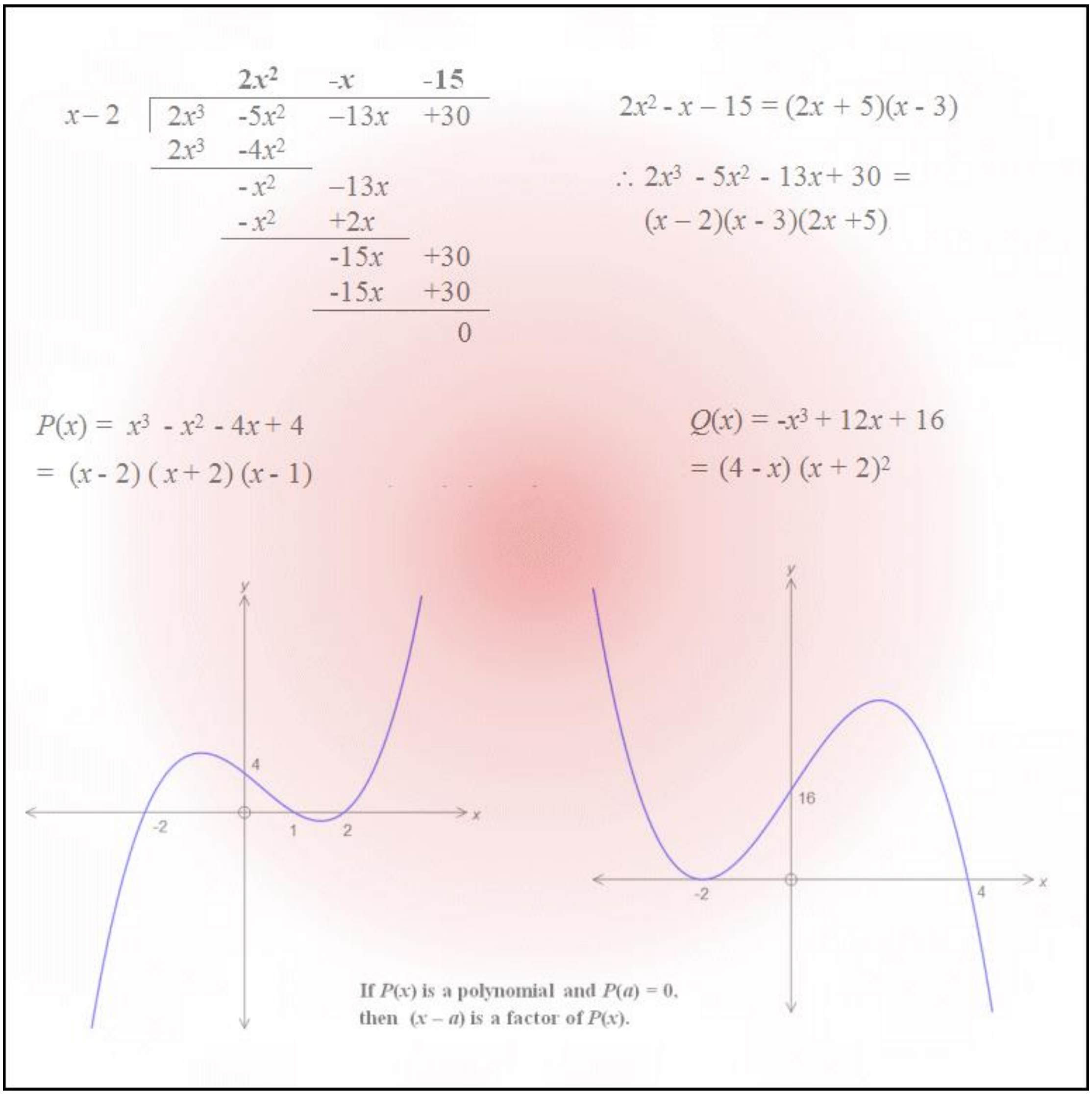
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 1 / AS

POLYNOMIALS



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POLYNOMIALS

A **polynomial** expression is one that takes the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots\dots\dots + a_0$$

where $a_n, a_{n-1} \dots\dots a_0$ are constants and n is a positive integer.

For example, $3x^3 - 5x + 6$ is a polynomial, where $a_3 = 3$, $a_2 = 0$, $a_1 = -5$ and $a_0 = 6$.

The **degree** of a polynomial is the highest power of x in it, the degree of $3x^3 - 5x + 6$ is 3.
A quadratic thus has a degree of 2 and a linear expression a degree of 1. (A constant can be said to have degree of 0).

Algebraic division.

Division of polynomials is analogous to that of integers. Thus if you work out 38 ÷ 5, you obtain a **quotient** of 7 and a **remainder** of 3. This relationship can be shown as (7 × 5) + 3 = 38. Also 38 is the **dividend** and 5 the **divisor**.

The long division format is the most common method used at AS level, and so will be featured here .

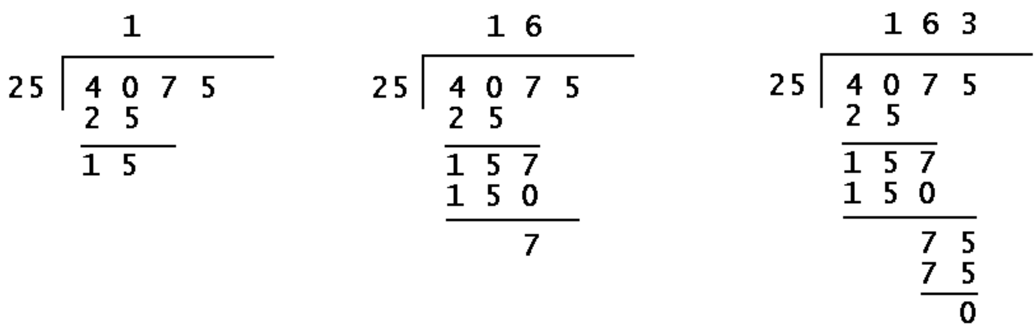
Pre- example (1): Find the value of 4075 ÷ 25.

Since 25 does not go into 4, we leave the space above the 4 blank. We can divide 25 into 40, so we put the answer, 1, above the 0, write the value of 1 × 25 below the 40, and subtract to find the remainder, 15. (First diagram)

Next, we bring down the next digit, 7, in the dividend and proceed to divide 25 into 157. The largest multiple of 25 below 157 is 25 × 6 or 150, so we write 150 below the 157, and subtract 150 from 157 to get 7. (Second diagram).

Then we bring down the next digit, 5, and proceed to divide 75 by 25. Now 75 is exactly 25 × 3, so we write 75 under the 75, with the final subtraction leaving a remainder of zero.

∴ 4075 ÷ 25 = 163.



Example (1): Divide $2x^2 + 9x - 5$ by $x + 5$.

Although this quadratic can be factorised quite easily, the method will be shown for illustration.

$$\begin{array}{l|lll}
\cline{2-4}
x+5 & 2x^2 & +9x & -5
\end{array}$$

We first look at the terms in the highest power of x in the dividend and the divisor. They are $2x^2$ in the dividend and x in the divisor. Dividing $2x^2$ by x gives $2x$.

We therefore put $2x$ in the quotient, and the product $(2x)(x + 5)$, namely $2x^2 + 10x$, underneath the terms $2x^2 + 9x$.

Note how all identical powers of x are in the same column each time – important ! .

$$\begin{array}{l|lll}
 & & \mathbf{2x} & \\
\cline{2-4}
x+5 & 2x^2 & +9x & -5 \\
 & \mathbf{2x^2} & \mathbf{+10x} &
\end{array}$$

Next, as in ordinary long division, we subtract and bring down the next term.

$$\begin{array}{l|lll}
 & & \mathbf{2x} & \\
\cline{2-4}
x+5 & 2x^2 & +9x & -5 \\
 & 2x^2 & +10x & \\
\cline{2-3}
 & & \mathbf{-x} & \mathbf{-5}
\end{array}$$

Now we have to divide x into $-x$ to obtain a result of -1. We thus bring down $(-1)(x + 5)$, or $(-x - 5)$ and put -1 in the quotient.

$$\begin{array}{l|lll}
 & & \mathbf{2x} & \mathbf{-1} \\
\cline{2-4}
x+5 & 2x^2 & +9x & -5 \\
 & 2x^2 & +10x & \\
\cline{2-3}
 & & -x & -5 \\
 & & \mathbf{-x} & \mathbf{-5} \\
\cline{3-4}
 & & & \mathbf{0}
\end{array}$$

Subtraction leaves no remainder here, and thus the quotient obtained by dividing $(2x^2 + 9x - 5)$ by $(x + 5)$ is $(2x - 1)$ exactly.

Note also how the dividend $(2x^2 + 9x - 5)$ has degree 2, and the divisor $(x + 5)$ and quotient $(2x - 1)$ have degree 1.

The degree of the dividend is always equal to the sum of the degrees of the divisor and the quotient.

Example(2): Divide $x^3 - 4x^2 - 9x + 36$ by $x + 3$.

$$\begin{array}{l|llll} x+3 & x^3 & -4x^2 & -9x & +36 \end{array}$$

Continue as in the previous example:

Dividing x^3 by x gives x^2, and $x^2\,(x + 3) = x^3 + 3x^2$, so we put that below the dividend and x^2 in the quotient.

$$\begin{array}{l|llll} & & \mathbf{x^2} & & \\ \hline x+3 & x^3 & -4x^2 & -9x & +36 \\ & \mathbf{x^3} & \mathbf{+\,3x^2} & & \end{array}$$

We then subtract to obtain a remainder of $-7x^2$ and bring down the next term, $-9x$.
Dividing $-7x^2$ by x gives $-7x$, and as $(-7x)\,(x + 3) = -7x^2 - 21x$, we put that below the dividend and $-7x$ in the quotient.

$$\begin{array}{l|llll} & & \mathbf{x^2} & \mathbf{-7x} & \\ \hline x+3 & x^3 & -4x^2 & -9x & +36 \\ & x^3 & +\,3x^2 & & \\ \cline{2-3} & & \mathbf{-\,7x^2} & \mathbf{-9x} & \\ & & \mathbf{-\,7x^2} & \mathbf{-21x} & \end{array}$$

Subtracting again, we have a remainder of $12x$ and bring down the last term, $+36$.
Dividing $-2x$ by x gives 12, and $(12)\,(x + 3) = 12x + 36$, so we put that below the dividend and 12 in the quotient.

$$\begin{array}{l|llll} & & \mathbf{x^2} & \mathbf{-7x} & \mathbf{+12} \\ \hline x+3 & x^3 & -4x^2 & -9x & +36 \\ & x^3 & +\,3x^2 & & \\ \cline{2-3} & & \mathbf{-\,7x^2} & \mathbf{-9x} & \\ & & \mathbf{-\,7x^2} & \mathbf{-21x} & \\ \cline{3-4} & & & \mathbf{12x} & \mathbf{+36} \\ & & & \mathbf{12x} & \mathbf{+36} \\ \cline{4-5} & & & & \mathbf{0} \end{array}$$

The final subtraction leaves no remainder, i.e. the division comes out exact, and so the quotient obtained by dividing $(x^3 - 4x^2 - 9x + 36)$ by $(x + 3)$ is $(x^2 - 7x + 12)$.

Missing powers in the dividend.

Example(3): Divide $x^3 - 5x - 2$ by $x + 2$.

The long division method requires a little care, because the term in x^2 is zero, but its place must still be included in the layout.

$$\begin{array}{l|llll} x+2 & x^3 & +\,0x^2 & -5x & -2 \end{array}$$

Continue as in the previous example:

Dividing x^3 by x gives x^2, and $x^2(x + 2) = x^3 + 2x^2$, so we put that below the dividend and x^2 in the quotient.

$$\begin{array}{l|llll} & & \mathbf{x^2} & & \\ \hline x+2 & x^3 & +\,0x^2 & -5x & -2 \\ & \mathbf{x^3} & \mathbf{+\,2x^2} & & \end{array}$$

We then subtract to obtain a remainder of $-2x^2$ and bring down the next term, $-5x$.
Dividing $-2x^2$ by x gives $-2x$, and multiplying $(-2x)(x + 2) = -2x^2 - 4x$, so we put that below the dividend and $-2x$ in the quotient.

$$\begin{array}{l|llll} & & \mathbf{x^2} & \mathbf{-2x} & \\ \hline x+2 & x^3 & +\,0x^2 & -5x & -2 \\ & x^3 & +\,2x^2 & & \\ \cline{2-3} & & \mathbf{-\,2x^2} & \mathbf{-5x} & \\ & & \mathbf{-\,2x^2} & \mathbf{-4x} & \end{array}$$

Subtracting again, we have a remainder of $-x$ and bring down the last term, -2.
Dividing $-x$ by x gives -1, and $(-1)(x + 2) = -x - 2$, so we put that below the dividend and -1 in the quotient.

$$\begin{array}{l|llll} & & \mathbf{x^2} & \mathbf{-2x} & \mathbf{-1} \\ \hline x+2 & x^3 & +\,0x^2 & -5x & -2 \\ & x^3 & +\,2x^2 & & \\ \cline{2-3} & & -\,2x^2 & -5x & \\ & & -\,2x^2 & -4x & \\ \cline{3-4} & & & \mathbf{-x} & \mathbf{-2} \\ & & & \mathbf{-x} & \mathbf{-2} \\ \cline{4-5} & & & & \mathbf{0} \end{array}$$

The quotient obtained by dividing $(x^3 - 5x - 2)$ by $(x + 2)$ is $(x^2 - 2x - 1)$.

Missing powers in the quotient.

Example(4): Divide x^3 $-3x^2$ - $5x$ + 15 by x- 3.

$x - 3$	x^3	$-3x^2$	$-5x$	$+15$

(Dividing x^3 by x gives x^2)

		$\mathbf{x^2}$		
$x - 3$	x^3	$-3x^2$	$-5x$	$+15$
	$\mathbf{x^3}$	$\mathbf{-3x^2}$		

Subtraction will leave us with a zero remainder in the x^2 term, so dividing $0x^2$ by x gives us $0x$.
The term in x in the quotient is zero, but we still place it in the quotient.

		$\mathbf{x^2}$	$\mathbf{+ 0x}$	
$x - 3$	x^3	$-3x^2$	$-5x$	$+15$
	x^3	$-3x^2$		
		$\mathbf{0x^2}$		

Because of the zero remainder in the last division, we finish by bringing down the next *two* terms to correspond with the two terms in the divisor.

(Dividing -5x by x gives -5)

		$\mathbf{x^2}$	$\mathbf{+ 0x}$	$\mathbf{-5}$
$x - 3$	x^3	$-3x^2$	$-5x$	$+15$
	x^3	$-3x^2$		
		$0x^2$		
			$\mathbf{-5x}$	$\mathbf{+15}$
			$\mathbf{-5x}$	$\mathbf{+15}$
				$\mathbf{0}$

Dividing (x^3 $-3x^2$ - $5x$ + 15) by (x- 3) gives a quotient of (x^2 - 5) .

Missing powers in the divisor.

Example(5): Divide $x^4 - 2x^3 - 7x^2 + 8x + 12$ by $x^2 - 4$.

Here we have a missing power of x in the divisor, but again its place must be included in the layout.

Notice that the dividend is of the 4th degree and the divisor a quadratic. The quotient will thus be of degree (4 – 2) or 2, i.e. a quadratic.

$x^2 + 0x - 4$	x^4	$-2x^3$	$-7x^2$	$+8x$	$+12$

Dividing x^4 by x^2 gives x^2, and $x^2(x^2 - 4) = x^4 - 4x^2$, so we put that below the dividend and x^2 in the quotient, making sure that the missing powers of x still have their places included in the working.

			x^2		
$x^2 + 0x - 4$	x^4	$-2x^3$	$-7x^2$	$+8x$	$+12$
	x^4	**$-0x^3$**	**$-4x^2$**		

We then subtract to obtain a remainder of $-2x^3 - 3x^2$ and bring down the next term, $8x$.

			x^2		
$x^2 + 0x - 4$	x^4	$-2x^3$	$-7x^2$	$+8x$	$+12$
	x^4	**$-0x^3$**	**$-4x^2$**		
		$-2x^3$	**$-3x^2$**	**$+8x$**	

Dividing $-2x^3$ by x^2 gives $-2x$, and as $(-2x)(x^2 - 4) = -2x^3 + 8x$, we put that below the dividend and $-2x$ in the quotient.

			x^2	**$-2x$**	
$x^2 + 0x - 4$	x^4	$-2x^3$	$-7x^2$	$+8x$	$+12$
	x^4	$-0x^3$	$-4x^2$		
		$-2x^3$	$-3x^2$	$+8x$	
		$-2x^3$	**$+0x^2$**	**$+8x$**	
			$-3x^2$	**$+0x$**	**$+12$**

Subtracting again, we have a remainder of $-3x^2$ and bring down the last term, +12.
Dividing $-3x^2$ by x^2 gives -3, and $(-3)(x^2 - 4) = -3x^2 + 12$, so we put that below the dividend and -3 in the quotient

			x^2	**$-2x$**	**-3**
$x^2 + 0x - 4$	x^4	$-2x^3$	$-7x^2$	$+8x$	$+12$
	x^4	$-0x^3$	$-4x^2$		
		$-2x^3$	$-3x^2$	$+8x$	
		$-2x^3$	$+0x^2$	$+8x$	
			$-3x^2$	$+0x$	$+12$
			$-3x^2$	**$+0x$**	**$+12$**
					0

Dividing $(x^4 - 2x^3 - 7x^2 + 8x + 12)$ by $(x^2 - 4)$ gives a quotient of $(x^2 - 2x - 3)$.

Example (6): Find the quotient and the remainder when dividing $x^3 - 7x^2 + 6x - 1$ by $x - 3$.

(Copyright OUP, *Understanding Pure Mathematics*, Sadler & Thorning, ISBN 9780199142590, Exercise 5G, Q.1c)

		$\mathbf{x^2}$		
$x - 3$	x^3	$-7x^2$	$+6x$	-1
	$\mathbf{x^3}$	$\mathbf{-3x^2}$		

		$\mathbf{x^2}$	$\mathbf{-4x}$	
$x - 3$	x^3	$-7x^2$	$+6x$	-1
	x^3	$-3x^2$		
		$\mathbf{-4x^2}$	$\mathbf{+6x}$	
		$\mathbf{-4x^2}$	$\mathbf{+12x}$	

		$\mathbf{x^2}$	$\mathbf{-4x}$	$\mathbf{-6}$
$x - 3$	x^3	$-7x^2$	$+6x$	-1
	x^3	$-3x^2$		
		$-4x^2$	$+6x$	
		$-4x^2$	$+12x$	
			$\mathbf{-6x}$	$\mathbf{-1}$
			$\mathbf{-6x}$	$\mathbf{+18}$
				$\boxed{\mathbf{-19}}$

This time, there is a final remainder, namely -19.

So $x^3 - 7x^2 + 6x - 1 = (x-3)(x^2 - 4x - 6) - 19$.

The Remainder Theorem.

In Example (4) above we divided $x^3 - 7x^2 + 6x - 1$ by $x - 3$ to give a quotient of $x^2 - 4x - 6$ and a remainder of -19.

Another way to find out the remainder is to substitute certain values for x.
Writing down $x^3 - 7x^2 + 6x - 1 = (x - 3)(Ax^2 + Bx + C) + D$, we can see that substituting $x = 3$, the right-hand side of the expression simplifies to D because the product of the brackets is zero.
This gives $3^3 - 7(3^2) + (6 \times 3) - 1 = 27 - 63 + 18 - 1 = -19$ as before.

Therefore, when a polynomial $P(x)$ is divided by $(x - a)$, the remainder is $P(a)$.

Example (7): Find the remainder when the polynomial $P(x) = x^3 - 7x^2 + 6x - 1$ is divided by

(a) $x + 1$; (b) $x - 2$; (c) $2x + 1$

In (a) the remainder is $P(-1) = -1 - 7 - 6 - 1 = -15$.
In (b) the remainder is $P(2) = 8 - 28 + 12 - 1 = -9$.
In (c) the remainder is $P(-0.5) = -0.125 - 1.75 - 3 - 1 = -5.875$.

For (c) the theorem can be generalised as:
when a polynomial $P(x)$ is divided by $(ax - b)$, the remainder is $P\left(\frac{b}{a}\right)$.

The Factor Theorem.

This is a special case of the remainder theorem when the remainder is zero. It states that:

If $P(x)$ is a polynomial and $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

Example (8):Show that (x-2) is a factor of $P(x) = x^3 - x^2 - 4x + 4$, and hence solve $P(x) = 0$.

(Copyright OCR 2004, MEI Mathematics Practice Paper C1-A, 2004, Q. 7)

Substituting $x = 2$, we find that $P(2) = 2^3 - 2^2 - (4\times2) + 4$ or $8 - 4 - 8 + 4 = 0$.
$\therefore$ (x-2) is a factor of $P(x)$.

We then factorise the expression completely:

$$\begin{array}{r|llll}
 & & \mathbf{x^2} & & \\ \cline{2-5}
x-2 & x^3 & -x^2 & -4x & +4 \\
 & \mathbf{x^3} & \mathbf{-2x^2} & &
\end{array}$$

$$\begin{array}{r|llll}
 & & \mathbf{x^2} & \mathbf{+x} & \\ \cline{2-5}
x-2 & x^3 & -x^2 & -4x & +4 \\
 & x^3 & -2x^2 & & \\ \cline{2-3}
 & & \mathbf{x^2} & \mathbf{-4x} & \\
 & & \mathbf{x^2} & \mathbf{-2x} &
\end{array}$$

$$\begin{array}{r|llll}
 & & \mathbf{x^2} & \mathbf{+x} & \mathbf{-2} \\ \cline{2-5}
x-2 & x^3 & -x^2 & -4x & +4 \\
 & x^3 & -2x^2 & & \\ \cline{2-3}
 & & x^2 & -4x & \\
 & & x^2 & -2x & \\ \cline{3-4}
 & & & \mathbf{-2x} & \mathbf{+4} \\
 & & & \mathbf{-2x} & \mathbf{+4} \\ \cline{4-5}
\end{array}$$

The quotient is therefore $x^2 + x - 2$.

The next step is to factorise the quotient, giving $(x + 2)(x - 1)$.

$x^3 - x^2 - 4x + 4 = (x - 2)(x + 2)(x - 1)$.

$\therefore$ The solutions of $P(x) = 0$ are $x = 1$, 2 and –2.

Again, a more generalised form of the Factor Theorem states :

If $P(x)$ is a polynomial and $P\left(\frac{b}{a}\right) = 0$, then $(ax - b)$ is a factor of $P(x)$.

Example (9): A polynomial is given by $Q(x) = 2x^3 - 5x^2 - 13x + 30$.

a) Find the value of $Q(-2)$ and $Q(2)$, and state one factor of $Q(x)$.
b) Factorise $Q(x)$ completely.

a) $Q(-2) = 2(-2)^3 - 5(-2)^2 - (13 \times (-2)) + 30 = -16 - 20 + 26 + 30 = 20$.
$Q(2) = 2(2)^3 - 5(2)^2 - (13 \times (2)) + 60 = 16 - 20 - 26 + 30 = 0$.

$\therefore$ One factor of $2x^3 - 5x^2 - 13x + 30$ is $(x - 2)$.

b) We then divide $(x - 2)$ into $2x^3 - 5x^2 - 13x + 30$ to obtain a quadratic quotient:

$$
\begin{array}{r|llll}
 & & \mathbf{2x^2} & \mathbf{-x} & \mathbf{-15} \\ \cline{2-5}
x-2 & 2x^3 & -5x^2 & -13x & +30 \\
 & 2x^3 & -4x^2 & & \\ \cline{2-3}
 & & -x^2 & -13x & \\
 & & -x^2 & +2x & \\ \cline{3-4}
 & & & -15x & +30 \\
 & & & -15x & +30 \\ \cline{4-5}
\end{array}
$$

The quotient is therefore $2x^2 - x - 15$.

Trial inspection and factorising gives $2x^2 - x - 15 = (2x + 5)(x - 3)$.

$\therefore$ $2x^3 - 5x^2 - 13x + 30$ factorises fully to $(x - 2)(x - 3)(2x + 5)$.

Example (10): A polynomial is given by $P(x) = x^3 - 2x^2 - 4x + k$ where k is an integer constant.

Find the values of k satisfying the following conditions:

i) the graph of $y = P(x)$ passes through the origin.

ii) the graph of $y = P(x)$ intersects the y-axis at the point (0,5).

iii) $(x - 3)$ is a factor of $P(x)$.

iv) the remainder when $P(x)$ is divided by $(x + 1)$ is 5.

(Copyright OCR 2004, MEI Mathematics Practice Paper C1-C, Q.11 altered)

In i), $P(0) = 0$ when the graph of $P(x)$ passes through the origin, therefore $0^3 - 2(0)^2 - 4(0) + k = 0$ and thus $k = 0$.

In ii), $P(0) = 5$, therefore $0^3 - 2(0)^2 - 4(0) + k = 5$ and thus $k = 5$.

In iii), $(x - 3)$ is a factor of $P(x)$ if $P(3) = 0$ by the Factor Theorem.
$\therefore\ 3^3 - 2(3)^2 - 4(3) + k = 0$
$\Rightarrow 27 - 18 - 12 + k = 0$
$\Rightarrow k = 3$.

In iv), $P(-1) = 5$ by the Remainder Theorem.

$\therefore\ (-1)^3 - 2(-1)^2 - 4(-1) + k = 5$
$\Rightarrow -1 - 2 + 4 + k = 5$
$\Rightarrow k = 4$.

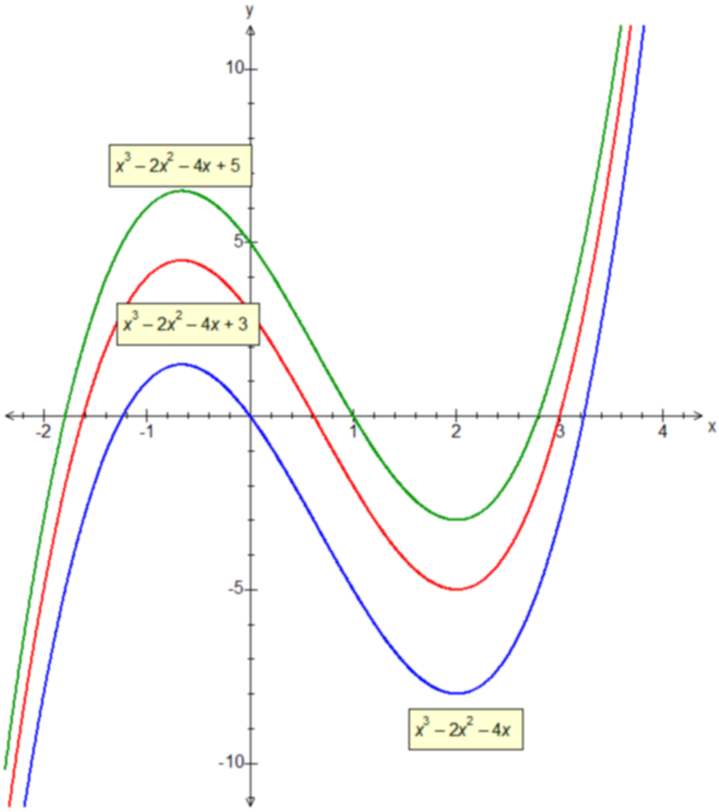
The solutions to parts i), ii) and iii) are shown graphically on the right.

Notice the following:

i) The graph of $x^3 - 2x^2 - 4x$ passing through the origin.

ii) the graph of $x^3 - 2x^2 - 4x + 5$ passing through the point (0,5). It also appears to pass through (1,0) – confirmed by substituting $x = 1$ in the expression, $\therefore$ $(x-1)$ is a factor as well.

iii) The graph of $x^3 - 2x^2 - 4x + 3$ passing through (3, 0).



Example(11): The polynomial $Q(x) = 3x^3 + 2x^2 - bx + a$ where a and b are integer constants.

It is given that $(x - 1)$ is a factor of $Q(x)$, and that division of $Q(x)$ by $(x + 1)$ gives a remainder of 10. Find the values of a and b.

If $(x - 1)$ is a factor of $Q(x)$, then $Q(1) = 0$ by the Factor Theorem.
Substituting $x = 1$, we have :

$3 + 2 - b + a = 0$
$\Rightarrow 5 - b + a = 0$
$\Rightarrow a - b = -5$

If $(x + 1)$ leaves a remainder of 10 when divided into $Q(x)$, then $Q(-1) = 10$ by the Remainder Theorem.

Substituting $x = -1$, we have:

$-3 + 2 + b + a = 10$
$\Rightarrow -1 + a + b = 10$
$\Rightarrow a + b = 11$

This leaves us with two linear simultaneous equations:

$a - b = -5$ $\quad A$
$a + b = 11$ $\quad B$

$2a = 6$ $\quad A+B$

Substituting $a = 3$ into equation B gives $b = 8$.

Hence $Q(x) = 3x^3 + 2x^2 - 8x + 3$.

(Question does not ask for the expression to be formally factorised.)

Example(12): The polynomial $P(x) = 6x^3 - 23x^2 + ax + b$ where a and b are integer constants.

It is given that division of $P(x)$ by $(x - 3)$ gives a remainder of 11, and that division of $P(x)$ by $(x + 1)$ gives a remainder of -21.

Find the values of a and b and hence factorise the expression.

If $(x - 3)$ leaves a remainder of 11 when divided into $P(x)$, then $P(3) = 11$ by the Remainder Theorem.

Substituting $x = 3$, we have $162 - 207 + 3a + b = 11$
$\Rightarrow -45 + 3a + b = 11$
$\Rightarrow 3a + b = 56$

Similarly, if $(x + 1)$ leaves a remainder of -21 when divided into $P(x)$, then $P(-1) = -21$ by the Remainder Theorem.

Substituting $x = -1$, we have $-6 - 23 - a + b = -21$
$\Rightarrow -29 - a + b = -21$
$\Rightarrow -a + b = 8$

This leaves us with two linear simultaneous equations:

$3a + b = 56$ $\quad A$
$-a + b = 8$ $\quad B$

$4a = 48$ $\quad A\text{-}B$

Substituting $a = 12$ into equation A gives $b = 20$.

$\therefore P(x) = 6x^3 - 23x^2 + 12x + 20.$

To factorise the equation, try substituting various values of x to find one that gives zero;
$P(1) = 6 - 23 + 12 + 20 = 15$, so $(x - 1)$ is not a factor by the Remainder Theorem.
$P(2) = 48 - 92 + 24 + 20 = 0$, so $(x - 2)$ is a factor.

We can then proceed to factorise the cubic:

		$\mathbf{6x^2}$	$\mathbf{-11x}$	$\mathbf{-10}$
$x-2$	$6x^3$	$-23x^2$	$+12x$	$+20$
	$6x^3$	$-12x^2$		
		$-11x^2$	$+12x$	
		$-11x^2$	$+22x$	
			$-10x$	$+20$
			$-10x$	$+20$

$\therefore P(x) = (x - 2)(6x^2 - 11x - 10).$

The quadratic quotient in turn factorises to $6x^2 - 11x - 10 = (3x + 2)(2x - 5)$

$\therefore P(x) = (x - 2)(3x + 2)(2x - 5).$

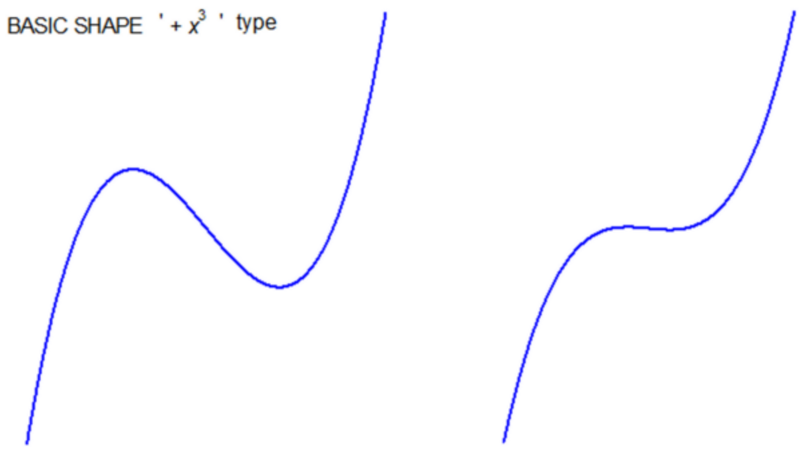
Sketching cubic graphs.

Examination questions might also ask for a sketch of a polynomial graph, usually no more complex than a cubic.

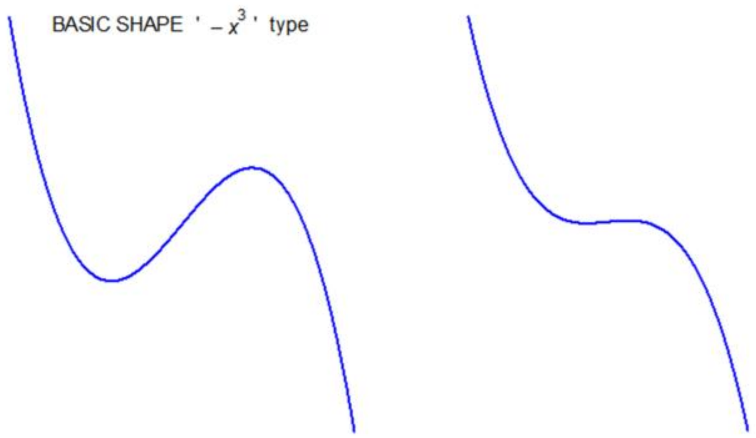
The main criteria for sketching a cubic graph are a) obtaining the correct general shape, b) finding the intercepts and c) , locating and finding any turning points if asked to do so.

The examples below will not require any work on finding turning points.

The basic shape of a cubic graph features a 'double bend' of varying severity. If the coefficient of x^3 is positive, then the curve follows a general lower left to upper right direction.



If the coefficient of x^3 is negative, then the curve follows a general upper left to lower right direction.



Finally, if the cubic has repeated factors, the x-intercept at that particular root is a tangent to the x-axis.

Example (13): The polynomial $P(x) = x^3 - x^2 - 4x + 4$ in Example (8) was factorised to

$P(x) = (x - 2)(x + 2)(x - 1)$.

i) Sketch the graph of $P(x)$.

Since the coefficient of x^3 is positive, the general shape of the graph is an increasing one from lower left to upper right, namely of the basic '+ x^3' type.

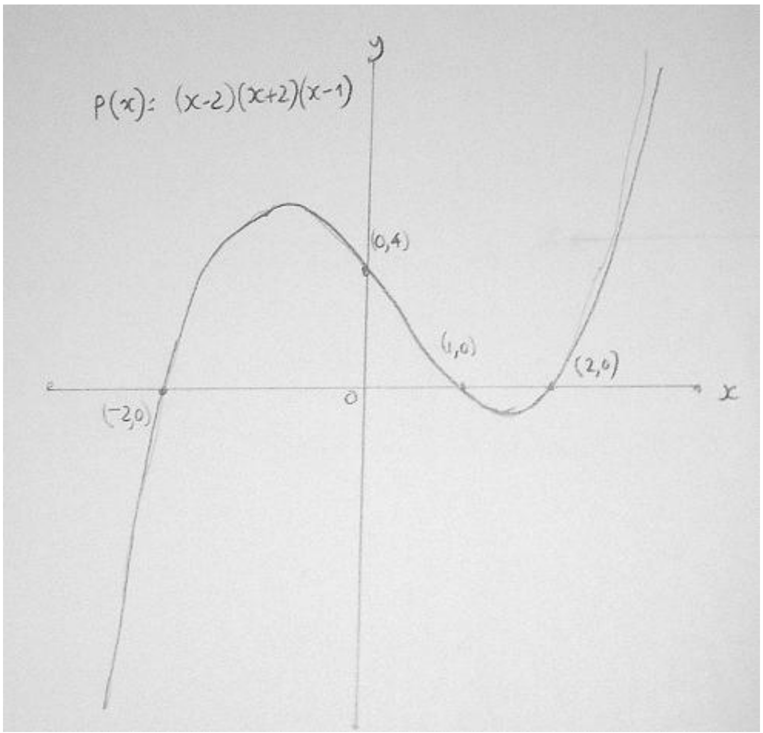
i) The x-intercepts correspond to the roots at x = -2, 1 and 2, and so we plot the points (-2, 0), (1, 0) and (2, 0).

When x = 0, y = 4, so we plot the y-intercept at (0, 4).

Finally, we connect the points with a basic '+ x^3' curve, with a local maximum at about x = -0.5 and a local minimum at about x = 1.5.

Although the section of the graph from x = 1 to x = 2 looks like part of a quadratic curve, it does not have the symmetry of a quadratic, so we cannot say that minimum has an x-coordinate of *exactly* 1.5.

The same caveat applies to the x-coordinate of the maximum – it is not halfway between -2 and 1.



Example (14): Show by the Factor Theorem that $(x + 4)$ is a factor of $P(x) = x^3 - 2x^2 – 15x + 36$. Hence factorise $P(x)$ completely and sketch its graph.

Substituting $x = -4$ gives $P(-4) = -64 – 32 + 60 + 36 = 0$, $\therefore$ $(x + 4)$ is a factor of $P(x)$.

		$\mathbf{x^2}$	$\mathbf{-6x}$	$\mathbf{+9}$
$x + 4$	x^3	$-2x^2$	$–15x$	$+36$
	x^3	$+4x^2$		
		$- 6x^2$	$–15x$	
		$- 6x^2$	$- 24x$	
			$9x$	$+36$
			$9x$	$+36$

The quotient, $x^2 - 6x + 9$, can be factorised to $(x-3)^2$.

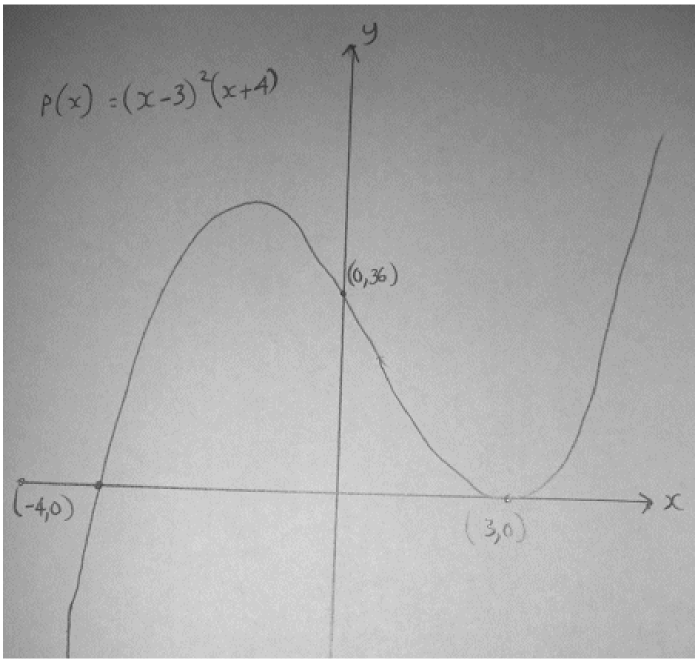
$\therefore$ $P(x) = x^3 - 2x^2 – 15x + 36$ factorises fully to $(x + 4)(x - 3)^2$.

From the above facts, we can deduce that the graph meets the x–axis at (-4, 0) and (3, 0).

Because $(x – 3)$ is a repeated factor, the x-intercept (3, 0) is also a tangent to the x-axis.

When $x = 0$, $y = 36$, so the graph meets the y-axis at (0, 36).

Finally, we connect the points with a basic '+ x^3' curve, with a local maximum at about $x = -1$ and a tangent to the x-axis (actually a local minimum) at $x = 3$.



Example (15): Sketch the graph of $Q(x) = -x^3 + 12x + 16$.
We are given that $(4 - x)$ is a factor.
Note: $4-x$ is the same as $-x + 4$.

The first thing to notice here is that the coefficient of x^3 is negative, and therefore the graph will be of the '$-x^3$' type.

Firstly we factorise $Q(x)$ completely:

		$\boldsymbol{x^2}$	$\boldsymbol{+4x}$	$\boldsymbol{+4}$
$-x + 4$	$-x^3$	$+0x^2$	$+12x$	$+16$
	$-x^3$	$+4x^2$		
		$-4x^2$	$+12x$	
		$-4x^2$	$+16x$	
			$-4x$	$+16$
			$-4x$	$+16$

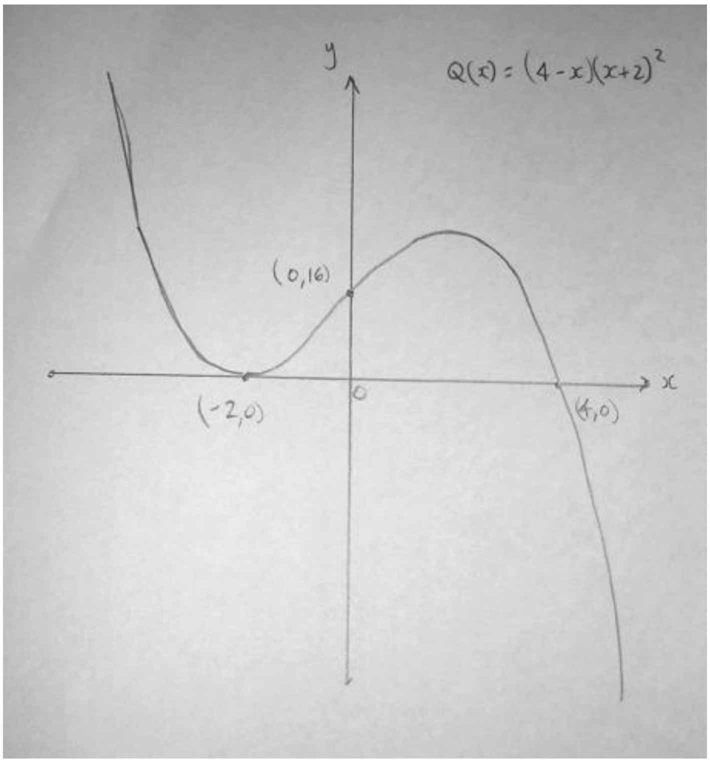
The quotient, $x^2 + 4x + 4$, can be factorised to $(x+2)^2$.

$\therefore$ $Q(x) = -x^3 + 12x + 16$ factorises fully to $(4 - x)(x + 2)^2$.

From the above, the graph meets the x–axis at (4, 0) and (-2, 0).
We have a repeated factor of $(x+2)$, and so the x-intercept (-2, 0) is also a tangent to the x-axis.

When $x = 0$, $y = 16$, so the graph meets the y-axis at (0, 16).

Finally, we connect the points with a basic '$-x^3$' curve, with a tangent to the x-axis (actually a local minimum) at $x = -2$ and a local maximum at about $x = 2$.



Roots of a Cubic – Summary.

A cubic equation of the factorised form $(ax + b)(cx + d)(ex + f) = 0$ can have three distinct roots, one repeated pair of roots, or a thrice-repeated root.
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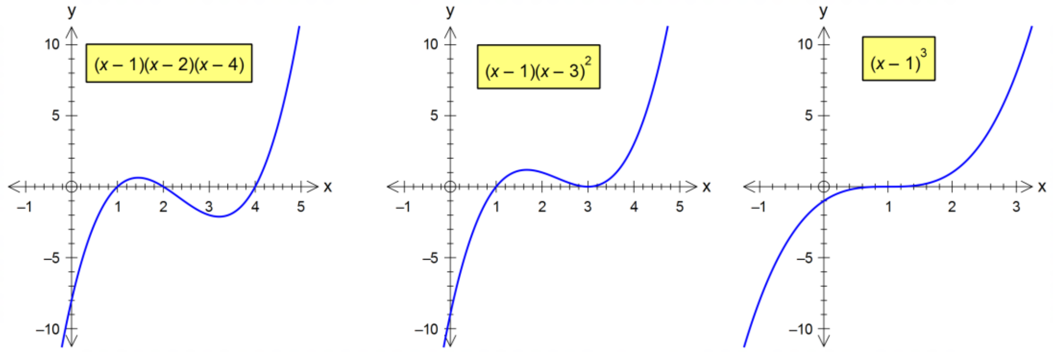
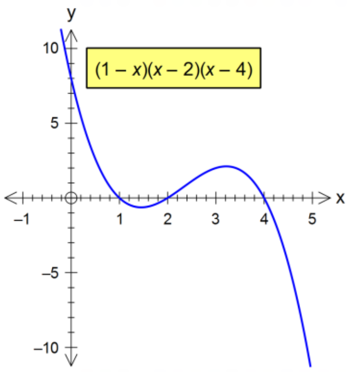


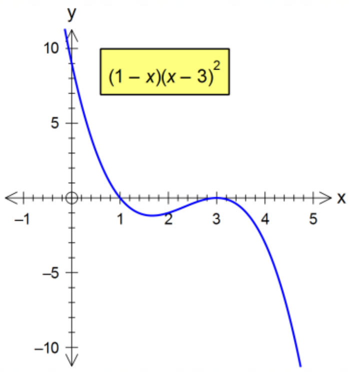
Hence the equation $(x - 1)(x - 2)(x - 4) = 0$ has three distinct solutions for x : i.e. $x = 1, 2$, or 4. The graph intersects the x-axis at three points – (1,0), (2,0) and (4,0). There are also two turning points – a maximum and a minimum.

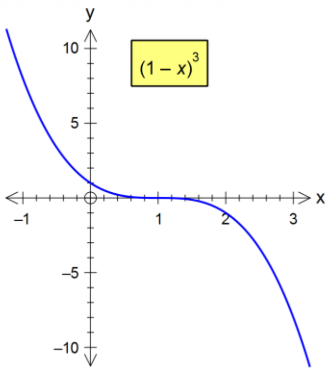
When we come to the equation $(x - 1)(x - 3)^2 = 0$, the quadratic factor means that we now have a repeated root of $x = 3$ as well as the distinct one $x = 1$. The graph intersects the x-axis at two points – (1,0) and (3,0), and is a tangent to the x-axis at the latter. There are still two turning points, however. .

Finally in the case of $(x - 1)^3 = 0$, we now have a thrice-repeated root of $x = 1$, and the graph now has only one turning point at (1,0), and that turning point is now a point of inflection rather than a maximum or a minimum.

All the examples above have a positive *x*-coefficient – a negative one gives a mirror-image graph



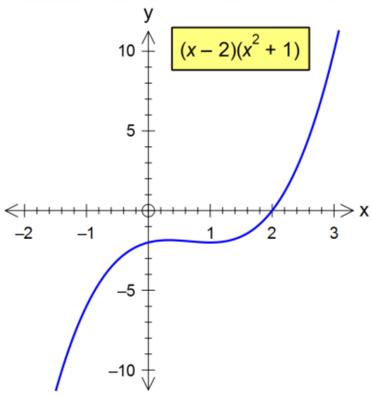




To 'negate' the graphs in this example, the factor (x-1) has been replaced with (1-x), effectively multiplying by -1 and reflecting the graphs in the x-axis. The roots are the same as on the original graphs, but any maxima and minima have also exchanged places.

Other cubic equations might factorise into the form $(ax + b)(cx^2 + dx + e) = 0$, where the quadratic factor has no real roots. This also gives rise to a cubic with only a single real root.

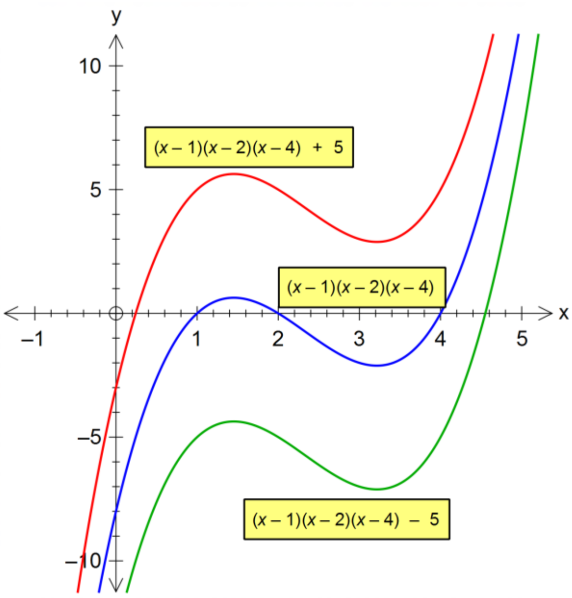
Although this example appears to have two shallow turning points, other examples might have none. This is discussed in the section on "Maxima and Minima".



We could also have the same situation if we translate any of the previous examples by a suitable number of units in the y-direction, as in Example (10).

For example, $(x - 1)(x - 2)(x – 4) + 5 = 0$ will only have one solution, since the local minimum will end up above the x-axis, and there will be only one root at $x \approx 0.2$.

Similarly, $(x - 1)(x - 2)(x – 4) - 5 = 0$ will only have one solution, since the local maximum will end up below the x-axis, with only one root at $x \approx 4.6$.



Alternative method of dividing / factorising polynomials.

Although the long division method is the most commonly used one for dividing and factorising polynomials, there is another method which can sometimes prove easier to use – called the method of equating coefficients.

Example (16):The polynomial $P(x) = 2x^3 + 3x^2 - 23x - 12$ has two linear factors of $(x - 3)$ and $(x + 4)$. Find the third linear factor.

Since $P(x)$ is a cubic, its factorised form is $(x - 3)(x + 4)(Ax + B)$ where A and B are constants.
The only way to obtain the term of $2x^3$ in $P(x)$ is to multiply x, x and Ax together from the factors.
By equating the x^3 terms, $A = 2$.

Similarly, the only way to obtain the term of -12 in $P(x)$ is to multiply -3, 4 and B together. Equating the constants gives $B = 1$.

Hence the third linear factor of $P(x)$ is $2x + 1$.

Example (17): Find the quotient and the remainder when the polynomial $P(x) = 6x^3 - 13x^2 + 16x - 3$ is divided by the polynomial $Q(x) = 2x^2 - 3x + 5$.

The degree of the divisor is 2, and so the quotient will be of degree 1.

Therefore $6x^3 - 13x^2 + 16x - 3 = (2x^2 - 3x + 5)(Ax + B) + (Cx + D)$.

Expanding, we have $6x^3 - 13x^2 + 16x - 3 = (2Ax^3 - 3Ax^2 + 5Ax) + (2Bx^2 - 3Bx + 5B) + Cx + D$.

Equating the x^3 terms, we have $2A = 6$, so $A = 3$.

Equating the x^2 terms, we have $2B - 3A = -13$, or $2B - 9 = -13$, or $2B = -4$, so $B = -2$.

Equating the x terms, we have $5A - 3B + C = 16$, or $15 + 6 + C = 16$, or $21 + C = 16$, so $C = -5$.

Equating the constants, we have $5B + D = -3$, or $-10 + D = -3$, so $D = 7$.

$\therefore$ The quotient is $Ax + B$ or $3x - 2$, and the remainder is $Cx + D$ or $-5x + 7$, or $7 - 5x$.

$\therefore$ $6x^3 - 13x^2 + 16x - 3 = (2x^2 - 3x + 5)(3x - 2) + (7 - 5x)$.

Corresponding long division method:

			$\mathbf{3x}$	**-2**
$2x^2 - 3x + 5$	$6x^3$	$-13x^2$	$+16x$	-3
	$6x^3$	$-9x^2$	$+15x$	
		$-4x^2$	x	-3
		$-4x^2$	$+6x$	-10
			$-5x$	$+7$

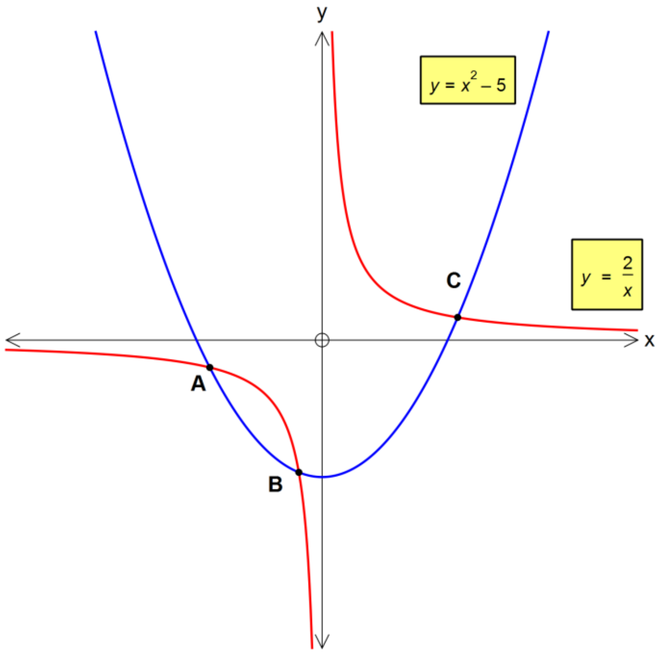
Example (18). The graphs of $y = x^2 - 5$ and $y = \frac{2}{x}$ are shown below. The two curves intersect at the points **A**. **B** and **C**.

i) Show algebraically that the x-coordinates of points **A**. **B** and **C** are the roots of the equation $x^3 - 5x - 2 = 0$.

ii) Point **A** has integer coordinates. Find them using the Factor Theorem.

iii) Hence find the coordinates of points **B** and **C**, giving your values in the form

$a + b\sqrt{2}$ where a and b are integers.



i) Starting with $x^2 - 5 = \frac{2}{x}$, we multiply both sides by x:

$x^3 - 5x = 2$, and so $x^3 - 5x - 2 = 0$. (We have turned the equation into a polynomial.)

ii) Substituting $x = -2$ into the resulting cubic gives $(-2)^3 - 5(-2) - 2 = -8 + 10 - 2 = 0$.
Hence the x-coordinate of **A** is -2 and the y-coordinate is -1 by substituting in either $x^2 - 5$ or $\frac{2}{x}$.

iii) From ii), we know that $(x+2)$ is a factor. Dividing $(x^3 - 5x - 2)$ by $(x + 2)$ gives us the quadratic quotient of $x^2 - 2x - 1$. (Full working in Example (3)).

The equation $x^2 - 2x - 1 = 0$ can be solved by completing the square (used here) or the general formula.

$$x^2 - 2x - 1 = 0 \;\Rightarrow\; (x - 1)^2 - 1 - 1 = 0 \;\Rightarrow\; (x - 1)^2 - 2 = 0 \;\Rightarrow\; (x - 1)^2 = 2 \;\Rightarrow\; (x - 1) = \pm\sqrt{2}$$

Hence $\mathbf{x = 1 \pm \sqrt{2}}$.

The x-coordinate of **B** is the negative one, i.e. $1-\sqrt{2}$, and that of **C** is the positive one, i.e. $1+\sqrt{2}$

Substituting in $y = \frac{2}{x}$ gives the y-coordinate of **B** as $\frac{2}{1-\sqrt{2}}$ which can be rationalised to

$$\frac{2}{1-\sqrt{2}} \times \frac{1+\sqrt{2}}{1+\sqrt{2}} = \frac{2+2\sqrt{2}}{-1} = -2-2\sqrt{2}.$$

Similarly the y-coordinate of **C** is $\frac{2}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{2-2\sqrt{2}}{-1} = -2+2\sqrt{2}$

$\therefore$ The curves intersect at **A** (-2 ,-1) , **B** $(1-\sqrt{2}, -2-2\sqrt{2})$ and **C** $(1+\sqrt{2}, -2+2\sqrt{2})$.

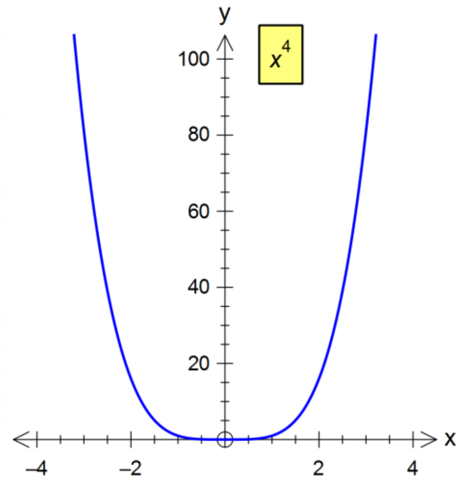
APPENDIX Quartic (fourth-power) graphs. New from 2017.

We shall have a look at fourth-degree or quartic graphs. The basic quartic graph of $y = x^4$ resembles that of $y = x^2$ but is shallower for $-1 < x < 1$ and steeper for other values of x. It is **not** a parabola.

It also has only one minimum point at the origin.

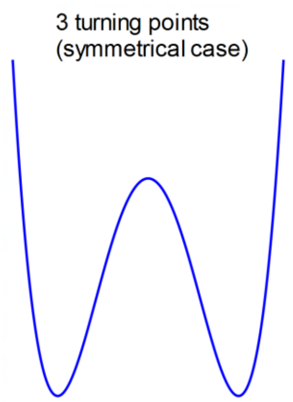
(The graph of $y = -x^4$ is a reflection of $y = x^4$ in the x-axis and has a maximum point at the origin.)

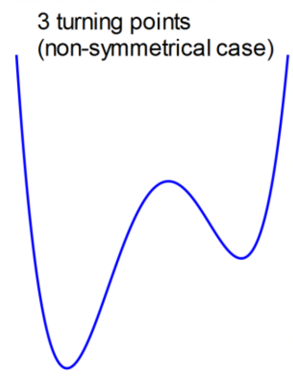
Other quartic graphs are a little more complicated, and we shall restrict ourselves to those where the coefficient of x^4 is positive.

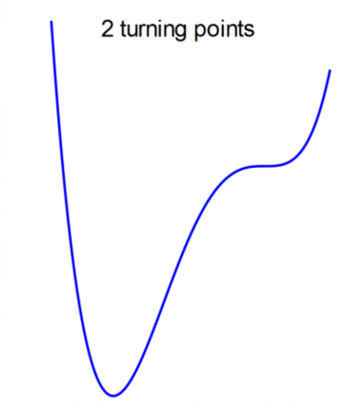


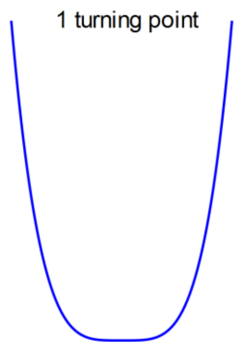
Quartic graphs can have up to three turning points and intersect the x-axis at up to four points (one more than cubics !), but like quadratics, they can also not intersect the x-axis at all. The following examples do not cover all cases, but give an idea of what to look for in the properties of selected quartic graphs. We shall restrict ourselves to fully-factorisable examples in this document, and not attempt to evaluate any turning points unless they coincide with roots. (The third example has 2 stationary points in the strict sense of the word, as one of them is a point of inflection.)

The basic shapes of quartic graphs with a positive x^4-coefficient look like those below. (For negative x^4 coefficients, reflect the graph in the x-axis.)

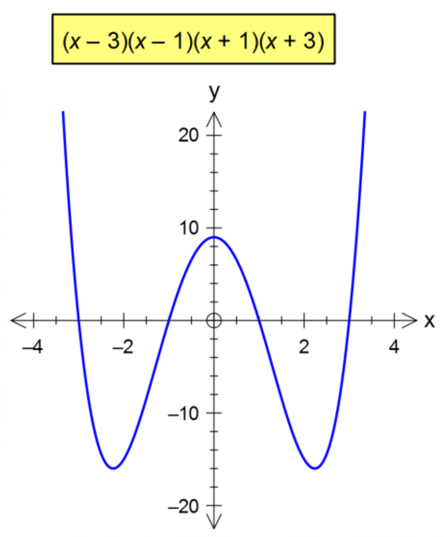


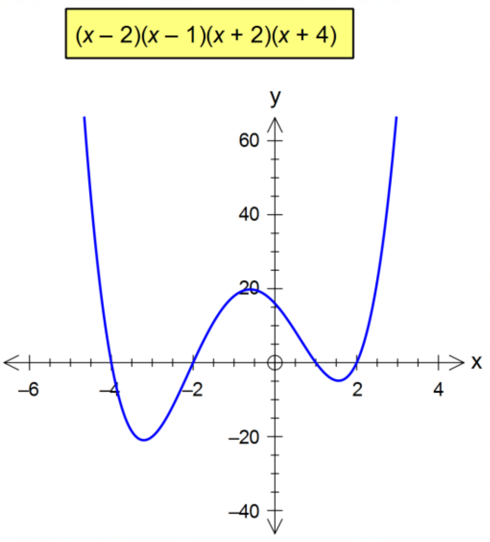






Four distinct roots.





These graphs have a W-shape, which can have a line of symmetry or not, depending on the differences between outer pairs of roots.

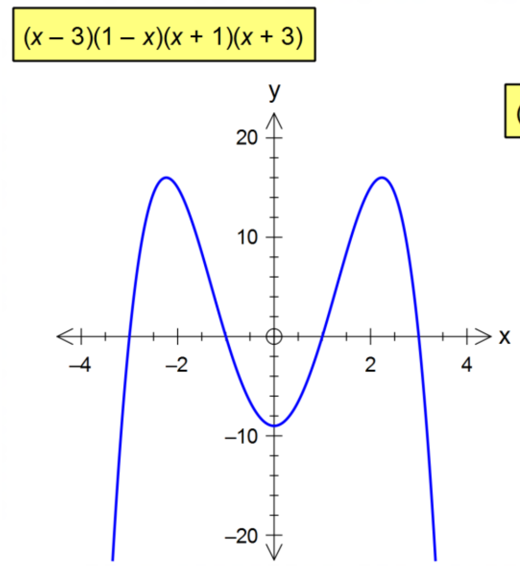
The graph of $y = (x-3)(x-1)(x+1)(x+3)$ has roots at 3, 1, -1 and -3, coinciding with factors taking a value of zero. The outer pairs of roots are evenly spaced, so there is also a line of symmetry about the y-axis, i.e. when x lies halfway between -1 and 1, i.e. $x = 0$.

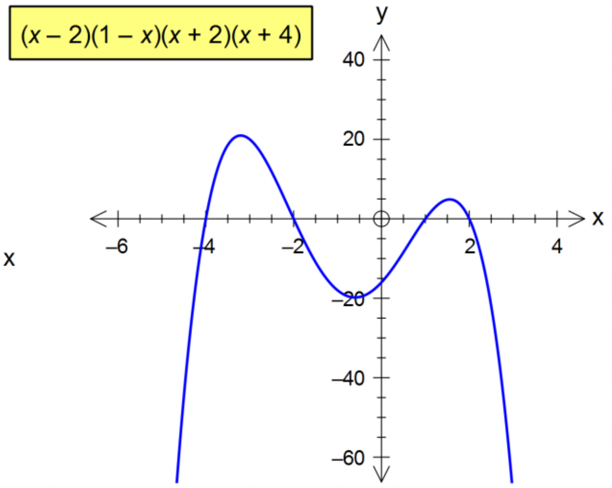
The turning points also take the sequence minimum-maximum-minimum as x increases.
Because of the symmetry of the graph, the x-coordinate of the local maximum is 0, but, as with the cubic cases earlier, the x-coordinates of the local minima are only approximately -2 and 2.

The graph of $y = (x-2)(x-1)(x+2)(x+4)$ has roots at 2, 1, -2 and -4, but this time the outer pairs are not spaced symmetrically. As a result, the graph has no vertical lines of symmetry.

The sequence of turning points is still the same though: through minimum to maximum through minimum again, and again, their x-coordinates are not halfway between the pairs of roots..

The graphs below are reflections of the previous ones in the *x*-axis, giving an M-shape rather than a W-shape. In both cases the factor of (x-1) has been replaced by (1-x), effectively multiplying the equation of the graph by -1. The roots are still all the same as before, but the sequence of turning points has been reversed to maximum-minimum-maximum with increasing x.





Three roots.

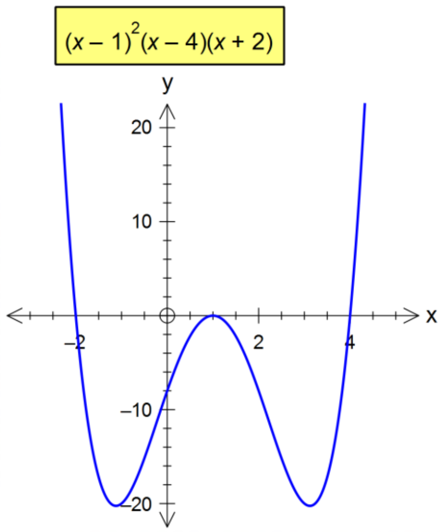
A quartic curve can have one pair of repeated roots, but the shape of its graph is same as that of the graph with four distinct roots. The main difference is that the curve is tangent to the x-axis at the turning point corresponding to the repeated root.

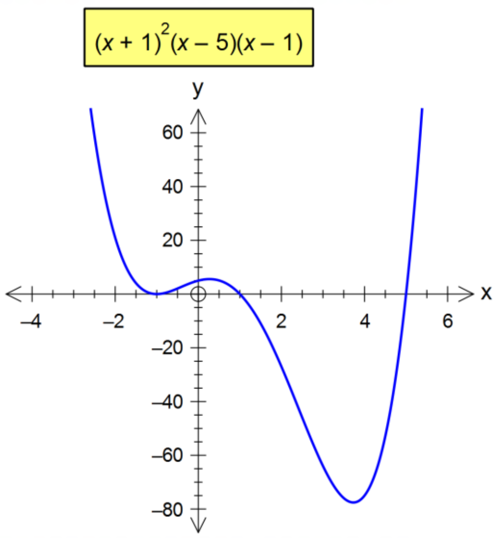
The two examples below both have a positive coefficient of x^4.

The graph of $y = (x-1)^2(x-4)(x+2)$ has roots at 1 (repeated), 4 and -2. Because the "middle" root of 1 is halfway between -2 and 4, there is also a line of symmetry about the y-axis - i.e. the line $x = 1$, with the local maximum point of (1,0) tangent to the x-axis. The turning points also take the sequence minimum-maximum-minimum as x increases.

The graph of $y = (x+1)^2(x-5)(x-1)$ has roots at -1, 5 and 1. The repeated root of -1 is one of the "outer" ones, and so the graph has no vertical lines of symmetry. The local minimum point of (-1, 0) is also tangent to the x-axis.

Again, in the case of a negative coefficient of x^4, the order of the maximum and minimum points will be interchanged.





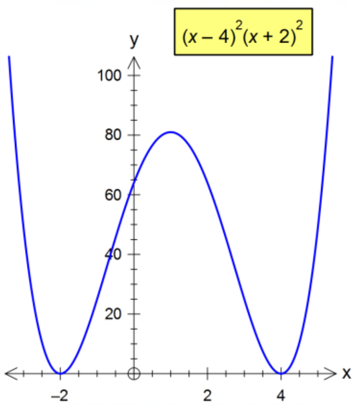
Two roots.

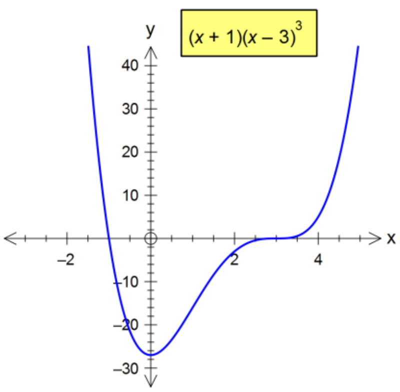
When it comes to the case of the quartic curve having two roots, there are two distinct cases:

We could have a product of two squares of linear terms giving rise to two twice-repeated roots as in the curve of $y = (x-4)^2 (x+2)^2$, or we could have the product of a linear and a cubic giving us a thrice-repeated root, as in $y = (x+1)(x-3)^3$.

The graph of $y = (x-4)^2 (x+2)^2$ has the same symmetrical shape as that of $y = (x-3)(x-1)(x+1)(x+3)$ shown earlier, but this time we have two pairs of repeated roots at $x = -2$ and $x = -4$, resulting in the graph being tangent to the x-axis at the points (4,0) and (-2,0). Again, the x-coordinate of the "middle" minimum point is halfway between 4 and -2, or 1. (Its y-coordinate is 81 if you want to check it out)

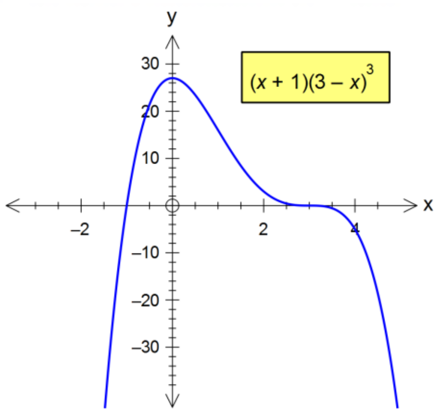
By contrast, the graph of $y = (x+1)(x-3)^3$ is quite different from all the previous cases. It has one non-repeated root at (-1,0) , one turning point (here a minimum at (0,-27)) and a point of inflection at (3, 0). The graph has a resemblance to that of $y = x^3$ in the neighbourhood of the point of inflection.
This point of inflection is on the "increasing" part of the curve, so that portion resembles the graph of $y = x^3$ and not $y = -x^3$.





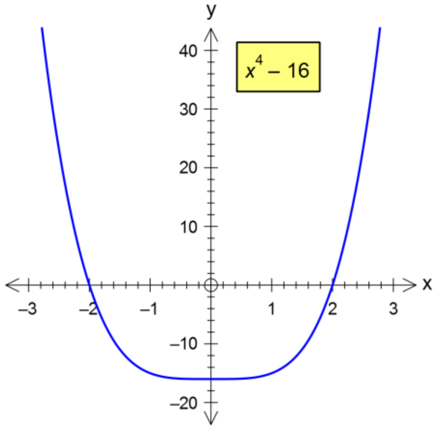
The graph on the right is that of $y = (x+1)(3-x)^3$, effectively multiplying the last graph's expression by -1, and making it have a negative coefficient of x^4.

It now has a maximum at (0, 27) and the point of inflection is now on the "decreasing" part of the curve, so the graph resembles $y = -x^3$ near the point of inflection at (3, 0).



These are not the only cases with two roots; for example the equation $x^4 - 16 = 0$ has two roots at 2 and -2, but the expression does not factorise into four linear factors.

Using difference of squares, $x^4 - 16 = (x^2 + 4)(x^2 - 4) = (x^2 + 4)(x - 2)(x + 2)$, leaving us with a product of two linear factors and a quadratic factor whose equation, $x^2 + 4 = 0$, has no real roots.

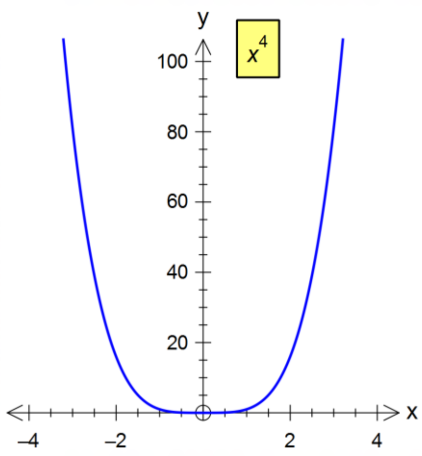


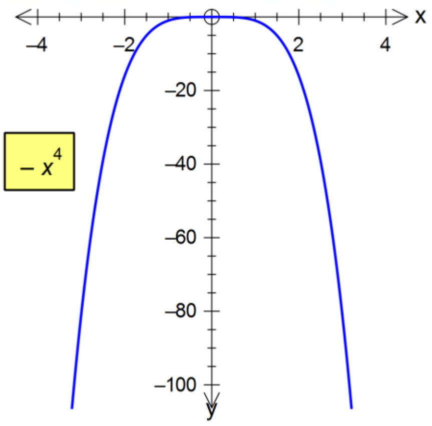
One root (four-times repeated)

The basic quartic function $y = x^4$ has only one turning point and root at (0,0)

This can be generalised to any x-translation and/or y-stretch of the basic function.

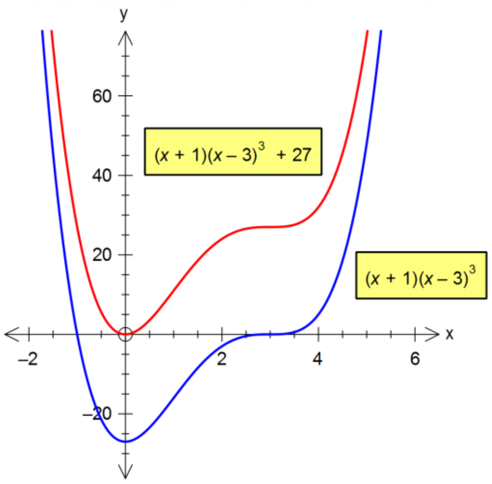
Hence the graph of any function of the form $y = a(bx + k)^4$ (where a,b and k are constants and a and b are not zero) has only one turning point and root at $x = \frac{-b}{a}$.





Other cases would include suitable translations of some of the earlier examples.

For example, if we were to translate the graph of $y = (x+1)\,(x-3)^3$ upwards by 27 units, we would see that the only root of $y = (x+1)\,(x-3)^3 + 27 = 0$ would be $x = 0$.



No roots.

There are no factorisable examples, but obvious cases include such functions as like $x^4 + 1 = 0$, or translations of the former examples e,g, $(x-3)(x-1)(x+1)(x+3) + 20$.

(See section on cubic graph types).