **r**, is $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$.

a.n =(**0**).(**k**) = 0, so the equation of the plane is

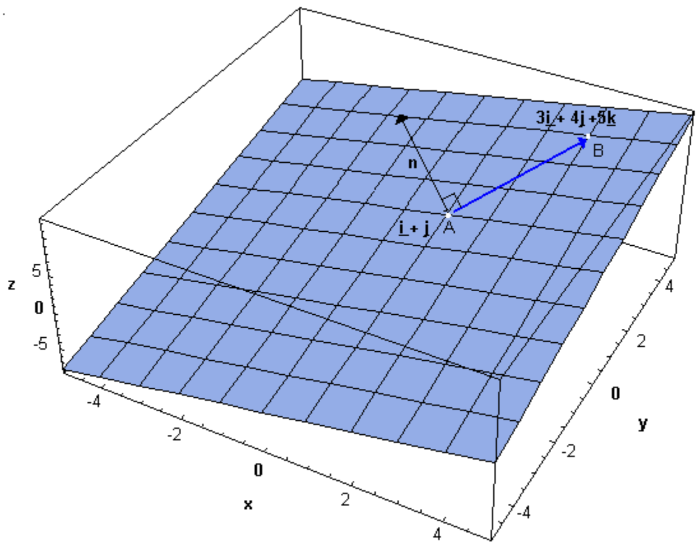
$$\mathbf{r.n} = \mathbf{a.n} \Rightarrow \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} . (\mathbf{k}) = 0.$$

Any vector in the *x*-*y* plane could have been used here: thus $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} . \mathbf{k} = 0$ is equally suitable.

The previous example was again rather trivial – this one is a slightly more difficult.

Example (4): Points A and B have position vectors of

$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ respectively.



Find a vector perpendicular to $\overrightarrow{AB}$ and hence the scalar product form of the equation of the plane containing points A and B.

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$.

The perpendicular vector **n** is of the form $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ where $\mathbf{n}.\overrightarrow{AB} = 0$.

We thus need to find values of *a*, *b* and *c* where $\begin{pmatrix} a \\ b \\ c \end{pmatrix} . \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = 0$.

$2a + 3b + 5c = 0 \rightarrow a = 1, b = 1, c = -1$ is one such set of values.
(There are infinitely many others - $a = 5$, $b = 0$, $c = -2$ is another case).

$\therefore \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is perpendicular to the plane containing $\overrightarrow{AB}$.

An equation of the plane can then be generalised by again finding the dot product, but this time between **a** and **n**:

$\mathbf{a}.\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 1 + 1 + 0 = 2$.

(We could have used **b** to arrive at the same result: $\mathbf{b}.\mathbf{n} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 3 + 4 - 5 = 2$).

One vector equation of the plane in scalar product form is therefore

$\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n} = 2 \rightarrow \mathbf{r}. \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 2$ for all vectors **r** in the plane.

Cartesian Equation of a Plane.

The scalar product form of the equation of a plane was given as **r.n =** *p* where **r** is the position vector of a point in the plane, **n** is a vector perpendicular to the plane and *p* is a number to be determined.

If **n** = $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and **r** = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then their dot product will be $ax + by + cz$ and the equation of the plane will be $ax + by + cz = p$.

In other words, the coefficients *a*, *b* and *c* will be equal to the **i-**, **j-** and **k-**components of the perpendicular vector **n**. To find *p*, we substitute for *x*, *y* and *z* in the vector **r** in the plane.

The plane in the last example had a vector equation of **r.** $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ = 2 for all vectors **r** in the plane.

The Cartesian equation is thus $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 2 \rightarrow = \boldsymbol{x + y - z = 2}$.

Example (5): A plane is perpendicular to vector **n** = $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, and the point with position vector

r = $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ lies on the plane. Find the Cartesian equation of the plane.

The equation of the plane is $2x - y + 4z = p$ (coefficients equal the **i-**, **j-** and **k-**components of **n**).

Substituting $x = 3$, $y = 1$ and $z = 2$ (the **i-**, **j-** and **k-**components of **r**) into $2x - y + 4z = p$ gives $p = 13$.

$\therefore$ the equation of the plane is $\boldsymbol{2x - y + 4z = 13}$.

Example (6): Give the Cartesian equation of the plane with vector equation **r. n** = 7, where
n = $\begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix}$.

The dot product **r. n** will be $-3x + 2y + 4z$, and the equation of the plane will be $\boldsymbol{-3x + 2y + 4z = 7}$.

Example (7): Points A and B have position vectors of **a** = $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and **b** = $\begin{pmatrix}3\\4\\5\end{pmatrix}$ respectively (same as in Example 4).

i) Show that the vector **n** = $\begin{pmatrix}5\\0\\-2\end{pmatrix}$ is perpendicular to $\overrightarrow{AB}$.

ii) Find the Cartesian equation of the plane containing points A and B.

$\overrightarrow{AB}$ = **b** – **a** = $\begin{pmatrix}3\\4\\5\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\3\\5\end{pmatrix}$.

Taking the scalar product, $\begin{pmatrix}5\\0\\-2\end{pmatrix} \boldsymbol{.} \begin{pmatrix}2\\3\\5\end{pmatrix}$ = 10 – 10 = 0 , ∴ **n** is perpendicular to $\overrightarrow{AB}$.

An equation of the plane can then be generalised by again finding the dot product of **a** and **n** :

a.n = $\begin{pmatrix}1\\1\\0\end{pmatrix} \boldsymbol{.} \begin{pmatrix}5\\0\\-2\end{pmatrix}$ = 5.

From this result, another vector equation of the plane in scalar product form is therefore

r.n = **a.n =** 5 $\Rightarrow$ **r.** $\begin{pmatrix}5\\0\\-2\end{pmatrix}$ = 5 for all vectors **r** in the plane.

Another Cartesian equation of the plane is thus $\begin{pmatrix}x\\y\\z\end{pmatrix} \boldsymbol{.} \begin{pmatrix}5\\0\\-2\end{pmatrix}$ = 5 $\Rightarrow$ = $\mathbf{5}\boldsymbol{x} \mathbf{- 2}\boldsymbol{z} \mathbf{= 5}$.

This result looks quite different from $x + y - z = 2$ obtained earlier, but substituting (x, y, z) = (1, 1, 0) (position vector of point A) satisfies both 1 + 1 –2 = 0 and 5(1)-2(0) = 5.

Similarly, substituting (x, y, z) = (3, 4, 5) (position vector of point B) satisfies both 3 + 4 –5 = 2 and 5(3)-2(5) = 5.

Example (8): Three points A, B and C have position vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\begin{pmatrix}3\\4\\5\end{pmatrix}$ and $\begin{pmatrix}-4\\3\\-3\end{pmatrix}$ respectively.

i) Find a vector perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and hence give the Cartesian equation of the plane containing the three points.

ii) Point P has position vector $\begin{pmatrix}-7\\8\\-1\end{pmatrix}$. Is it coplanar with points A, B and C ?

iii) Point Q has position vector $\begin{pmatrix}8\\10\\-4\end{pmatrix}$. Is it coplanar with points A, B and C ?

i) Firstly, we find the direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$\overrightarrow{AB}$ = **b** – **a**

$= \begin{pmatrix}3\\4\\5\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\3\\5\end{pmatrix}$.

$\overrightarrow{AC}$ = **c** – **a**

$= \begin{pmatrix}-4\\3\\-3\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}-5\\2\\-3\end{pmatrix}$.

Next, we find a vector $\begin{pmatrix}p\\q\\r\end{pmatrix}$ such that $\overrightarrow{AB} . \begin{pmatrix}p\\q\\r\end{pmatrix} = 0$ and $\overrightarrow{AC} . \begin{pmatrix}p\\q\\r\end{pmatrix} = 0$.

$\begin{pmatrix}2\\3\\5\end{pmatrix} . \begin{pmatrix}p\\q\\r\end{pmatrix} = 0 \quad \Rightarrow 2p + 3q + 5r = 0.$ (**A**)

$\begin{pmatrix}-5\\2\\-3\end{pmatrix} . \begin{pmatrix}p\\q\\r\end{pmatrix} = 0 \quad \Rightarrow -5p + 2q - 3r = 0.$ (**B**)

Firstly, we eliminate one of the variables *p*, *q* and *r* – here *p* is eliminated.

$10p + 15q + 25r = 0$	5**A**
$-10p + 4q - 6r = 0$	2**B**
$19q + 19r = 0$	5**A**-2**B**

From this result, $q + r = 0$, so $q = -r$. We can therefore let $q = 1$ and $r = -1$ (the simplest case).

Substituting for *q* and *r* in either dot product equation gives us *p:* (first equation chosen here)

$$\begin{pmatrix}2\\3\\5\end{pmatrix} \cdot \begin{pmatrix}p\\1\\-1\end{pmatrix} = 0 \Rightarrow 2p + 3 - 5 = 0 \Rightarrow 2p - 2 = 0 \Rightarrow 2p = 2 \text{ and } p = 1.$$

The perpendicular vector is therefore $\begin{pmatrix}1\\1\\-1\end{pmatrix}$, and the corresponding Cartesian equation is

$x + y - z = d$ where *d* is to be determined.

To find *d*, we substitute the coordinates for any of points A, B, or C :

Thus (for point A) 1 + 1 – 0 = 2; (check: for point B) 3 + 4 – 5 = 2

The Cartesian equation of the plane containing the three points is thus $\boldsymbol{x + y - z = 2}$**.**

ii) Substituting the coordinates for point P into the Cartesian equation gives -7 +8 – (-1) = 2.
This final value of 2 is consistent, so point P is coplanar with A, B and C.

iii) Substituting the coordinates for point Q into the Cartesian equation gives 8 + 10 – (-4) = 22.
This final value of 22 is inconsistent, so point P is not coplanar with A, B and C.

Example (9): A triangle has vertices at A (2, 1, 3), B (5, 3, 6) and C (6, -1, -2).

Find a vector perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and hence give the Cartesian equation of the plane containing the triangle. Give the result in the form $ax + by + cz = d$ where a, b, c and d are integers.

Firstly, we find the direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$\overrightarrow{AB}$ = **b** – **a**

$$= \begin{pmatrix} 5 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}.$$

$\overrightarrow{AC}$ = **c** – **a**

$$= \begin{pmatrix} 6 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix}.$$

Next, we find a vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ such that $\overrightarrow{AB} . \begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0$ and $\overrightarrow{AC} . \begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0$.

$$\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} . \begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0 \;\Rightarrow 3p + 2q + 3r = 0. \;(\mathbf{A})$$

$$\begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix} . \begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0 \Rightarrow 4p - 2q \; -5r = 0. \;(\mathbf{B})$$

We can eliminate q by adding equations **A** and **B**.

$3p + 2q + 3r = 0$	**A**
$4p - 2q \; -5r = 0$	**B**
$7p - 2r = 0$	**A+B**

From here, $7p = 2r$. We can therefore let $p = 4$ and $r = 14$ (the simplest case to make q an integer).

Substituting for p and r in either dot product equation gives us q: (first equation chosen here)

$$\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} . \begin{pmatrix} 4 \\ q \\ 14 \end{pmatrix} = 0 \;\Rightarrow 12 + 2q + 42 = 0 \;\Rightarrow\; 2q + 54 = 0 \;\Rightarrow\; q = -27.$$

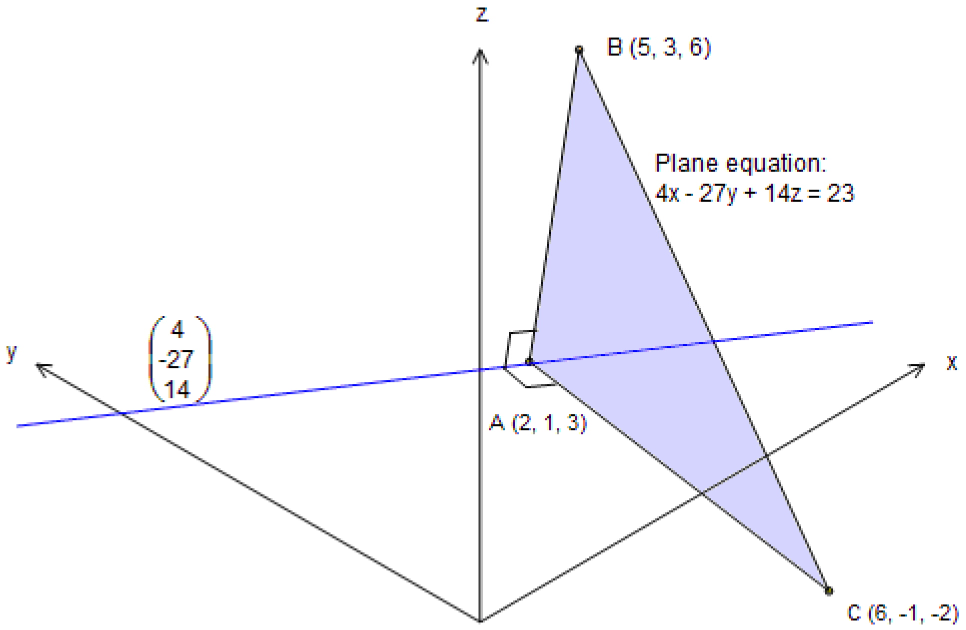
One perpendicular vector is therefore $\begin{pmatrix} 4 \\ -27 \\ 14 \end{pmatrix}$.

The corresponding Cartesian equation of the plane is $4x - 27y + 14z = d$ where d is to be determined.

To find d, we substitute the coordinates for any of points A, B, or C :

Thus (for point A) 4(2) -27(1) + 14(3) = 8 – 27 +42 = 23.
(check: for point B) 4(5) -27(3) + 14(6) = 20 – 81 + 84 = 23.

The Cartesian equation of the plane containing the three points is thus **$4x - 27y + 14z = 23$.**



Finding the Cartesian equation of a plane from parametric vector form.

This method is analogous to earlier methods for finding Cartesian equations of lines from vector form, but this time we need to find two parameters rather than one.

Example(10): Find the Cartesian equation of the plane with parametric vector equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Expressing $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and equating **i**-, **j**- and **k**-components we have

$x = 3 + 2\lambda + \mu$ A (equating **i** – components)
$y = -\lambda + \mu$ B (equating **j** – components)
$z = 1 + \mu$ C (equating **k** – components)

Starting with equation C, we obtain $\mu = z - 1$.
Substituting for μ in B gives $y = -\lambda + z - 1 \Rightarrow \lambda = z - 1 - y$.

Finally, substituting for both λ and μ in A gives $x = 3 + 2(z - 1 - y) + (z - 1)$
$\Rightarrow x = 3 + 2z - 2 - 2y + z - 1$
$\Rightarrow x + 2y - 3z = 0.$

The intersection of a line and a plane.

There are three possibilities:

- The line may intersect the plane at a single point
- The line may be parallel to the plane and never intersect
- The line may be entirely inside the plane.

Example (11): A laser beam is aimed from the point (12, 10, 10) in the direction $\begin{pmatrix}-2\\-2\\-3\end{pmatrix}$ towards a plane surface containing the point (1, 4, 2). The vector $\begin{pmatrix}3\\-5\\15\end{pmatrix}$ is perpendicular to the plane.

i) Find the coordinate of the point where the laser beam meets the surface of the plane.

ii) What would happen if the laser beam were aimed from the same point and towards the same plane, but in the direction $\begin{pmatrix}-5\\-6\\-1\end{pmatrix}$?

iii) What would happen if the laser beam were aimed at the same plane and in the direction $\begin{pmatrix}-5\\-6\\-1\end{pmatrix}$ as in ii), but this time from the point (1, 1, 1) ?

i) We need to find the vector form of the path of the beam and the Cartesian equation of the plane first:

The path of the laser beam (in vector form) is $\begin{pmatrix}12\\10\\10\end{pmatrix} + \lambda \begin{pmatrix}-2\\-2\\-3\end{pmatrix}$.

The Cartesian equation of the plane is $3x - 5y + 15z = p$ where (x, y, z) is the position vector of a point on the plane. We are given that the point (1, 4, 2) lies on the plane.

Substituting for x, y and z gives $p = 3 - 20 + 30 = 13$, and so the Cartesian equation of the plane is **$3x - 5y + 15z = 13$.**

The position vector of a general point on the laser beam's path is given as $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}12-2\lambda\\10-2\lambda\\10-3\lambda\end{pmatrix}$.

To find where the line meets the plane, we therefore substitute for x, y and z in the equation of the plane, find the value of the parameter λ, and finally substitute for λ in the equation for the line.

Thus $3(12-2\lambda) - 5(10-2\lambda) + 15(10-3\lambda) = 13 \Rightarrow 36 - 6\lambda - 50 + 10\lambda + 150 - 45\lambda = 13$
$\Rightarrow 136 - 41\lambda = 13 \Rightarrow 123 - 41\lambda = 0 \Rightarrow \lambda = 3$.

Substituting $\lambda = 3$ in the equation of the laser beam's path gives a point with position vector

$$\begin{pmatrix} 12 \\ 10 \\ 10 \end{pmatrix} + 3 \begin{pmatrix} -2 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix}.$$

$\therefore$ the laser beam meets the plane at the point (6, 4, 1).

ii) The equation of the plane is identical to that in part i), but this time a general point on the laser beam's path has a position vector of $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12-5\lambda \\ 10-6\lambda \\ 10-\lambda \end{pmatrix}$.

Substitution for x, y and z in the equation of the plane gives $3(12-5\lambda) - 5(10-6\lambda) + 15(10-\lambda) = 13$
$\Rightarrow 36 - 15\lambda - 50 + 30\lambda + 150 - 15\lambda = 13 \quad \Rightarrow 136 = 13$.

The λs seem to have cancelled out and left an inconsistent result. There is no solution, therefore the line and the plane do not meet and the line is parallel to the plane.

Another way of showing the result would be to use dot products.

The vector $\mathbf{n} = \begin{pmatrix} 3 \\ -5 \\ 15 \end{pmatrix}$ is perpendicular to the plane, and if the direction vector of the laser beam is also perpendicular to **n**, then the laser beam's ray is parallel to the plane.

In fact, $\begin{pmatrix} -5 \\ -6 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 15 \end{pmatrix} = -15 + 30 - 15 = 0$, $\therefore$ the laser beam is parallel to the plane and never meets it.

iii) The equation of the plane is identical to that in both previous parts, and this time a general point on the laser beam's path has position vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-5\lambda \\ 1-6\lambda \\ 1-\lambda \end{pmatrix}$.

Substitution for x, y and z in the equation of the plane gives $3(1-5\lambda) - 5(1-6\lambda) + 15(1-\lambda) = 13$
$\Rightarrow 3 - 15\lambda - 5 + 30\lambda + 15 - 15\lambda = 13 \quad \Rightarrow 13 = 13$.

Again, the λs seem to have cancelled out but this time we have a consistent result, true for all λ.

This corresponds to the line actually being on the plane itself. Again, the dot product of the laser's direction vector and the normal is zero as in ii), but we could show that the point (1, 1, 1) is on the plane simply by substituting into x, y and z in the equation of the plane to give $3 - 5 + 15 = 13$.

The (acute) angle between a line and a plane.

We use a variant of the formula for the angle between two lines

$$\cos\theta = \frac{\mathbf{p.q}}{|\mathbf{p}||\mathbf{q}|} \quad \text{modified to} \quad \cos\theta = \frac{|\mathbf{a.n}|}{|\mathbf{a}||\mathbf{n}|}$$

where **a** is the direction vector of the line and **n** is the direction vector of the normal (perpendicular) to the plane.

The modulus function is used to ensure an acute-angle solution.

Note also that the angle is that of the **normal** to the plane – to obtain the angle between line and plane we must subtract the result from 90° or modify the formula to $\sin\theta = \frac{|\mathbf{a.n}|}{|\mathbf{a}||\mathbf{n}|}$

since sin θ° = cos (90-θ)°.

Example (13): A laser beam is aimed in the direction $\mathbf{a} = \begin{pmatrix} -2 \\ -2 \\ -3 \end{pmatrix}$ and meets a plane surface perpendicular to the vector $\mathbf{n} = \begin{pmatrix} 3 \\ -5 \\ 15 \end{pmatrix}$.

Find the angle between the laser beam and the plane.

We substitute into the modified dot product formula $\sin\theta = \frac{|\mathbf{a.n}|}{|\mathbf{a}||\mathbf{n}|}$

$$\sin\theta = \frac{(-2)(3)+(-2)(-5)+(-3)(15)}{\sqrt{3^2+(-5)^2+15^2}\times\sqrt{(-2)^2+(-2)^2+(-3)^2}}$$

$$\Rightarrow \sin\theta = \frac{41}{\sqrt{259}\sqrt{17}}$$

$\Rightarrow$ θ = 38.2° to 1 d.p.

The intersection of two planes.

There are three possibilities:

- The planes may intersect, having a line in common
- The planes may be parallel and never intersect
- The planes may be coincident.

When the equations of two planes are given in scalar product form or Cartesian form it is easy to find out whether they intersect or not.

If two planes have equations of $\mathbf{r}.\begin{pmatrix}1\\1\\0\end{pmatrix} = 2$ and $\mathbf{r}.\begin{pmatrix}1\\1\\0\end{pmatrix} = 7$ respectively, then it can be seen at once that they are parallel, since they are both perpendicular to the same normal.

Example (13): Two planes have Cartesian equations of $4x + 8y - 12z = 20$ and $-3x - 6y + 9z = 15$. respectively. Show that they are parallel.

Both equations can be factorised out to give $4(x + 2y - 3z) = 20$ and $3(x + 2y - 3z) = 15$
The bracketed terms are equal, therefore the planes are parallel as each can be written in the form $\mathbf{r}.\begin{pmatrix}1\\2\\-3\end{pmatrix} = p$.

The planes are not only parallel, but also coincident, since each is a multiple of $x + 2y - 3z = 5$.

Example (14): Find the equation of the line of intersection of the two planes
$7x - 4y + 3z = -3$ and $4x + 2y + z = 4$. Give the result both in Cartesian and vector form.

$7x - 4y + 3z = -3$	*A*
$4x + 2y + z = 4$	*B*
$5x + 10y = 15$	*3B – A* (eliminating *z*)

The last equation can be rewritten as $x + 2y = 3 \rightarrow x = 3 - 2y$.

$7x - 4y + 3z = -3$	*A*
$4x + 2y + z = 4$	*B*
$15x + 5z = 5$	*A +2B* (eliminating *y*)

The last equation can be rewritten as $3x + z = 1 \Rightarrow x = \frac{1-z}{3}$.

The Cartesian equations of the line are thus $x = 3 - 2y = \frac{1-z}{3}$.

We must now rewrite the equations in the form $\frac{x-a}{d} = \frac{y-b}{e} = \frac{z-c}{f} (= \lambda)$,

i.e. $\frac{x-0}{1} = \frac{y-\frac{3}{2}}{-\frac{1}{2}} = \frac{z-1}{-3}$, to give the Cartesian equation of the line.

To express the equation of the line in vector form , we just read off the values *a* to *f* in the Cartesian equation, and the position vector is $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, with direction vector $\lambda \begin{pmatrix} d \\ e \\ f \end{pmatrix}$.

$\therefore$ The vector equation of the line is $\begin{pmatrix} 0 \\ \frac{3}{2} \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -3 \end{pmatrix}$.

This expression is correct, but by setting $\lambda = 1 + 2\mu$, we can write the equation without fractions:

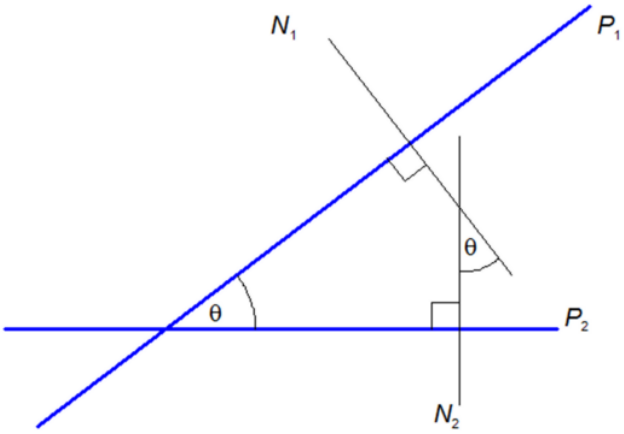
$\begin{pmatrix} 0 \\ \frac{3}{2} \\ 1 \end{pmatrix} + (1 + 2\mu) \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -3 \end{pmatrix}$.

$\Rightarrow$ position vector $\begin{pmatrix} 0 \\ \frac{3}{2} \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -3 \end{pmatrix}$, direction vector $2\mu \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -3 \end{pmatrix}$.

$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ -6 \end{pmatrix}$.

The angle between two planes.

The acute angle between two planes P_1 and P_2
is the same as the angle between their normals N_1 and N_2, as the diagram on the right shows.



To find the acute angle between two planes, we therefore find the angle between the direction vectors of their normals, i.e. the angle satisfying

$$\cos \theta = \frac{|\mathbf{p.q}|}{|\mathbf{p}||\mathbf{q}|}$$

where **p** and **q** are the direction vectors of the normals. (We use the modulus function to ensure the acute angle)

Example (15): Find the acute angle between the two planes whose Cartesian equations are $6x + 2y + 5z = 4$ and $x – 4y + 3z = 9$.

From the Cartesian equations, we can obtain the direction vectors of the normals at once:

$\mathbf{n_1} = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}$ and $\mathbf{n_2} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$.

From there we substitute into the equation for the angle between two lines:

$$\cos\theta = \frac{6+(-8)+15}{\sqrt{6^2+2^2+5^2}\times\sqrt{1^2+(-4)^2+3^2}} \Rightarrow \cos\theta = \frac{13}{\sqrt{65}\sqrt{26}}$$

$$\Rightarrow \cos\theta = \frac{13}{(\sqrt{5}\sqrt{13})(\sqrt{2}\sqrt{13})} \Rightarrow \cos\theta = \frac{1}{(\sqrt{10})}$$

$$\therefore\ \theta = 71.6°.$$

Example (16): Find the acute angle between the two planes whose scalar product equations are

$\mathbf{r.}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 7$ and $\mathbf{r.}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 10$.

We just find the angle between the normals to obtain

$$\cos\theta = \frac{2+1+3}{\sqrt{1^2+(-1)^2+1^2}\times\sqrt{2^2+(-1)^2+3^2}} \Rightarrow \cos\theta = \frac{6}{\sqrt{3}\sqrt{14}}$$

$$\Rightarrow \cos\theta = \frac{6}{(\sqrt{6}\sqrt{7}))} \Rightarrow \cos\theta = \frac{6\sqrt{6}}{6\sqrt{7}} \Rightarrow \cos\theta = \frac{\sqrt{6}}{\sqrt{7}}.$$

$\therefore\ \theta = 22.2°$.

Notice how the right-hand sides of the equations in each case do not affect the result.

In Example 11, we found that the acute angle between the planes with Cartesian equations $6x + 2y + 5z = 4$ and $x - 4y + 3z = 9$ was 71.6°.

The same result would have held if the planes' equations were (say) $6x + 2y + 5z = 10$ and $x - 4y + 3z = 2$.

Distance of a plane from the origin.

(Unit vectors will be referred to in the underlined form $\underline{\hat{n}}$ rather than boldface due to software issues).

If a plane has the equation **r.** $\underline{\hat{n}}$ = d, where $\underline{\hat{n}}$ is a *unit* vector normal to the plane, then d is the perpendicular distance of the plane from the origin.

Example (17): A plane contains a point A, whose position vector is **a** = $\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$. This plane is also normal to the vector $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$. Find the perpendicular distance of the plane from the origin.

Firstly we must write the vector equation in scalar product form:

$$\mathbf{r}.\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \mathbf{a}.\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \Rightarrow \mathbf{r}.\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}.\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \Rightarrow \mathbf{r}.\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 3 + 8 - 4 = 7.$$

The vector $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ is normal to the plane, but it is not of unit length. In fact its length is $\sqrt{(1^2 + 2^2 + (-2)^2)}$ or 3 units.

We therefore divide the equation by 3 throughout to obtain

$$\mathbf{r}.\ \tfrac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \tfrac{7}{3}$$

giving the perpendicular distance of the plane from the origin as $\frac{7}{3}$ units.

Distance between parallel planes.

The method used in the previous example can also be adapted to obtain the distance between parallel planes.

Supposing one plane had the equation $\mathbf{r}.\begin{pmatrix}1\\2\\-2\end{pmatrix} = 7$, and another one had the equation $\mathbf{r}.\begin{pmatrix}1\\2\\-2\end{pmatrix} = 13$. They would be normal to the same vector, namely $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ and therefore parallel.

The same would apply to a plane with equation $\mathbf{r}.\begin{pmatrix}1\\2\\-2\end{pmatrix}$ -5, except that in this case the plane would be on the other side of the origin to $\mathbf{r}.\begin{pmatrix}1\\2\\-2\end{pmatrix} = 7$.

Example (18): Find the distance between the planes with equations
$\mathbf{r}.\begin{pmatrix}1\\2\\-2\end{pmatrix} = 7$ and $\mathbf{r}.\begin{pmatrix}1\\2\\-2\end{pmatrix} = -5$.

We found the distance of the plane $\mathbf{r}.\begin{pmatrix}1\\2\\-2\end{pmatrix} = 7$ from the origin in the last question.

We therefore again we divide the equation $\mathbf{r}.\begin{pmatrix}1\\2\\-2\end{pmatrix} = -5$ by 3 (the length of the normal) and obtain

$\mathbf{r}.\ \frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix} = -\frac{5}{3}$ units, i.e. the plane is $\frac{5}{3}$ units from the origin, but on the opposite side to the plane with equation $\mathbf{r}.\begin{pmatrix}1\\2\\-2\end{pmatrix} = 7$.

$\therefore$ the distance between the planes is $\frac{7}{3} - (-\frac{5}{3})$ units, or 4 units.