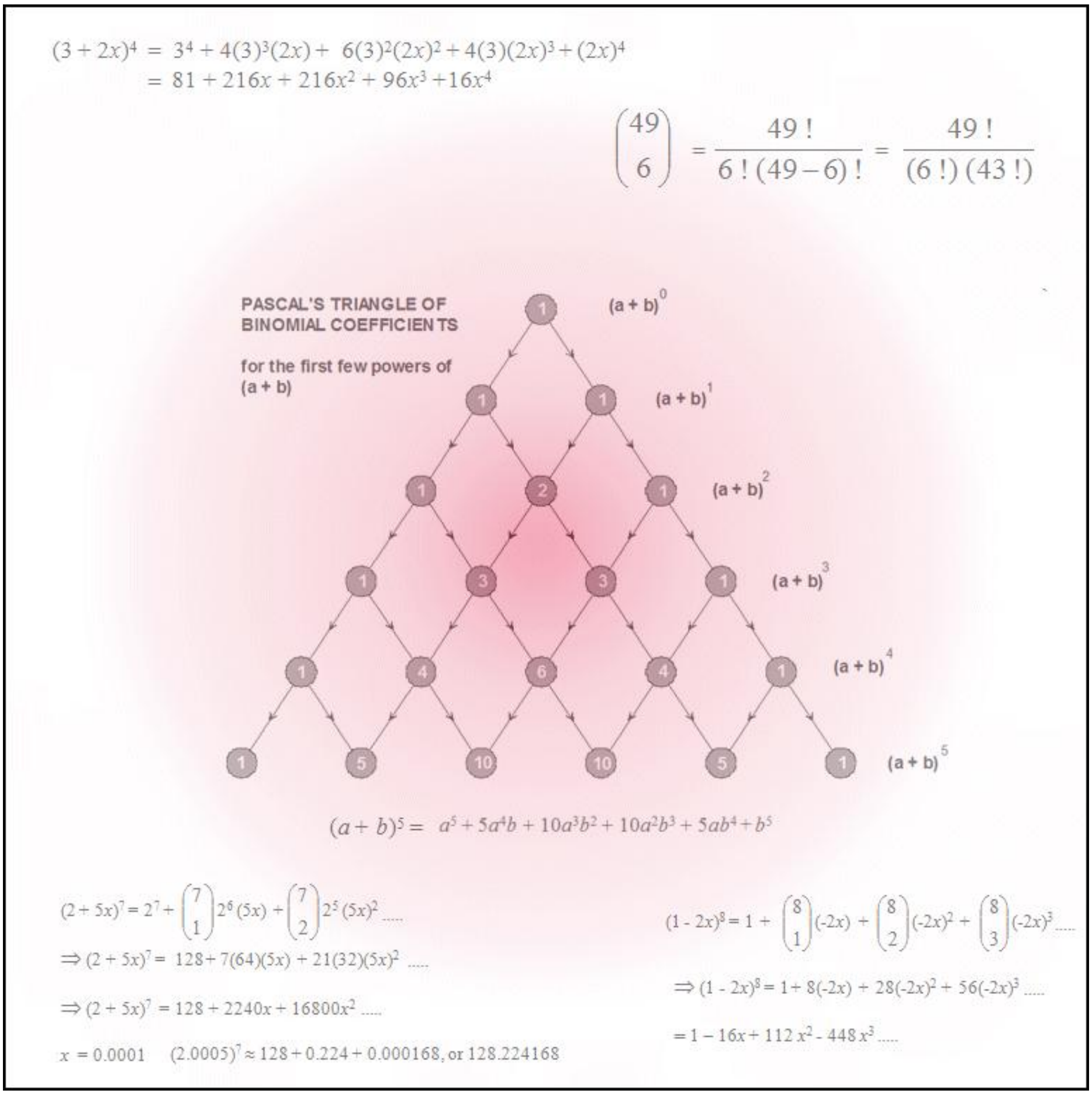
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 1 / AS

THE BINOMIAL SERIES



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THE BINOMIAL SERIES

Background on factorials and combinations.

The **factorial** of a number is the product obtained by multiplying together all the positive integers from the number down to 1. It is denoted by $n!$

Thus 1! = 1; 2! = 2 × 1 or 2; 3! = 3 × 2 × 1 or 6; 4! = 4 × 3 × 2 × 1 or 24, and so forth.

The first few factorials are:

n	1	2	3	4	5	6	7	8	9	10	11	12
***n*!**	**1**	**2**	**6**	**24**	**120**	**720**	**5040**	**40320**	**362880**	**3628800**	**39916800**	**479001600**

Most calculators have buttons to evaluate higher factorials, and the factorial of 0 is taken as 1.

A **combination** is the number of ways a number of objects can be selected from a larger number. (Rearrangements of the same objects are not counted as different).

Take for example the four suits in a pack of cards, spades ♠, hearts ♥, diamonds ♦ and clubs ♣.

There is only one way in which none of the suits can be selected.

One suit can be selected in 4 different ways; ♠, ♥, ♦ or ♣.

Two suits can be selected as follows: there are 4 ways of selecting the first, and 3 ways of selecting the second (one has gone), giving 12 possibilities. However, this repeats certain combinations (e.g. ♠♥ and ♥♠) and we must divide by 2 to exclude mere rearrangements. There are thus 6 ways of selecting two suits from four.

Three suits can be selected in a similar way: 4 ways of selecting the first, 3 ways of selecting the second (one has gone) and 2 ways of selecting the third, giving 24 possibilities. However, each combination of 3 suits repeats itself 3! or 6 times because of rearrangements. Hence there are 4 ways of selecting three suits from four – the same as that obtained by selecting one suit from 4. This is because one suit can be chosen, but also omitted, in 4 different ways. This explains the left-right symmetry of Pascal's triangle.

All four suits can be selected in only one way, although there are 4! or 24 rearrangements.

The notation for selecting r objects from a sample of n objects is often denoted by $\binom{n}{r}$ or ${}^nC_r$.

The form $\binom{n}{r}$ will be used throughout in this document, although the ${}^nC_r$ form is more commonly seen on calculators.

It is related to factorials as $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ or $\dfrac{n(n-1)....(n-r+1)}{r!}$

$$\binom{n}{3} = \frac{n!}{3!(n-3)!} \text{ or } \frac{n(n-1)(n-2)}{3!}$$

In addition, it is worth knowing that $\binom{n}{r} = \binom{n}{n-r}$

Example (1): Work out $\binom{7}{2}$ and $\binom{10}{5}$

$$\binom{7}{2} = \frac{7\,!}{2\,!\,(7-2)\,!} = \frac{7\,!}{(2\,!)\,(5\,!)} = \frac{7\times6\times5\times4\times3\times2\times1}{(2\times1)(5\times4\times3\times2\times1)}$$

(Note how the 5! cancels out on top and bottom)

$$\Rightarrow \binom{7}{2} = \frac{7\times6}{2\times1} = 21$$

This corresponds to the 21 ways in which two objects can be chosen from seven.

$$\binom{10}{5} = \frac{10\,!}{5\,!\,(10-5\,)!} = \frac{10\,!}{(5\,!)\,(5\,!)}$$

$$\Rightarrow \binom{10}{5} = \frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1} = \frac{10\times9\times7\times6}{5\times3} = 2\times3\times7\times6 = 252.$$

This is the number of ways in which 5 objects can be chosen from 10.
The calculator-friendly form of this expression is ${}^{10}C_5$.

Example (2): In the National Lottery, 6 balls are chosen at random out of 49. Express the number of possible combinations in terms of factorials, and evaluate the result.

We are choosing 6 balls from 49, so the number of possible combinations is $\binom{49}{6}$

$$= \frac{49\,!}{6\,!\,(49-6)\,!} = \frac{49\,!}{(6\,!)\,(43\,!)} = 13{,}983{,}816.$$

Calculator-friendly form : ${}^{49}C_6$.

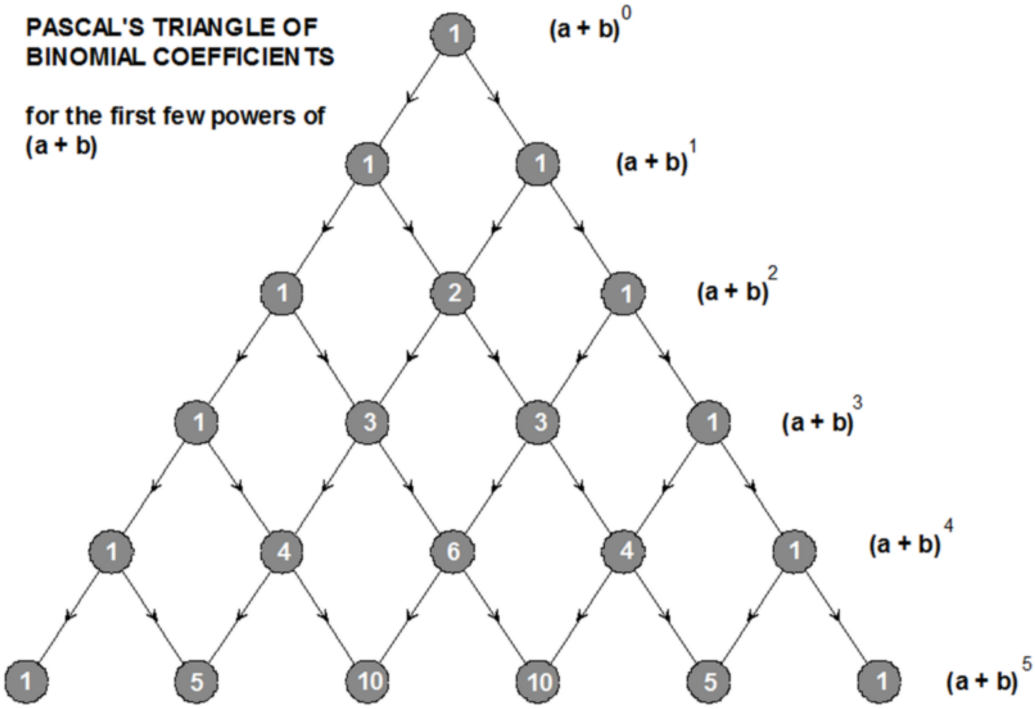
Example (2a): From October 2015, 6 balls are chosen at random out of 59 in the National Lottery. Express the number of possible combinations in terms of factorials, and evaluate the result.

We are choosing 6 balls from 59, so the number of possible combinations is $\binom{59}{6}$

$$= \frac{59\,!}{6\,!\,(59-6)\,!} = \frac{59\,!}{(6\,!)\,(53\,!)} = 45{,}057{,}474.$$

Calculator-friendly form : ${}^{59}C_6$.
Since most calculators have this function installed, we will not use these factorial forms in the rest of the document.

These combinations also appear in **Pascal's Triangle** of binomial coefficients.



Each term in one line of the triangle can be obtained by adding the terms directly above it. The values in each line are the values of $\binom{n}{r}$, as r goes from 0 to n from left to right. The line labelled $(a + b)^5$ has the values of $\binom{5}{0} = 1$, $\binom{5}{1} = 5$, $\binom{5}{2} = 10$ and so on to $\binom{5}{5}$. These values are **binomial coefficients.**

A **binomial** expression has two terms such as $(a + b)$, and an expansion of the form $(a + b)^n$ is called a **binomial expansion**.

The first few powers of $(a + b)$ expand out as follows:

$(a + b)^2$ is $a^2 + 2ab + b^2$
$(a + b)^3$ is $(a^2 + 2ab + b^2)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$
$(a + b)^4$ is $(a^3 + 3a^2b + 3ab^2 + b^3)(a + b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Each term in the binomial expansion is the product of three quantities:
(a power of a) × (a power of b) × (a binomial coefficient).

(The first term lacks a power of b since it is treated as b^0, which is 1. The same argument applies to the last term but there a is absent.)

There are five things to notice in the pattern:

a) the power of a is reduced by 1 going left to right
b) the power of b is increased by 1 from left to right
c) the powers of a and b always add up to the power, n, of the expression.
d) the binomial coefficients follow the order of the rows in Pascal's triangle.
e) a particular binomial coefficient is $\binom{n}{b}$ where b is the power of b and n is the power of $(a+b)$.

Since a and b add up to n, the binomial coefficient $\binom{n}{a}$ is the same as $\binom{n}{b}$.

The general binomial expansion for $(a + b)^n$ is therefore

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \binom{n}{3} a^{n-3}b^3 + \ldots + b^n$$

or

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3}b^3 + \ldots + b^n$$

Special cases of the expansion include

$$(1+ax)^n = 1 + \binom{n}{1}(ax) + \binom{n}{2}(ax)^2 + \binom{n}{3}(ax)^3 + \ldots + (ax)^n$$

or

$$(1+ax)^n = 1 + n(ax) + \frac{n(n-1)}{2!}(ax)^2 + \frac{n(n-1)(n-2)}{3!}(ax)^3 + \ldots + (ax)^n$$

This expansions are simpler since all powers of 1 are equal to 1 itself.

The corresponding expansions for $(1 - ax)^n$ are similar, except that the terms in odd powers of x have their signs reversed.

$(1- ax)^n = 1 - \binom{n}{1}(ax) + \binom{n}{2}(ax)^2 - \binom{n}{3}(ax)^3 + \ldots$(Last term is $+ (ax)^n$ if even n, $- (ax)^n$ if odd n) or

$$(1-ax)^n = 1 - n(ax) + \frac{n(n-1)}{2!}(ax)^2 - \frac{n(n-1)(n-2)}{3!}(ax)^3 + ..$$

After all this theory, here are some examples:

Example(3): Expand $(a + b)^5$.

The coefficients are given as 1, 5, 10, 10, 5 and 1 in Pascal's triangle, and thus $(a + b)^5$ =

$$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Example(4): Expand $(1 + x)^5$.

This result is similar to the previous one, with 1 for a and x for b, but can be simplified as all powers of 1 are equal to 1 itself:

$(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$.

Example(5): Expand $(3 + 2x)^4$.

This is a fourth-power expression, and so the coefficients are 1, 4, 6, 4 and 1.
Here we must substitute $a = 3$ and $b = 2x$ in the term $(a + b)$.

Therefore $(3 + 2x)^4 = 3^4 + 4(3)^3(2x) + 6(3)^2(2x)^2 + 4(3)(2x)^3 + (2x)^4$
or $81 + 216x + 216x^2 + 96x^3 + 16x^4$.

Do not fall into this error trap: $(2x)^3 = 8x^3$ and not $2x^3$ (remember to cube the 2 as well as the x !).

Sometimes a question might only ask for part of the binomial expansion of an expression:

Example(6): Give the terms up to and including the term in x^2 in the expansion of $(2 + 5x)^7$. From the result, find the value of $(2.0005)^7$ to 6 decimal places.

Substituting $a = 2$ and $b = 5x$ the expansion begins

$$(2 + 5x)^7 = 2^7 + \binom{7}{1} 2^6 (5x) + \binom{7}{2} 2^5 (5x)^2$$

$$\Rightarrow (2 + 5x)^7 = 128 + 7(64)(5x) + 21(32)(5x)^2$$

$$\Rightarrow (2 + 5x)^7 = 128 + 2240x + 16800x^2$$

Substituting $x = 0.0001$, we have $(2.0005)^7 \approx 128 + 0.224 + 0.000168$, or 128.224168 to 6 decimal places.

Example(7): Give the terms up to and including the term in x^3 in the expansion of $(1 - 2x)^8$ and thus find $(0.998)^8$ to 8 decimal places.

Substituting $a = 1$ and $b = (-2x)$, the expansion begins

$$(1 - 2x)^8 = 1 + \binom{8}{1}(-2x) + \binom{8}{2}(-2x)^2 + \binom{8}{3}(-2x)^3$$

(notice how the powers of 1 all disappear as in example 4)

$$\Rightarrow (1 - 2x)^8 = 1 + 8(-2x) + 28(-2x)^2 + 56(-2x)^3$$

$= 1 - 16x + 112x^2 - 448x^3$ (note how the terms of this series oscillate between positive and negative).

Substituting $x = 0.001$, we have $(0.998)^8 \approx 1 - 0.016 + 0.000112 - 0.000000448$, or 0.98411155 to 8 decimal places.

Example(8): Give the term in x^4 in the expansion of $(4 - 5x)^{10}$.

The term will be $\binom{10}{4} \times 4^{10-4} \times (-5x)^4 = 210 \times 4^6 \times (-5x)^4$

$= 210 \times 4096 \times 625\,x^4 = 537{,}600{,}000\,x^4$.

Example (9):

i) Give the term in x^3 in the expansion of $(2 + 3x)^9$.

ii) Use the result in i) to find the term in x^6 in the expansion of $\left(2+\frac{3x^2}{4}\right)^9$

i) The required term is $\binom{9}{3}\times 2^{9-3}\times(3x)^3 = 84\times 2^6\times(3x)^3$
$= 84 \times 64 \times 27\ x^3 = 145{,}152\ x^3$.

ii) Since $\frac{3x^2}{4} = 3\left(\frac{x^2}{4}\right)$, we can replace the x in the original expansion with $u = \frac{x^2}{4}$.

From part (i), the term in u^3 in the expansion of $(2 + 3u)^9$ is $145{,}152\ u^3$, but because $u^3 = \frac{x^6}{64}$,

this can be re-expressed in terms of x as $\frac{145{,}152\ x^6}{64} = 2268x^6$.

Example(10): Give the constant term in the expansion of $\left(3x+\left(\frac{2}{x}\right)\right)^6$.

Using the fact that $\frac{2}{x}$ is the same as $2x^{-1}$, the required term will have the powers of x 'cancelling out'

to give a multiple of x^0, in other words, a constant.

Since we are dealing with a sixth-power expansion, this term will therefore be the one containing $(3x)^3$ and $\left(\frac{2}{x}\right)^3$ in it. Now, $(3x)^3 = 27x^3$ and $\left(\frac{2}{x}\right)^3 = \frac{8}{x^3}$ or $8x^{-3}$.

The required term in the expansion is therefore
$\binom{6}{3}\times 27x^3\times 8x^{-3} = 20\times 27x^3\times 8x^{-3}$
$= 4320\ x^0 = 4320$.

Other questions on the binomial theorem might be more algebraic in nature.

Example(11): A binomial expansion of $(1 – ax)^5$ begins with the terms $1 – kx + 12kx^2$.

Find the values of k and a.

Expanding, $(1 - ax)^5 = 1 - 5(ax) + \binom{5}{2}(ax)^2..... = 1 – 5ax + 10a^2x^2 ...$

Hence $k = 5a$ and $12k = 10a^2$.

Dividing, we have $\frac{10a^2}{5a} = \frac{12k}{k} \Rightarrow 2a = 12$.

Therefore $a = 6$ and $k = 5a = 30$.

The expansion in question is thus $(1 - 6x)^5 = 1 – 30x + 360x^2 ...$

Example(12): The binomial expansion of $(a + bx)^n$ begins with the terms $32 + kx + 3kx^2$.

Find the values of k, a, b and n.

The expansion of $(a + bx)^n$ begins

$$(a + bx)^n = a^n + \binom{n}{1} a^{n-1}(bx) + \binom{n}{2} a^{n-2}(bx)^2 \ldots.$$

or

$$(a + bx)^n = a^n + na^{n-1}(bx) + \frac{n(n-1)}{2!} a^{n-2}(bx)^2 \ldots..$$

The first term must therefore satisfy $a^n = 32$, but we know that $2^5 = 32$.
Hence $a = 2$ and $n = 5$.

The expansion is therefore of $(2 + bx)^5$, so we can now write

$$(2 + bx)^5 = 2^5 + (5)(2^4)(bx) + \binom{5}{2} (2^3)(bx)^2 \ldots..$$

$$\Rightarrow (2 + bx)^5 = 32 + 80bx + 80b^2x^2 \ldots..$$

Going back to the original sum, $80b = k$ and $80b^2 = 3k$.

Dividing, $\frac{80b^2}{80b} = \frac{3k}{k} \Rightarrow b = 3$ and hence $k = 240$.

The expansion in question is thus $(2 + 3x)^5 = 32 + 240x + 720x^2 \ldots$