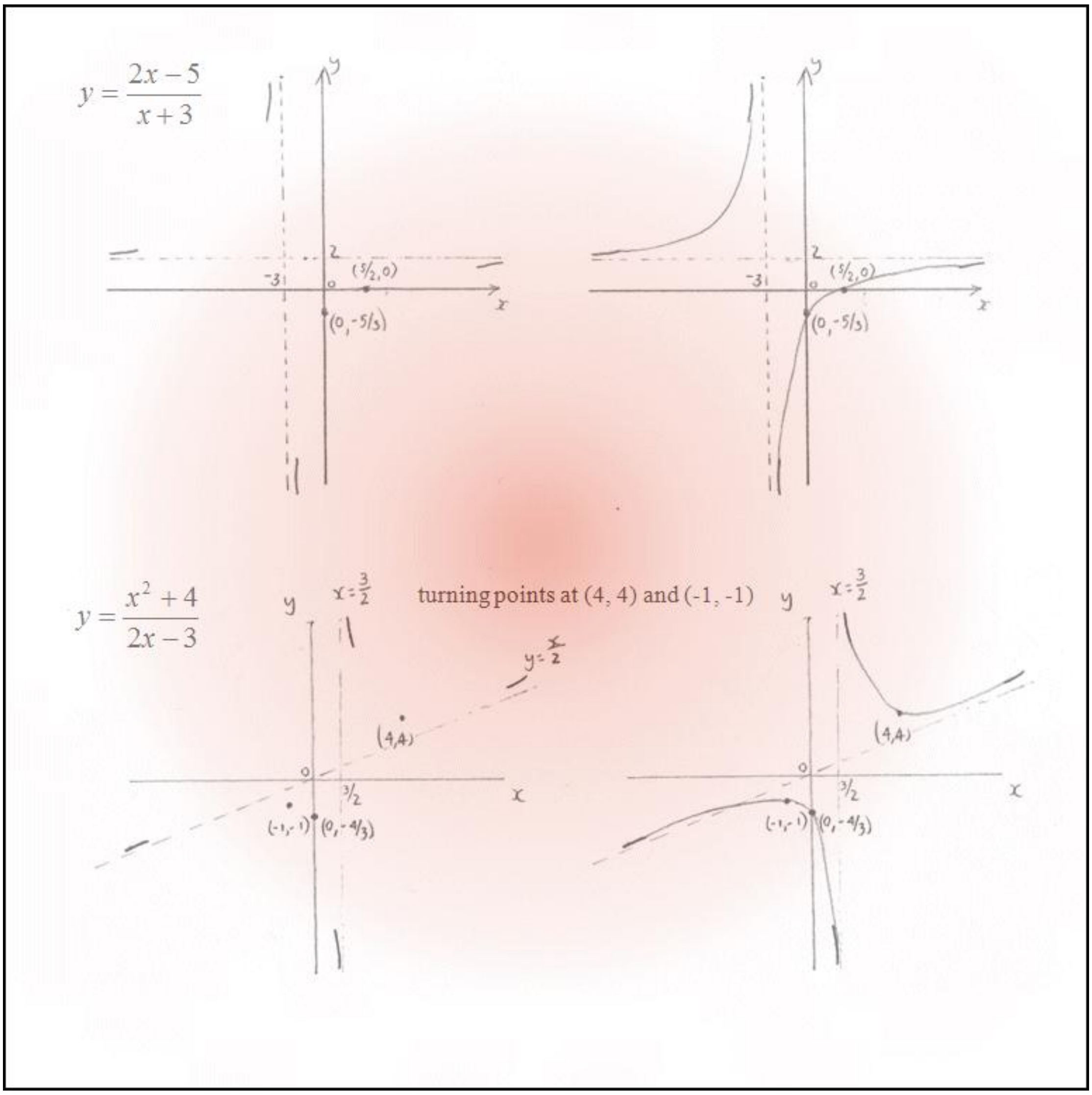
M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level MEI

OCR MEI: **C4**

MORE CURVE SKETCHING



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Further Curve Sketching.

This topic has been discussed under 'Quadratics' and 'Polynomials' in Core 1, and under 'Curve Sketching' in Core 2. This covered the methods of sketching other types of functions, such as cubics or reciprocals.

Again, the document 'Transformations of Graphs' showed how one graph could be obtained from another using transformations.

This section will deal more miscellaneous functions, but the basic steps are still the same.

Read the question.
The question asks for a sketch graph, and the mark scheme will give a clue about the detail involved. Do not waste time finding complicated turning points if the sketch is only worth 2 marks !

Find out where the curve intersects the axes.
Substitute $x = 0$ to find where the curve meets the y-axis; substitute $y = 0$ to find where it intersects the x-axis. If the required graph is that of a product of two functions, then the curve intersects the x-axis when either term of the product is zero.

Check for symmetry and periodicity.
If the function is even (i.e. it is a sum of even powers of x, including constant terms), then its graph will have reflective symmetry in the y-axis.
If the function is odd, (i.e. it is a sum of odd powers of x only) then its graph will have rotational symmetry of order 2 about the origin.

Note also that $\sin x$ is odd with a period of 2π (360°), and $\cos x$ is even, also with a period of 2π.

A sum / difference of two odd functions is odd.
A sum / difference of two even functions is even.
A product / quotient of two odd functions, or of two even functions, is even.
A product / quotient of an even function and an odd function is odd.

Remember that the vast majority of functions are neither odd nor even.

Check the behaviour of the function for large values of *x* and *y* (both positive and negative).
If the function is of rational or reciprocal type, then is value will tend to a limit without actually approaching it. This limit is known as an asymptote – a line that the curve would approach ever more closely, but never actually meet.

For a polynomial, as x becomes increasingly large, the function takes on the characteristics of the term of the highest power of x. (the highest power of x will tend to dominate or swamp the lower ones).

Check for discontinuities, and the behaviour of the function near them.
These will occur in graphs of rational or reciprocal functions, for values of x that make the denominator equal to zero. These discontinuities also give rise to **asymptotes** (see last sentence). Most graphs of this type will occur in unconnected parts, with the asymptotes as barriers between them.

Check for stationary points if the question asks for them.
If the question doesn't ask you to find them, it's either because the question is only worth a few marks, or the calculus becomes too time-consuming and messy.

Find out any points where the gradient is zero (differentiate once), and determine if they are maximum or minimum points (differentiate again). Also note the special case of repeated factors in the polynomial; if a factor $(x\text{-}a)$ is repeated, then the x-axis is tangent to the stationary point at $x = a$.

Join up.
Connect all points and arcs with a smooth curve (or curves, if the function is discontinuous).
This part is generally only worth one mark, so don't lose sleep over the smoothness or accuracy !

The earlier section on curve sketching covered polynomials and reciprocal graphs of the basic $\frac{1}{x}$ type. There are four other groups of functions to look at here.

Functions of the form $f(x) \pm g(x)$.

Example (1): Sketch the graph of $y = \frac{1}{x^2} - 9$. (There are no stationary points.)

<u>Intercepts</u>. When $x = 0$, y is undefined, and so the y-axis is an asymptote to the graph. ***(Mark the y-axis as an asymptote).***

To find the x-intercept(s), we solve $\frac{1}{x^2} = 9$, giving $x^2 = \frac{1}{9}$ or $x = \pm\frac{1}{3}$.

(Plot the points ($\frac{1}{3}$, 0) and (-$\frac{1}{3}$, 0).

<u>Symmetry</u>. Both $\frac{1}{x^2}$ and the constant function –9 are even, therefore $y = \frac{1}{x^2} - 9$ is an even function, with symmetry about the y-axis. (The x-intercepts gave a clue !)

<u>Behaviour for large x</u>. When x is large (both positive and negative, due to symmetry), $\frac{1}{x^2}$ is positive but ever-closer to 0 and thus $\frac{1}{x^2} - 9$ becomes closer to –9 from above. (Asymptote at the line $y = -9$).

(Draw an asymptote at y = -9.)
(Draw near-horizontal arcs at the left and right of the graph, just above the line y = -9.)

<u>Discontinuities and behaviour near them.</u>
When x is just greater than 0, y is large and positive., and this also holds true when x is just less than 0, by the symmetry of the graph.

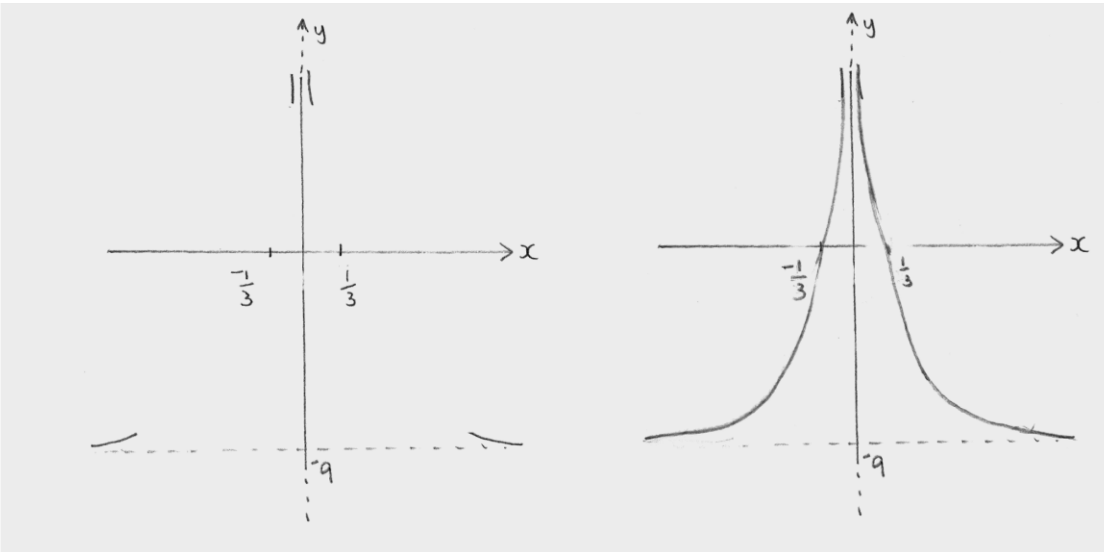
(Draw near-vertical arcs at the top of the graph on each side of the y-axis).

<u>Stationary points.</u> Not applicable here.

<u>Join up.</u>
(Connect the arcs and points on each side of the asymptote at the y-axis with a smooth curve).

The graph is actually a y-shift of that of $\frac{1}{x^2}$ by –9 units

For a better sketch, we could also have plotted for $x = 1$ and $x = -1$ to get the 'corners' right, but examiners only check the *general* shape.



Example (2): Sketch the graph of $y = x - \frac{4}{x}$.

Intercepts. When $x = 0$, y is undefined, so the y-axis is an asymptote to the graph, i.e. there is no y-intercept. ***(Mark the y-axis as an asymptote).***

To find the x-intercept(s), we solve $x - \frac{4}{x} = 0 \Rightarrow x^2 - 4 = 0$, giving or $x = \pm 2$.

(Plot the points (2, 0) and (-2, 0)).

Symmetry. The function $y = x$ is an odd function, as is $y = \frac{4}{x}$; the sum of two odd functions is odd, and so the graph of $y = x - \frac{4}{x}$ will have rotational symmetry about the origin.

Behaviour for large x.
When x is large and positive, the fractional part will be small and positive, so y will approach x ever more closely from below. By symmetry, a large negative value of x will have y approaching x from above.

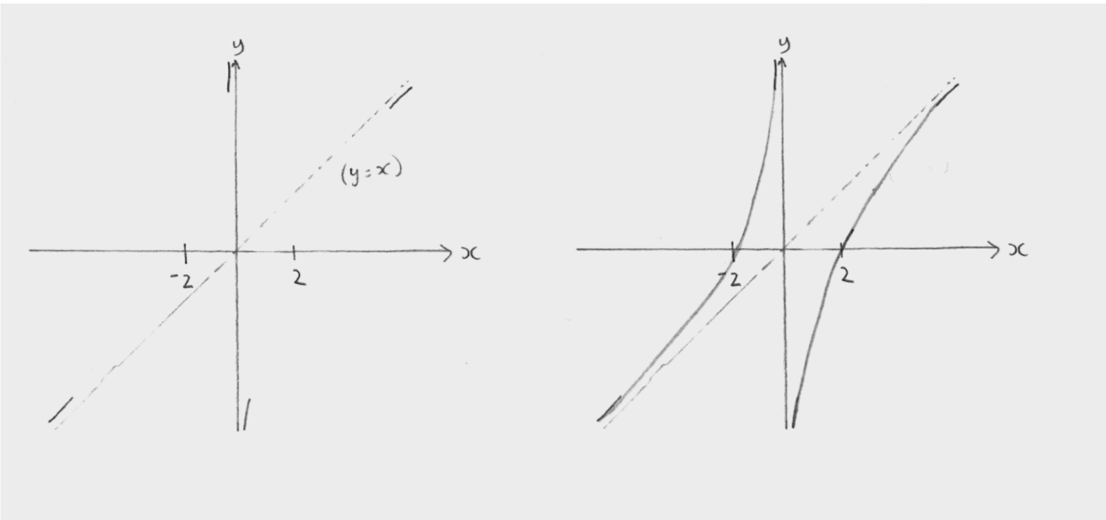
(Draw an asymptote at the line y = x).
(Draw a short arc below that line, and parallel to it, for large x).
(Draw a similar arc above that line, for small x).

Discontinuities and behaviour near them.
As shown under 'Intercepts', the y-axis is an asymptote to the graph.
When x is close to 0 and positive, y is large and negative, and when x is near 0 and negative, y is large and positive.

(Draw a near-vertical arc at the top of the graph to the left of the y-axis, and another one at the bottom, right of the y-axis).

Stationary points. We find that $\frac{d}{dx}\left(x - \frac{4}{x}\right) = \left(1 + \frac{4}{x^2}\right)$.
This derivative cannot take a value of zero for any real x, so there are no stationary points.

Join up.
(Connect the arcs and points on each side of the asymptote at the y-axis with a smooth curve).



Example (3): Sketch the graph of $y = \frac{x}{2} + \frac{1}{2x}$.

Intercepts. When $x = 0$, y is undefined, so the y-axis is an asymptote to the graph, i.e. there is no y-intercept. ***(Mark the y-axis as an asymptote).***

To find the x-intercept(s), we solve $\frac{x}{2} + \frac{1}{2x} = 0 \Rightarrow x^2 + 1 = 0$. This equation has no solution, and so there is no x-intercept either.

Symmetry. Both component functions are odd, so the graph of $y = \frac{x}{2} + \frac{1}{2x}$ will itself be odd and have rotational symmetry about the origin.

Behaviour for large x.
For large positive x, $\frac{1}{2x}$ will be small and positive, so y will approach $\frac{x}{2}$ ever more closely from above. By symmetry, y will approach $\frac{x}{2}$ from below when x is large and negative.

(Draw an asymptote at the line y = $\frac{x}{2}$).
(Draw a short arc above that line, and parallel to it, for large x).
(Draw a similar arc below that line, for small x).

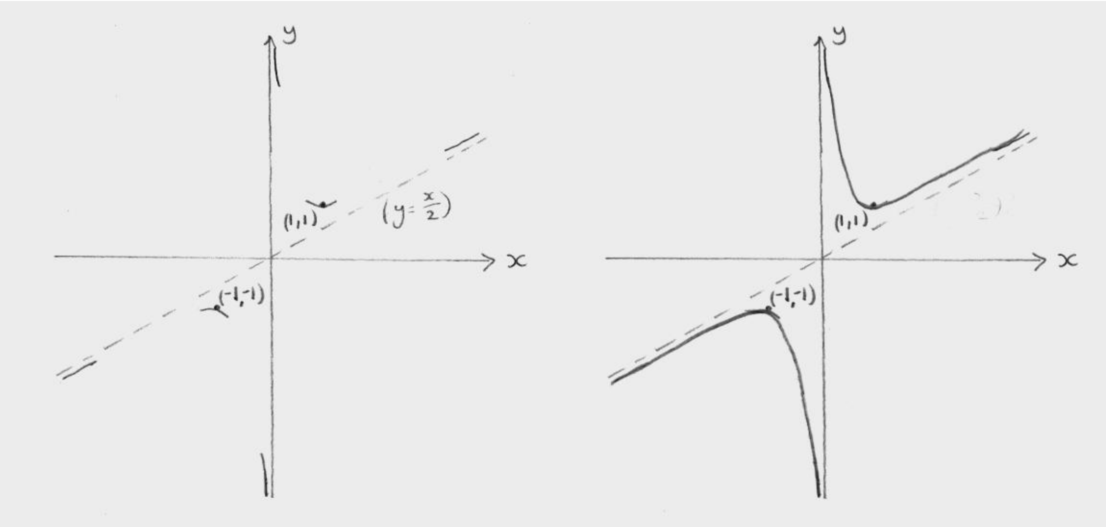
Discontinuities and behaviour near them.
The y-axis is an asymptote to the graph. When x is close to 0 and positive, y is large and positive; when x is close to 0 and negative, y is large and negative.

(Draw a near-vertical arc at the top of the graph to the right of the y-axis, and another one at the bottom left of the y-axis).

Stationary points. We find that $\frac{d}{dx}\left(\frac{x}{2} + \frac{1}{2x}\right) = \left(\frac{1}{2} - \frac{1}{2x^2}\right) = \frac{1}{2}\left(1 - \frac{1}{x^2}\right)$.

This derivative takes a value of zero when $\frac{1}{x^2} = 1$, i.e. when $x = \pm 1$. The stationary points are thus (1, 1) and (-1, -1). ***(Plot the points (1, 1) and (-1, -1)).***

Join up.
(Connect the arcs and points on each side of the asymptote at the y-axis with a smooth curve).



Functions of the form $f(x)g(x)$.

Example (4): Sketch the graph of $y = (2-x)e^x$ for $-3 \le x \le 3$, including any turning points.

Intercepts. When $x = 0$, $y = 2$. ***(Plot the point (0,2)).***
For the product to be zero, either e^x must be zero or $(2-x)$ must be zero.
The function e^x has an asymptote at $y = 0$, i.e. it is always > 0. The product $(2-x)e^x$ therefore only takes zero value when $(2-x) = 0$, or when $x = 2$. ***(Plot the point (2,0)).***

Symmetry. Neither $(2-x)$ nor e^x is even nor odd, so no help here.

Behaviour for large x.
When $x > 2$, $(2-x)$ becomes increasingly negative and e^x increasingly positive, therefore the product becomes large and negative.
(Draw a steep downward arc at the bottom right of the graph, i.e. where x = 3.)

When $x < 0$, $(2-x)$ becomes increasingly positive, but e^x increasingly closer to zero whilst still positive.
(Draw a shallow upward arc just above the x-axis at the left of the graph, i.e. where x = -3.)

Discontinuities and behaviour near them. Both $(2-x)$ and e^x are continuous, and so is their product.

Stationary points.

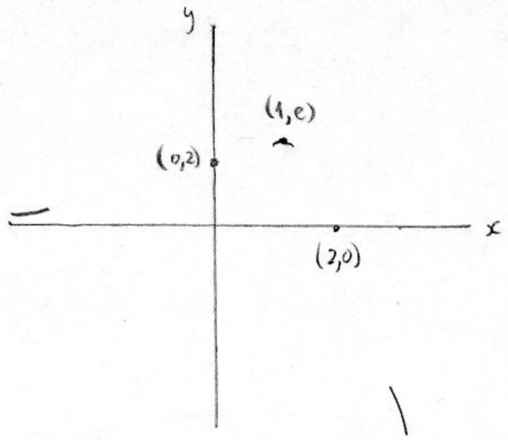
Differentiation by the product rule gives $\frac{dy}{dx} = (2-x)e^x - e^x$ or $(1-x)e^x$.
This derivative is zero when $x = 1$ (remember e^x is never zero), and substituting in the original function gives $y = (2-1)e^1$ or e.

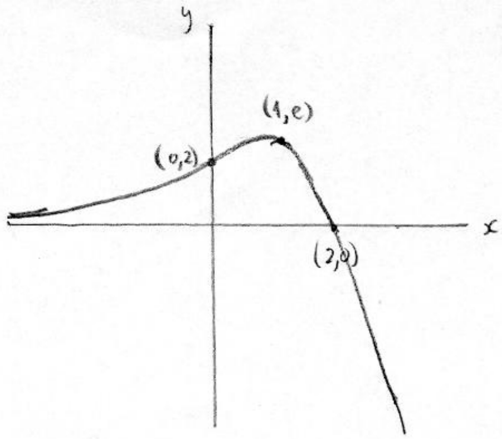
The turning point is at $(1, e)$, and further differentiation gives $\frac{d^2y}{dx^2} = \frac{d}{dx}(1-x)e^x$

$\Rightarrow$ $(1-x)e^x - e^x$ or $-xe^x$. When $x = 1$, the second derivative is $-e$, i.e. negative, and so $(1, e)$ is a maximum point.

(Draw a short 'crest' arc at the point (1, e)).

Join up.
(Connect up all points and arcs with a smooth curve.)





Example(5): Sketch the graph of the following function: $f(x) = 0$ for $x = 0$; $x(\ln x - 1)$ for x > 0, including any turning points.

Intercepts. The y-intercept is trivially (0, 0).

We next need to find the x-intercepts, if any. The origin is one; so is the point where $\ln x - 1 = 0$, namely $(e, 0)$. ***(Plot the point (e,0)).***

Symmetry. $y = x$ is an odd function, but $y = (\ln x - 1)$ is neither, so the product is neither odd nor even.

Behaviour for large x. When $x > e$, both x and $(\ln x - 1)$ are positive and increasing.
(Draw an upward arc at the top right of the graph.)

***Discontinuities and behaviour near them*.** Both x and $(\ln x - 1)$ are continuous.

Stationary points.

Differentiation by the product rule gives $\frac{dy}{dx} = (\ln x - 1) + 1$ or simply $\ln x$.

This derivative is zero when $x = 1$, and substituting in the original function gives $y = 1(\ln 1 - 1)$or -1.

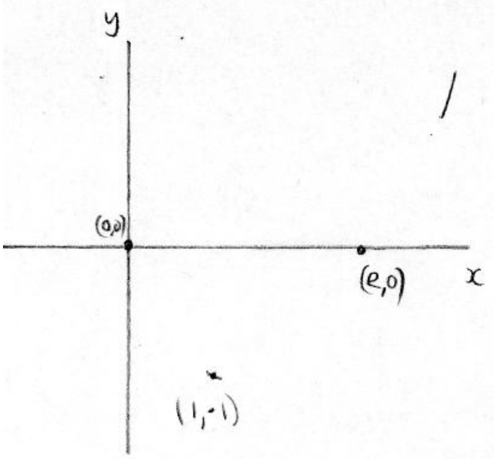
The turning point is at (1, -1) , and further differentiation gives $\frac{d^2y}{dx^2} = \frac{d}{dx}(\ln x) = \frac{1}{x}$.

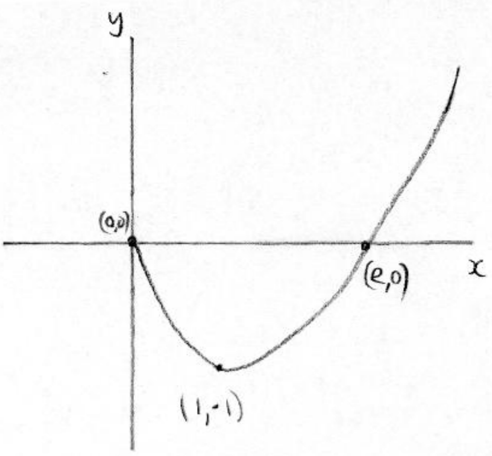
When $x = 1$, the second derivative is 1, i.e. positive, and so (1, -1) is a minimum point.

(Draw a short 'trough' arc at the point (1,-1)).

When $x < 1$, the function is negative and decreasing.
When $1 < x < e$, the function is negative but increasing .

Join up.
(Join points (0, 0) and the minimum at (1, -1) by a curve)
(Join points (1, -1) and (e, 0) by a curve with increasing gradient, and continue the curve to pass through the upward arc at top right.)

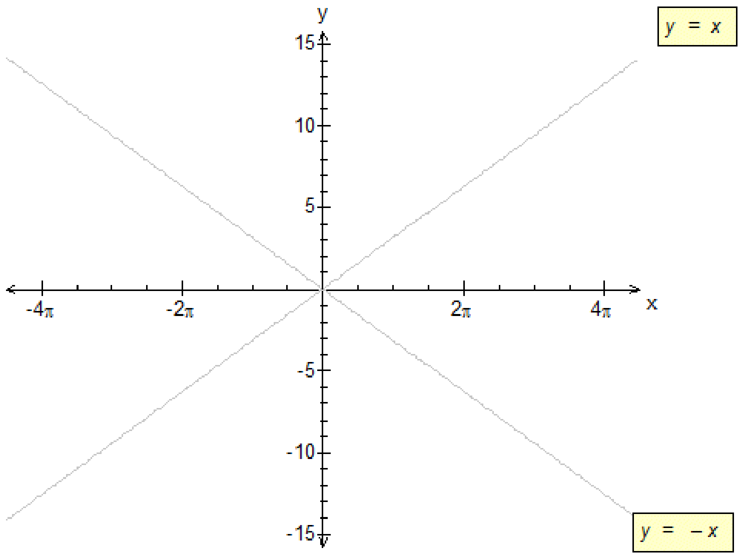




Example (6):

i) Write out a table of values for sin x for $0 \le x \le 4\pi$ in steps of $\pi/2$.
ii) Hence sketch the graph of $y = x \sin x$ for $-4\pi \le x \le 4\pi$.

The functions $y = x$ and $y = -x$ have been drawn for reference.



Hint: use properties of odd and even functions for completing the graph for negative x.

The table of required values for sin x is as follows;

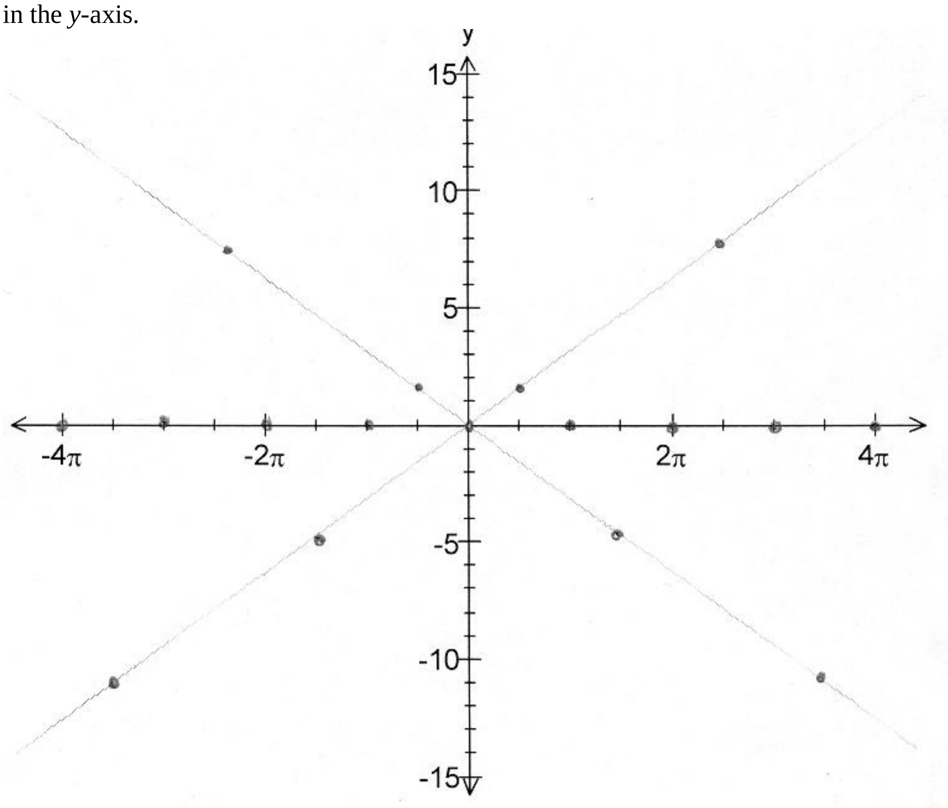
x	0	$\pi/2$	π	$3\pi/2$	2π	$5\pi/2$	3π	$7\pi/2$	4π
sin x	0	1	0	-1	0	1	0	-1	0

Intercepts. Firstly, we find the x-intercepts in the range, i.e. values of x where $y = 0$. ***(Plot the origin and all the multiples of π on the x-axis.)***

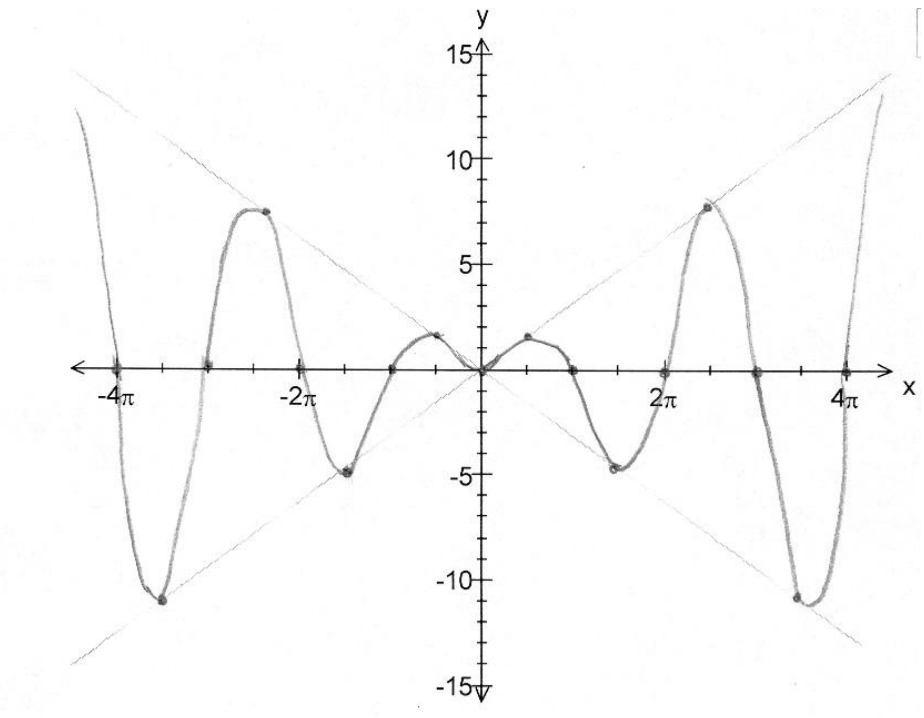
We then see that $\sin x = 1$, and hence $x \sin x = x$, when $x = \pi/2$ and $5\pi/2$.
(Plot the points corresponding to those two x-values on the line y = x.)

Similarly we see that $\sin x = -1$, and hence $x \sin x = -x$, when $x = 3\pi/2$ and $7\pi/2$.
(Plot the points corresponding to those two x-values on the line y = -x.)

So far, we have plotted the points for $0 \le x \le 4\pi$, but we need the complete it for negative *x*.
Instead of plotting a table, we can use the properties of odd and even functions. Both the sine function and the function $y = x$ are odd, therefore their product is even. We thus reflect the existing set of points in the *y*-axis.

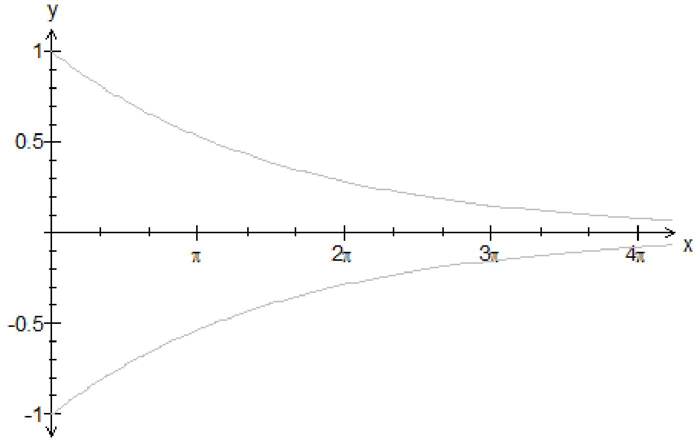


Join up.
We can therefore complete the graph of the even function $x \sin x$ for $-4\pi \le x \le 4\pi$ by joining the points with a sine curve.



Example (7):
i) Write out a table of values for cos x for $0 \le x \le 4\pi$ in steps of $\pi/2$.
ii) Hence sketch the graph of $e^{-0.2x}\cos x$ for $0 \le x \le 4\pi$.

The functions $e^{-0.2x}$ (above the x-axis) and $-e^{-0.2x}$ (below the x-axis) have been drawn for reference.



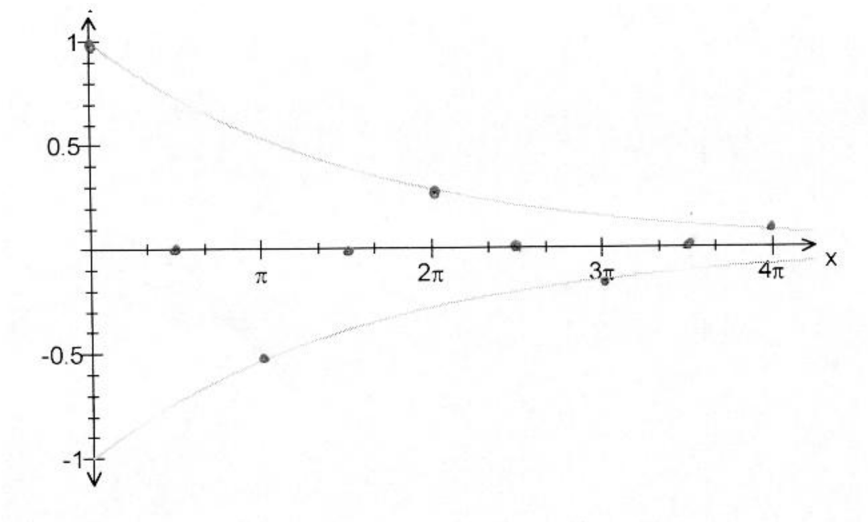
The table of required values for cos x is as follows;

x	0	$\pi/2$	π	$3\pi/2$	2π	$5\pi/2$	3π	$7\pi/2$	4π
cos x	1	0	-1	0	1	0	-1	0	1

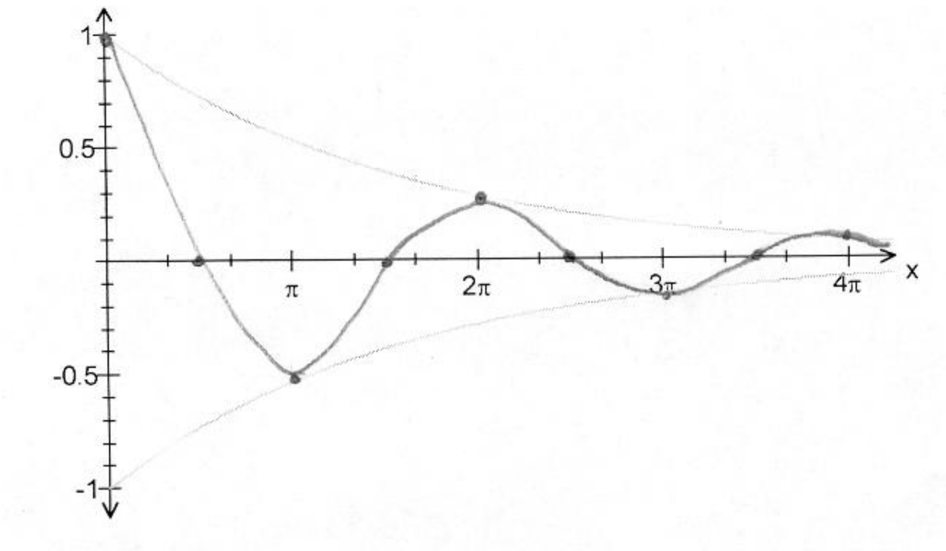
Intercepts. Firstly, we find the x-intercepts in the range, i.e. values of x where $y = 0$.
(Plot those four points on the x-axis.)

We then find the values of x satisfying cos $x = 1$; here they are 0, 2π and 4π. For those values of x, $e^{-0.2x}\cos x = e^{-0.2x}$, and so they will lie on the curve $e^{-0.2x}$.
(Plot the points corresponding to those three x-values on the upper curve.)

The values of x satisfying cos $x = -1$ are π and 3π. For those values of x, $e^{-0.2x}\cos x = -e^{-0.2x}$, and so they will lie on the curve $-e^{-0.2x}$.
(Plot the points corresponding to those two x-values on the lower curve.)



Join up.
(Finally, join all the points with a 'sine' curve.)



These functions of the form $e^{-kt}\cos t$ or $e^{-kt}\sin t$ where t represents time are important in mechanics, as they represent 'damping' or the decay of oscillations of an object such as a pendulum.

Functions of the form $\frac{f(x)}{g(x)}$.

Such graphs require a little more thought when sketching, but again it is a case of following the hints.

Example (8): Sketch the graph of $y = \frac{2x-5}{x+3}$, including asymptotes.

Intercepts. When $x = 0$, $y = -\frac{5}{3}$; also when $x = \frac{5}{2}$, $y = 0$. ***(Plot the points).***

Symmetry. Function neither odd nor even – no help.

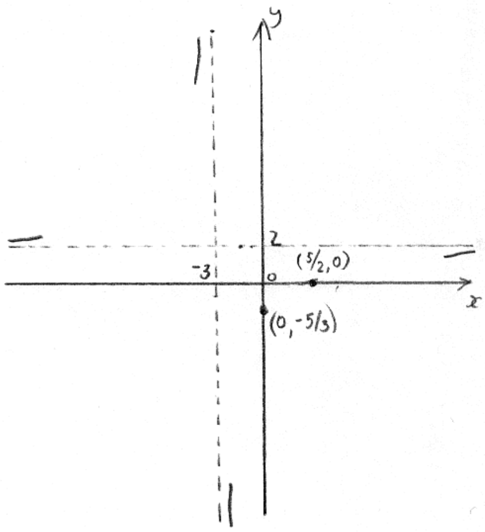
Behaviour for large x.
When x becomes large (+ve and –ve), then the constant terms become relatively insignificant, and the graph will approach that of $y = \frac{2x}{x}$ or merely $y = 2$.
When x becomes large and positive, y gets closer to 2 from below, e.g. (97, 1.89).
(Draw a short near-horizontal arc at right just below the asymptote x= 2).

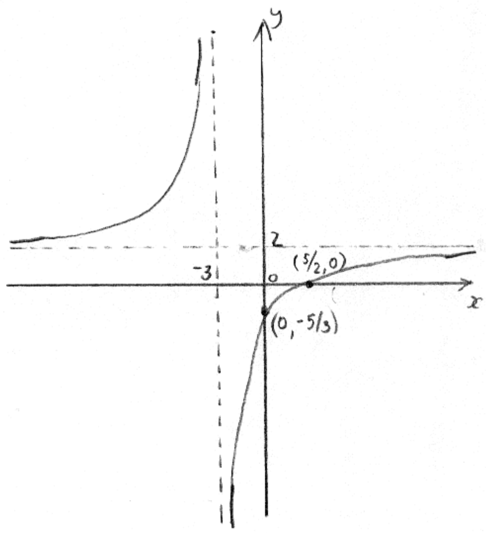
When x becomes large and negative, y approaches 2 from above, e.g. (-103, 2.11). ***(Draw a short near-horizontal arc at left just above the asymptote x = 2).***
(Draw asymptote at y = 2.)

Discontinuities and behaviour near them.
The function is undefined when $x + 3 = 0$, i.e. when $x = -3$. ***(Draw in the asymptote at x = -3).***
When x is slightly larger than –3, we have a $\frac{'+'}{'-'}$ with a small denominator, so y is large and negative.
(Draw a short near-vertical line at the bottom just to the right of the asymptote at x = -3).
When x is slightly smaller than –3 , we have a $\frac{'-'}{'-'}$ with a small denominator, and so y is large and positive.
(Draw a short near-vertical line at the top just to the left of the asymptote at x = -3).

Stationary points. Not applicable here.

Join up.
(Draw a curve connecting the the two short arcs in the the lower right of the two asymptotes, passing through points (5/2,0) and (0, -5/3). The curve bends quite sharply as it passes through those points). (Draw a similar curve connecting the two short arcs to upper left of the two asymptotes).





Example (9): Sketch the graph of $y = \dfrac{x^2+4}{2x-3}$, including asymptotes, for $-10 \le x \le 10$.

Hint: there are turning points at (4, 4) and (-1, -1).

Intercepts. We find the intercepts first. The numerator can never be zero, so there is no *x*-intercept. When $x = 0$, $y = -\frac{4}{3}$ ***(Plot the point (0,*** $-\frac{4}{3}$ ***)).***

Symmetry. Function neither odd nor even – no help.

Behaviour for large x. When *x* becomes large (both positive and negative), then the graph will approach that of $y = \dfrac{x^2}{2x}$ or $y = \dfrac{x}{2}$. ***(Draw a dotted or faint line y = x/2.)***

When *x* is large and positive, we can test it to find which side of the line it is: when $x = 10$, $y \approx 6.2$, which is greater than $\frac{x}{2}$, here 5. ***(Draw a short arc parallel to y = x/2, and above it.)***
When *x* is large and positive, we again test it as per the last case; when $x = -10$, $y \approx -4.6$, which is also greater than $\frac{x}{2}$, here -5. ***(Draw a short arc parallel to y = x/2, and above it.)***

Discontinuities and behaviour near them.
The function is undefined when $2x - 3 = 0$, i.e. when $x = \frac{3}{2}$. ***(Draw in the asymptote at x =*** $\frac{3}{2}$ ***).***

When *x* is just above $\frac{3}{2}$, we have a $\dfrac{'+'}{'+'}$ with a small denominator, so *y* is large and positive.

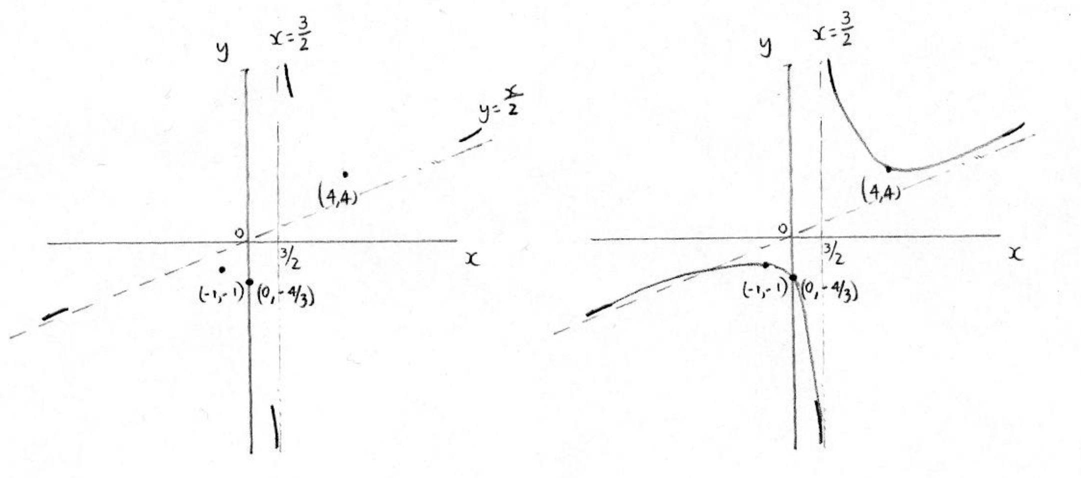
(Draw a short near-vertical arc at the top of the graph just to the right of the asymptote at x = $\frac{3}{2}$ ***).***

When *x* is just below $\frac{3}{2}$, we have a $\dfrac{'+'}{'-'}$ with a small denominator, and so *y* is large and negative.

(Draw a short near-vertical arc at the bottom of the graph just to the left of the asymptote at x = $\frac{3}{2}$ ***).***

Stationary points. Given the details above, we can infer that (4, 4) is a local minimum in the upper right branch, and that (-1, -1) is a local maximum in the lower left branch.
(Plot the turning points at (-1, -1) and (4, 4.)

Join up. ***(Join up the arcs and points on each side of the asymptote at x =*** $\frac{3}{2}$ ***, to form the two separate parts of the graph).***



Note that the line $y = \dfrac{x}{2}$ is not an asymptote, since the lower left limb appears to cross it !

Example (10): Sketch the graph of $y = \dfrac{x^3 - 8}{5x - 2}$, including asymptotes, for $-6 \le x \le 6$.
Hint: There is a turning point at approximately (-1.4, 1.2).

Intercepts. Substituting $x = 0$ gives the y-intercept of (0, 4). The numerator is zero when x^3-8 = 0, i.e. when $x = 2$. The x-intercept is therefore (2, 0). ***(Plot the points (0, 4) and (2, 0)).***

Symmetry. Function neither odd nor even – no help.

Behaviour for large x.
When x becomes large (both positive and negative), then the highest powers of both the top and bottom dominate, and the graph will tend to that of $y = \dfrac{x^3}{5x}$ or $y = \dfrac{x^2}{5}$.

(Sketch a faint or dotted curve $y = x^2/5$. This should go through the origin, (5, 5) and (-5, 5)).
(Plot short arcs coincident with the curve $y = x^2/5$, preferably near x = 6 and x = -6).

Discontinuities and behaviour near them.
The function is undefined when $5x - 2 = 0$, i.e. when $x = \frac{2}{5}$. ***(Draw in the asymptote at $x = \frac{2}{5}$).***

When x is just above $\frac{2}{5}$, we have a $\dfrac{'-'}{'+'}$ with a small denominator, so y is large and negative.

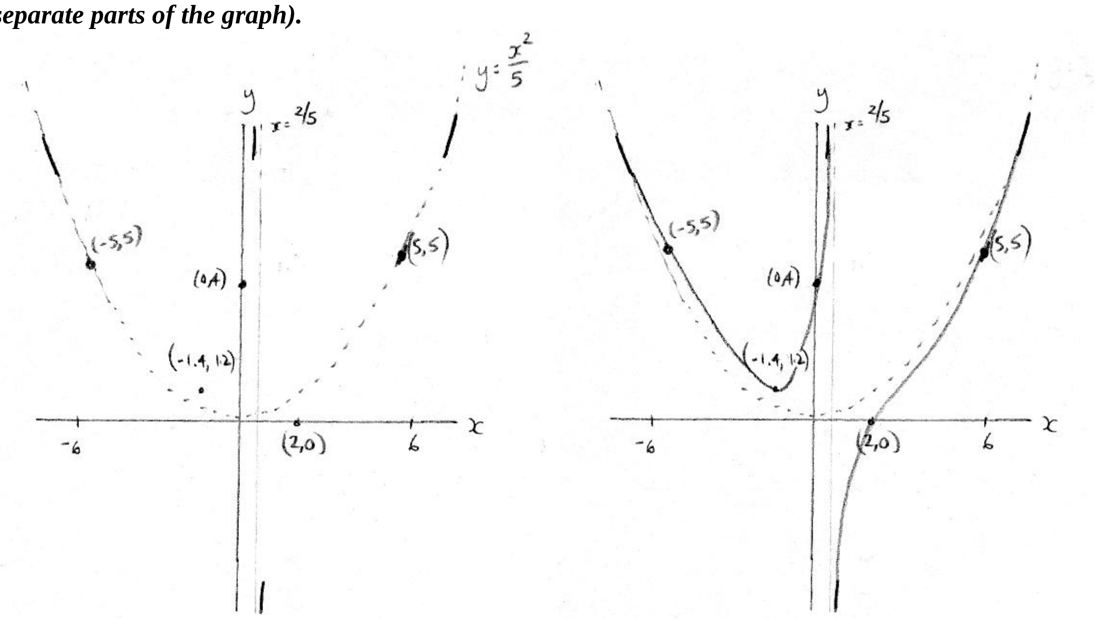
(Draw a short near-vertical line at the bottom of the graph just to right of the asymptote at $x = \frac{2}{5}$).

When x is just below $\frac{2}{5}$, we have a $\dfrac{'-'}{'-'}$ with a small denominator, so y is large and positive.

(Draw a short near-vertical line at the top of the graph just to left of the asymptote at $x = \frac{2}{5}$).

Stationary points. We are given a turning point at approximately (-1.4, 1.2), and by the context we can deduce that it is a local minimum. ***(Plot the point (-1.4, 1.2) and draw a 'trough' arc).***

Join up. ***(Connect up the arcs and points on each side of the asymptote at $x = \frac{2}{5}$, to form the two separate parts of the graph).***



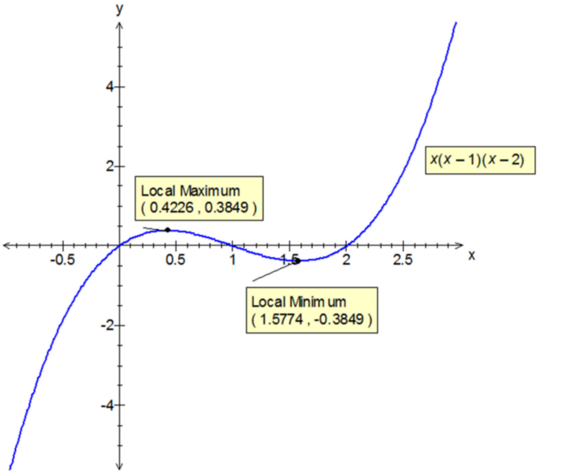
Functions of the form $\dfrac{1}{f(x)}$.

We begin this section with a study of the graph of a function and its reciprocal.

Example (11):
The graph of the right is of the function $f(x) = x(x-1)(x-2)$.

It crosses the x-axis when $x = 0$, 1 and 2, and crosses the y-axis at (0, 0).

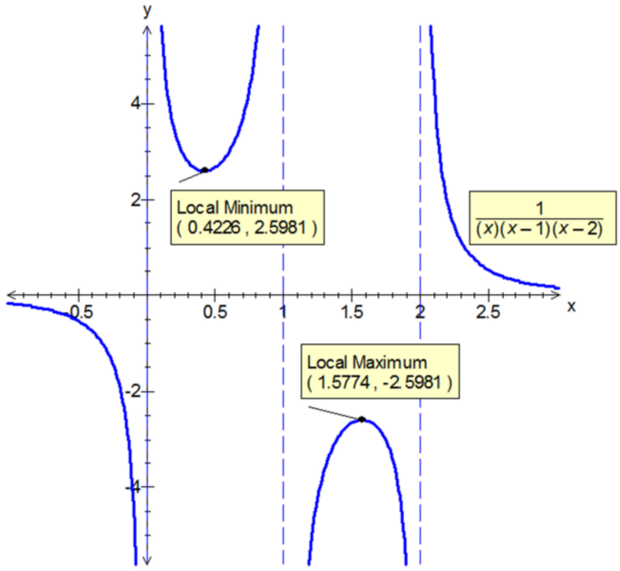
There are also two turning points: a local maximum near (0.4, 0.4), and a local minimum near (1.6, -0.4).



Below it is the graph of $\dfrac{1}{f(x)}$.

(This is **not** the same thing at all as the *inverse* function $f^{-1}(x)$!)

Notice the behaviour of the two graphs as x varies.



The x-intercepts of the graph of $f(x)$ become vertical asymptotes on the reciprocal graph. (Here they correspond to $x = 0$, 1 and 2.)

The result is that the reciprocal graph consists of four distinct sections, separated by the asymptotes.

The local maximum on the graph of $f(x)$ becomes a local minimum on the reciprocal graph, near (0.4, 2.5). The values of the two functions have a product of 1 as expected.

The local minimum on the graph of $f(x)$ becomes a local maximum on the reciprocal graph, near (1.6, -2.5).

When y is close to zero on the graph of $f(x)$, it is large on the reciprocal graph (but of the same sign).

When y is large on the graph of $f(x)$, it is close to zero on the reciprocal graph (but of the same sign).

When $f(x)$ is an ascending function, the reciprocal function is a descending function and vice versa.

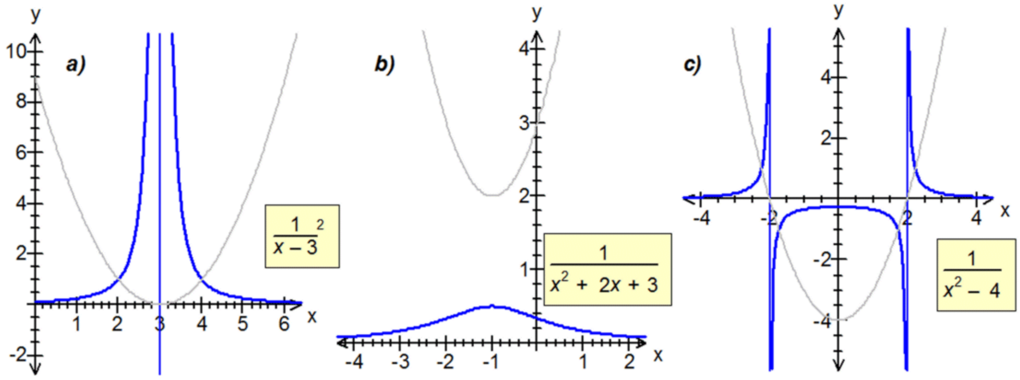
The two graphs meet when $f(x) = 1$.

The last example was fairly complicated, but all reciprocal functions follow the same rules, namely

- The x-intercepts of the graph of $f(x)$ become vertical asymptotes on the reciprocal graph.
- A local minimum on the graph of $f(x)$ becomes a local maximum on the graph of $\frac{1}{f(x)}$ (and vice versa).
- If $f(x) = a$, then $\frac{1}{f(x)} = \frac{1}{a}$ provided a is not zero.
- When $f(x)$ is an increasing function, $\frac{1}{f(x)}$ is a decreasing function and vice versa.
- When $f(x)$ is close to zero, $\frac{1}{f(x)}$ is very large and vice versa. (The signs are the same).

Reciprocals of Quadratic Functions.

The graphs of the reciprocals of quadratic functions have their own distinctive shapes depending on the number of roots of the parent quadratic (shown superimposed but greyed out).



Vertical asymptote at x = 3
Horizontal asymptote at y = 0

Maximum at (-1, ½)
Horizontal asymptote at y = 0

Vertical asymptotes at x = 2, x = -2
Local maximum at (0, -1/4)
Horizontal asymptote at y = 0

Diagram a) is the "one root" case, showing the graphs of $(x-3)^2$ and $\dfrac{1}{(x-3)^2}$.

The root is $x = 3$, and so the reciprocal graph has an asymptote at $x = 3$ and two separate sections on either side of it. The asymptote is also a line of symmetry.

Because the quadratic has a minimum at (3, 0), the reciprocal graph has no turning point since division by zero is undefined.

(The graph is in fact that of $\dfrac{1}{x^2}$ transformed by an x-shift of +3 units.)

Diagram b) is the "no roots" case, showing the graphs of $x^2 + 2x + 3$ and its reciprocal.

Expressing the quadratic in completed square form gives $(x + 1)^2 + 2$. From there, we can deduce that the minimum point of the quadratic is (-1, 2).

The quadratic has no roots, and therefore the reciprocal graph has no asymptotes. There is a local maximum at (-1, ½), though, and there is a line of symmetry at $x = -1$.

Diagram c) shows the "two roots" case, where the original quadratic has two roots. The denominator factorises into $(x + 2)(x - 2)$, giving roots of $x = -2$ and $x = 2$.

The two roots become asymptotes on the reciprocal graph, which thus consists of three sections:

- an increasing one when $x < -2$ (corresponding to the the original quadratic decreasing)
- an 'inversion' of the original for $-2 < x < 2$ with a maximum at $(0, -\frac{1}{4})$ corresponding to the original minimum of (0, -4)
- a decreasing one when $x > 2$ (corresponding to the the original quadratic increasing)

Note: all the above cases illustrate the behaviour of quadratic functions and their reciprocals where the coefficient of x^2 is *positive.*

(For negative coefficients of x^2, the graphs would be reflected in the x-axis, and the terms "increasing / decreasing" and "maximum / minimum" interchanged.)

Example (12): Sketch the graph of $y = (x-3)(x-5)$, and hence the graph of $y = \dfrac{1}{(x-3)(x-5)}$.

Part i) – sketching the graph of $y = (x-3)(x-5)$. (Full details not given).

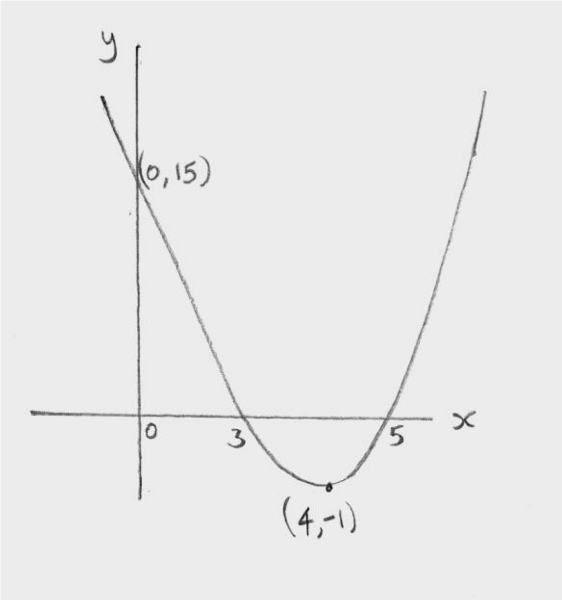
Intercepts. The x-intercepts are at (3, 0) and (5, 0); the y-intercept is at (0,15).

Symmetry. The graph is a parabola, so its line of symmetry is at $x = 4$ (the mean of the two roots).

Large x. When x is large (positive or negative), y is large and positive.

Discontinuities. (None.)

Stationary points. This being a quadratic, there is one turning point on the line of symmetry, i.e. at (4, -1). Because the term in x^2 is positive, we deduce that (4, -1) is a local minimum.



Part ii) – sketching the reciprocal graph.

The two roots of the original (3 and 5) give rise to the asymptotes $x = 3$ and $x = 5$ on the reciprocal graph. ***(Draw vertical asymptotes at x = 3 and x = 5).***

It is best to regard the graph as three separate sections, separated by the two asymptotes.

$x < 3$

The original function $(x – 3)(x – 5)$ is positive and decreasing, passing through (0, 15).
The reciprocal function is therefore positive and *increasing*, passing through $(0, \frac{1}{15})$. i.e just above zero.
(Sketch an increasing curve starting just above and parallel to the x-axis, passing through (0, $\frac{1}{15}$) and curving sharply upwards towards the left of the line x = 3.)

$3 < x < 5$

The original function $(x – 3)(x – 5)$ is negative and decreasing for $3 < x < 4$, has a minimum point at (4, -1), and is negative and increasing for $4 < x < 5$.

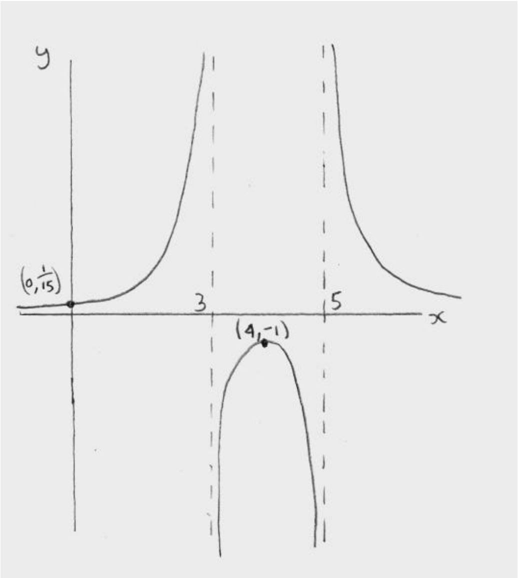
The reciprocal function is therefore negative and *increasing* for $3 < x < 4$, has a *maximum* point at (4, -1), and is negative and *decreasing* for $4 < x < 5$. (Note: -1 is a reciprocal of itself).

(Sketch an inverted 'U' curve, starting at the lower right of the line x= 3 near the bottom of the graph, increasing up to the maximum point at (4, -1), and then decreasing towards the lower left of the line x = 5.)

$x > 5$

(This is a reflection of the section for $x < 3$ in the line $x = 4$.)
The original function $(x – 3)(x – 5)$ is positive and increasing.
The reciprocal function is therefore positive and *decreasing*.
(Sketch a decreasing curve starting high up and just to the right of the line x = 5, and then curving sharply downwards towards the x-axis and just above it.)

The entire graph resembles that of Diagram c).



Example (13): Sketch the graph of $y = (x-4)(1-x)$, and hence the graph of $y = \dfrac{1}{(x-4)(1-x)}$.

Part i) – sketching the graph of $y = (x-4)(1-x)$. (Full details not given).

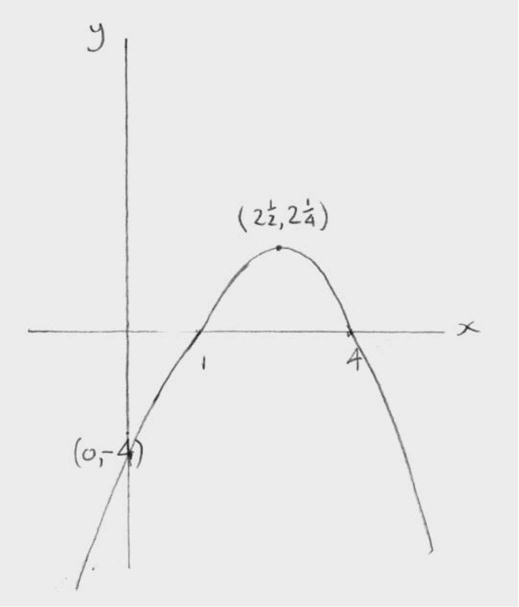
Intercepts. The x-intercepts are at (1, 0) and (4, 0); the y-intercept is at (0, -4).

Symmetry. The graph is a parabola, so its line of symmetry is at $x = 2\frac{1}{2}$ (the mean of the two roots).

Large x. When x is large positive or negative), y is large and negative.

Discontinuities. (None.)

Stationary points. There is one turning point on the line of symmetry, i.e. at $(2\frac{1}{2}, 2\frac{1}{4})$. Because the term in x^2 is negative, that point is a local maximum.



Part ii) – sketching the reciprocal graph.

The two roots of the original (1 and 4) give rise to the asymptotes $x = 1$ and $x = 4$ on the reciprocal graph. ***(Draw vertical asymptotes at x = 1 and x = 4).***

***x* < 1**
The original function $(x – 4)(1 - x)$ is negative and increasing, passing through (0, -4).
The reciprocal function is therefore negative and *decreasing*, passing through $(0, -\frac{1}{4})$.

(Sketch a decreasing curve starting just below and parallel to the x-axis, passing through (0, $-\frac{1}{4}$) and curving sharply downwards towards the left of the line x = 1.)

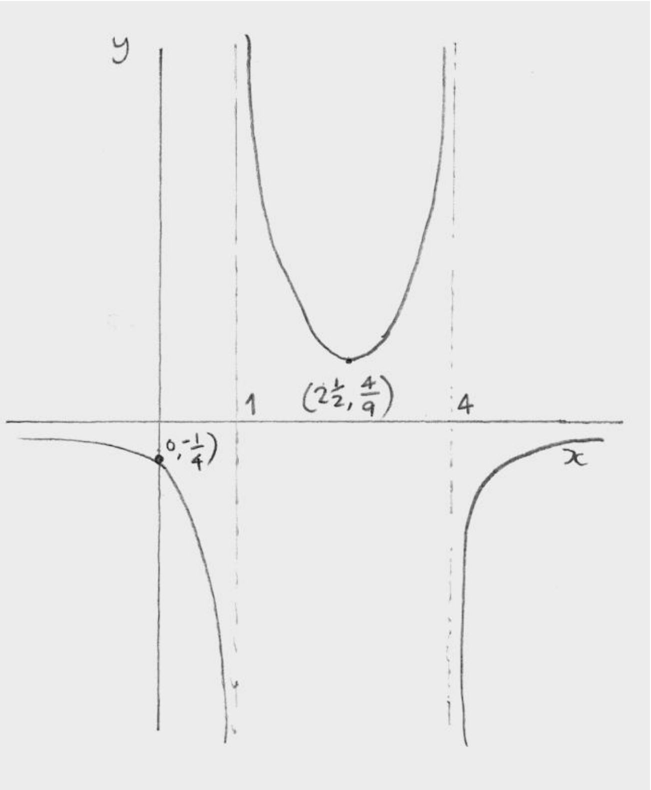
1 < *x* < 4
The original function $(x – 4)(1 - x)$ is positive and increasing for $1 < x < 2\frac{1}{2}$, has a maximum point at $(2\frac{1}{2}, 2\frac{1}{4})$, and is positive and decreasing for $2\frac{1}{2} < x < 4$.

The reciprocal function is therefore positive and *decreasing* for $1 < x < 2\frac{1}{2}$, has a *minimum* point at $(2\frac{1}{2}, \frac{4}{9})$, and is positive and *increasing* for $2\frac{1}{2} < x < 4$.

(Sketch a U curve starting upper right of the line x =1 near the top of the graph, down through the minimum at ($2\frac{1}{2}, \frac{4}{9}$), and then back up towards the upper left of the line x = 4.)

***x* > 4**
(This is a reflection of the section for $x < 1$ in the line $x = 2\frac{1}{2}$.)
The original function $(x – 4)(1 - x)$ is negative and decreasing.
The reciprocal function is therefore positive and *increasing*.
(Sketch an increasing curve starting low and just to the right of the line x = 4, and then curving sharply upwards towards the x-axis and just below it.)

The entire graph resembles that of Diagram c) reflected in the *x*-axis.



Example (14): Sketch the graph of $y = x^2 - 4x + 7$ and hence that of its reciprocal.

Part i) – sketching the graph of $y = x^2 - 4x + 7$. (Full detail not given).

The quadratic does not factorise here, so completing the square gives $(x - 2)^2 + 3$.

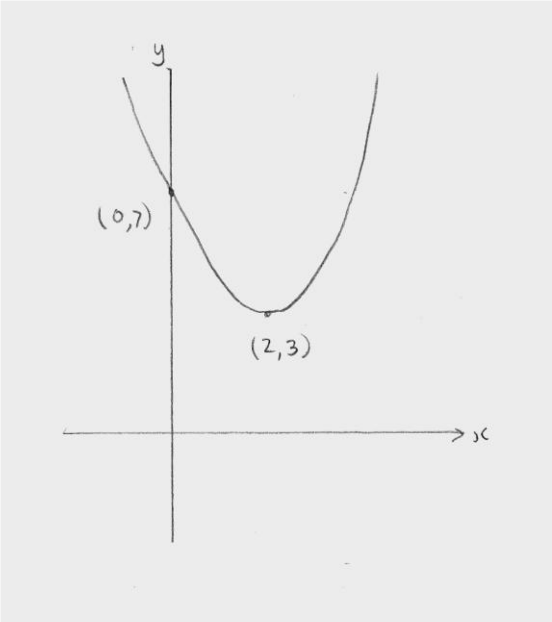
Intercepts. The function is always positive, so there are no *x*-intercepts. The *y*-intercept is at (0, 7).

Symmetry. The graph has a line of symmetry at $x = 2$.

Large x. When *x* is large, *y* is large and positive.

Discontinuities. (None.)

Stationary points. There is one turning point on the line of symmetry, i.e. at (2, 3).
The term in x^2 is positive, and hence that point is a local minimum.



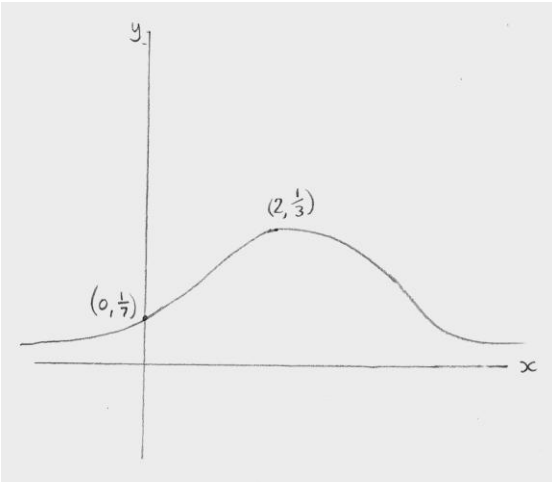
Part ii) – sketching the reciprocal graph.

Because the original graph has no roots, the reciprocal graph will have no asymptotes.

The original function is decreasing for $x < 2$ and increasing for $x > 2$.
The reciprocal function will be *increasing* for $x < 2$ and *decreasing* for $x > 2$.

The original function has a local minimum at (2, 3).
The reciprocal function has a local *maximum* at $(2, \frac{1}{3})$.

The original function becomes indefinitely large as *x* becomes large (positive or negative).
The reciprocal function *tends to zero* as *x* becomes large.

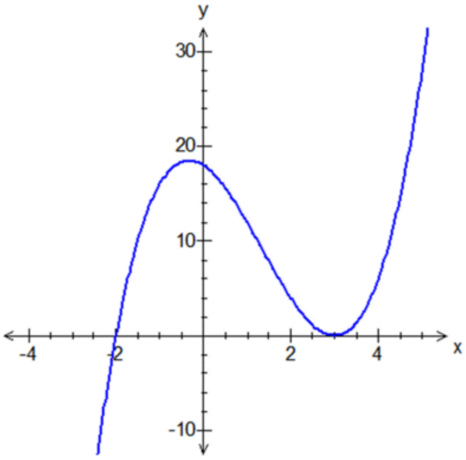


Example (15): The graph of $(x-3)^2(x+2)$ is given here.
Sketch the graph of its reciprocal.

(Do not attempt to find the exact coordinates of the local maximum at $x = -\frac{1}{3}$).

The original graph has two roots, at $x = -2$ and $x = 3$.

The reciprocal graph will therefore have asymptotes at $x = -2$ and $x = 3$. ***(Draw vertical asymptotes at x = -2 and x = 3).***



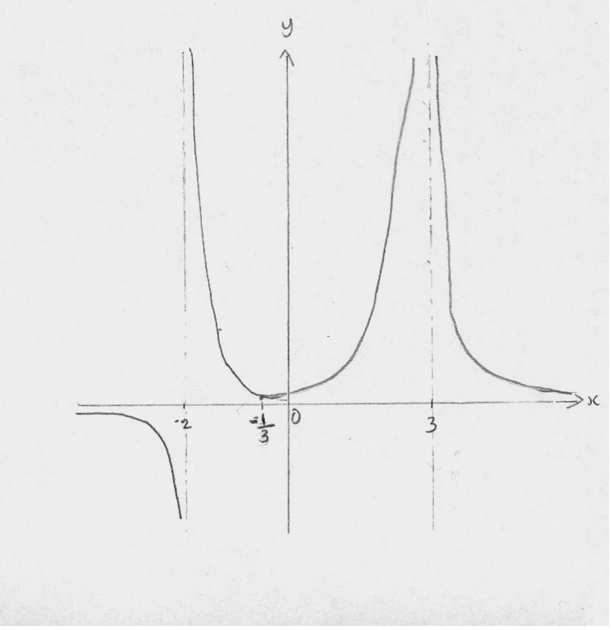
The three sections of the reciprocal graph will have the following properties:

<u>x < -2</u>
The original function is negative and increasing, so the reciprocal function is therefore negative and *decreasing*, beginning just below the *x*-axis. ***(Draw a decreasing curve which starts just below the x-axis and then shoots off to the bottom of the graph just left of the asymptote at x = -2.)***

<u>-2 < x < 3</u>
The original function is positive and increasing for $-2 < x < -\frac{1}{3}$, has a maximum point at roughly $(-\frac{1}{3}, 20)$, and is positive and decreasing for $-\frac{1}{3} < x < 3$.

The reciprocal function is positive and *decreasing* for $-2 < x < -\frac{1}{3}$, has a *minimum* point at roughly $(-\frac{1}{3}, \frac{1}{20})$, and is positive and *increasing* for $-\frac{1}{3} < x < 3$.

(Draw a U curve which starts to the upper right of the asymptote x = -2, reaches a minimum at $(-\frac{1}{3}, \frac{1}{20})$, and then ends up to the upper left of the asymptote x = 3.)



<u>x > 3</u>
The original function is positive and increasing, so the reciprocal function is therefore positive and *decreasing*, finally ending just above the *x*-axis.

(Draw a decreasing curve which starts off to the upper right of the asymptote at x = 3 and then levels off sharply to just above the x-axis.)