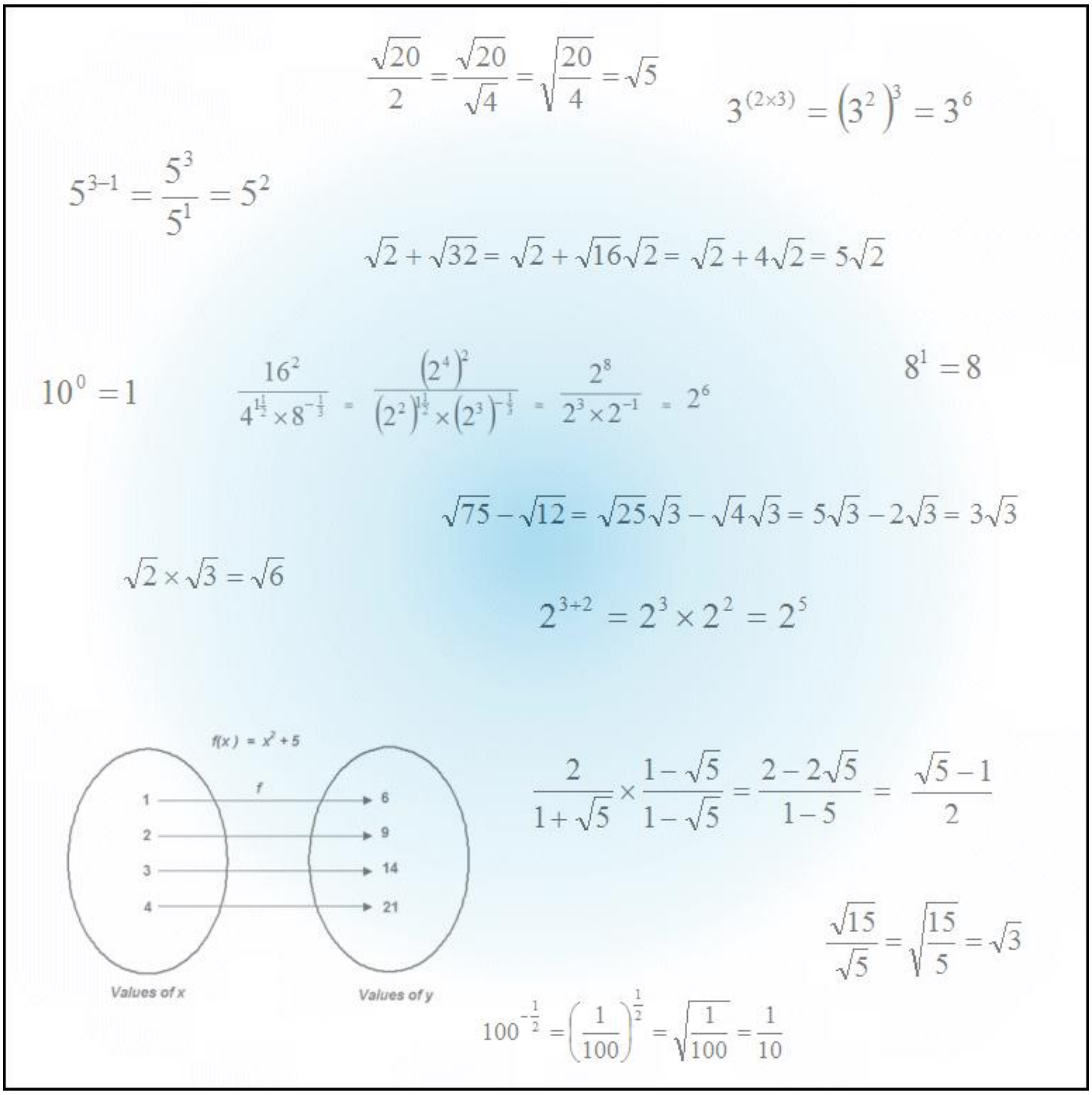
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 1 / AS

INDICES, SURDS AND FUNCTIONS (Revision)



Version: 2.4 Date: 21-10-2014

INDICES (Including revision from GCSE)

The basic laws of indices are as follows, applicable to all positive numbers *a*.

Multiplication: Addition of indices corresponds to multiplication of actual numbers.

$$a^{m+n} = a^m \times a^n$$

Example (1): $2^{3+2} = 2^3 \times 2^2 = 2^5$, or 8 × 4 = 32.

$$\left(\frac{2}{3}\right)^{1+2} = \left(\frac{2}{3}\right)^1 \times \left(\frac{2}{3}\right)^2 = \left(\frac{2}{3}\right)^3$$, or $\frac{2}{3} \times \frac{4}{9} = \frac{8}{27}$.

Division: Subtraction of indices corresponds to division of actual numbers.

$$a^{m-n} = \frac{a^m}{a^n}$$

Example (2): $5^{3-1} = \frac{5^3}{5^1} = 5^2$, or $\frac{125}{5} = 25$.

Brackets: When we multiply indices, we take a "power of a power".

$$a^{mn} = \left(a^m\right)^n$$

Example (3): $3^{(2\times3)} = \left(3^2\right)^3 = 3^6$, or $9^3 = 729$.

Negative Indices.

Any number raised to a negative power is the reciprocal of the same number raised to the corresponding positive power.

$$a^{-m} = \frac{1}{a^m}$$

Example (4): $2^{-3} = \frac{1}{2^3}$, or $\frac{1}{8}$.

$$\left(\frac{1}{4}\right)^{-2} = 4^2 \text{ or } 16$$

Here we convert the expression with a negative power to an expression with a positive power by turning the original fraction upside down.

$$\left(\frac{2}{5}\right)^{-2} = \left(\frac{5}{2}\right)^{2} \text{, or } \frac{25}{4}.$$

Zero index: Any positive number raised to the zero power is equal to 1.

$$a^0 = 1$$

Example (5): $10^0 = 1$

Power of one: Any positive number raised to the power of 1 is simply the number itself.

$$a^1 = a$$

Example (6): $8^1 = 8$

One to any power: The number 1 raised to any power is equal to 1.

$$1^n = 1$$

Example (7): $1^{12} = 1$

Fractional Indices – Roots.

Any positive number raised to the reciprocal of a power *m* is equivalent to the m^{th} root of that number.

$$a^{\frac{1}{m}} = \sqrt[m]{a}$$

Examples (8): $16^{\frac{1}{2}} = \sqrt{16}$ or 4.

$125^{\frac{1}{3}} = \sqrt[3]{125}$ or 5.

Harder Fractional and Negative Indices.

$a^{\frac{m}{n}}$ is the same as $a^{\frac{1}{n}}$ raised to the power of *m*, as the bracket laws ("power of a power") show.

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

Examples (9): $64^{\frac{5}{6}}$ is the same as $(64^{\frac{1}{6}})^5$ - here it is 2^5 or 32.

$$\left(\frac{9}{16}\right)^{\frac{3}{2}} = \left(\left(\frac{9}{16}\right)^{\frac{1}{2}}\right)^3 = \sqrt{\left(\frac{9}{16}\right)}^3 = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$$

$a^{-\frac{m}{n}}$ is the same as the reciprocal of $a^{\frac{1}{n}}$ raised to the power of *m*, as the bracket and reciprocal laws above show.

$$a^{-\frac{m}{n}} = \left(\frac{1}{a}\right)^{\frac{m}{n}} = \left(\left(\frac{1}{a}\right)^{\frac{1}{n}}\right)^m$$

Examples (10):

$$100^{-\frac{1}{2}} = \left(\frac{1}{100}\right)^{\frac{1}{2}} = \sqrt{\frac{1}{100}} = \frac{1}{10};$$

$$81^{-\frac{3}{4}} = \left(\frac{1}{81}\right)^{\frac{3}{4}} = \left(\left(\frac{1}{81}\right)^{\frac{1}{4}}\right)^3 = \left(\sqrt[4]{\frac{1}{81}}\right)^3 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

$$\left(\frac{125}{64}\right)^{-\frac{2}{3}} = \left(\frac{64}{125}\right)^{\frac{2}{3}} = \left(\sqrt[3]{\frac{64}{125}}\right)^2 = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$$

Some questions will ask for variations on the above rules, such as combining regular and index arithmetic.

Examples (11): Simplify i) $2x^{\frac{3}{2}} \times 3\sqrt{x}$ and ii) $18x^{-3} \div 2x^2$.

In i) we multiply the numbers and add the indices:

$$2x^{\frac{3}{2}} \times 3\sqrt{x} = 2 \times 3 \times x^{\frac{3}{2}} \times x^{\frac{1}{2}} = 6x^2.$$

In ii) we divide the numbers and subtract the indices:

$$\left(\frac{18x^{-3}}{2x^2}\right) = \left(\frac{18}{2}\right) \times \left(\frac{x^{-3}}{x^2}\right) = 9x^{-5} \text{ or } \frac{9}{x^5}$$

Other questions might ask for an unknown index or base:

Examples (12): Solve i) $2^x = \frac{1}{32}$ and ii) $x^{-½} = 7$.

In i) we know that 32 = 2^5, and using $2^{-m} = \frac{1}{2^m}$, we deduce that x = -5.

In ii), squaring both sides gives $x^{-1} = 49$ and hence $x = \frac{1}{49}$.

Still other questions might have a mixture of bases:

Examples (13): Simplify i) $9^{3\frac{1}{2}} \times 3^3$ ii) $\frac{25^2}{5^{-3}}$ and iii) $\frac{16^2}{4^{1\frac{1}{2}} \times 8^{-\frac{1}{3}}}$

The first step when dealing with this type of question is to reduce all terms in the expression *to the same base*. For example, $2^3 \times 4^2$ is definitely not 8^5 !

In i) we use 9 =3^2 to rewrite the expression as $\left(3^2\right)^{3\frac{1}{2}} \times 3^3 = 3^7 \times 3^3 = 3^{10}$.

In ii), we use 25 =5^2 : $\frac{\left(5^2\right)^2}{5^{-3}} = \frac{5^4}{5^{-3}} = 5^7$.

In iii), we use 4 = 2^2 , 8 =2^3 and 16 = 2^4 :

$$\frac{16^2}{4^{1\frac{1}{2}} \times 8^{-\frac{1}{3}}} = \frac{\left(2^4\right)^2}{\left(2^2\right)^{1\frac{1}{2}} \times \left(2^3\right)^{-\frac{1}{3}}} = \frac{2^8}{2^3 \times 2^{-1}} = 2^6.$$

SURDS (Revision)

A **surd** is a mathematical expression that includes irrational roots in it, usually square roots.
The following rules apply to surds, providing we restrict ourselves to real square roots, i.e. those of positive numbers.

1) When *a* and *b* are both positive, $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$; $\sqrt{a}\times b=\sqrt{ab^2}$.

Examples (14) : Express the following as the square root of a single number:

i) $\sqrt{2}\times\sqrt{3}$; ii) $\sqrt{2}\times\sqrt{8}$

i) Since $2\times3=6$, $\sqrt{2}\times\sqrt{3}=\sqrt{6}$.
ii) Similarly $\sqrt{2}\times\sqrt{8}=\sqrt{16}=4$. Here, two surds produce an integer when multiplied

Example (15) : Express $\sqrt{2}\times3$ as the square root of a single number:

$\sqrt{2}\times3=\sqrt{2\times3^2}=\sqrt{18}$.

We must square the 3 before enclosing it in the square root sign; $\sqrt{2}\times3$ is definitely not $\sqrt{6}$.

We could alternatively re-express 3 as $\sqrt{9}$ and treat the sum as $\sqrt{2}\times\sqrt{9}=\sqrt{18}$.

Examples (16) : Express the following in the form $a\sqrt{b}$ where *a* and *b* are integers and *a* is as large as possible: i) $\sqrt{48}$; ii) $\sqrt{50}$; iii) $2\sqrt{45}$

This time, it is a matter of finding the largest perfect square factor of the number inside the square root sign.

i) Of the factors of 48, the largest perfect square is 16, so we can re-express as follows :

$\sqrt{48}=\sqrt{16\times3}=\sqrt{16}\sqrt{3}=4\sqrt{3}$

ii) The largest square factor of 50 is 25, so $\sqrt{50}=\sqrt{25\times2}=\sqrt{25}\sqrt{2}=5\sqrt{2}$.

iii) The expression $2\sqrt{45}$ can be simplified further because 45 has a factor of 9:

$2\sqrt{45}=2\sqrt{9\times5}=2\sqrt{9}\sqrt{5}=6\sqrt{5}$.

2) When *a* and *b* are both positive, $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$; $\frac{\sqrt{a}}{b} = \sqrt{\frac{a}{b^2}}$; $\frac{a}{\sqrt{b}} = \sqrt{\frac{a^2}{b}}$.

Examples (17): Express the following as the square root of a single number:

i) $\frac{\sqrt{15}}{\sqrt{5}}$; ii) $\frac{\sqrt{27}}{\sqrt{3}}$

i) We replace the two separate square root signs with a single one around the whole fraction:
$\frac{\sqrt{15}}{\sqrt{5}} = \sqrt{\frac{15}{5}} = \sqrt{3}$;

ii) We do the same here: $\frac{\sqrt{27}}{\sqrt{3}} = \sqrt{\frac{27}{3}} = \sqrt{9} = 3$. Again the result comes out as an integer.

Example (18): Express i) $\frac{\sqrt{20}}{2}$ as the square root of a single number.

$\frac{\sqrt{20}}{2} = \sqrt{\frac{20}{4}} = \sqrt{5}$

Here, the 2 must first be squared before enclosing it in the square root sign;

$\frac{\sqrt{20}}{2}$ is definitely not $\sqrt{10}$. (This case often crops up when solving quadratic equations).

We could alternatively re-express 2 as $\sqrt{4}$ and treat the sum as

$\frac{\sqrt{20}}{2} = \frac{\sqrt{20}}{\sqrt{4}} = \sqrt{\frac{20}{4}} = \sqrt{5}$.

Again, we could "spot the square factor" as in Examples (16); $\frac{\sqrt{20}}{2} = \frac{\sqrt{4\times 5}}{2} = \frac{2\sqrt{5}}{2} = \sqrt{5}$.

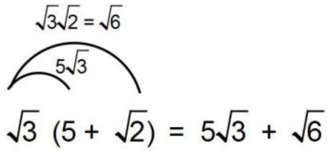
Expansion of surd expressions.

Surd expressions can be expanded like other algebraic expressions.

Examples (19): Expand and simplify the following :

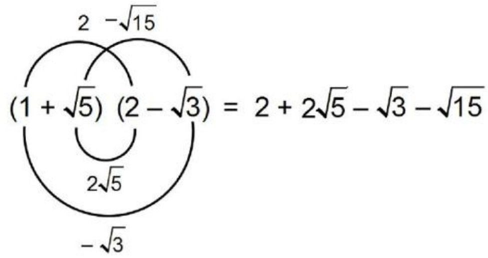
i) $\sqrt{3}(5+\sqrt{2})$; ii) $(1+\sqrt{5})(2-\sqrt{3})$; iii) $(1+\sqrt{2})(3+\sqrt{8})$; iv) $(5+\sqrt{2})^2$; v) $(4-\sqrt{7})^2$

i) $\sqrt{3}(5+\sqrt{2}) = 5\sqrt{3}+\sqrt{6}$.



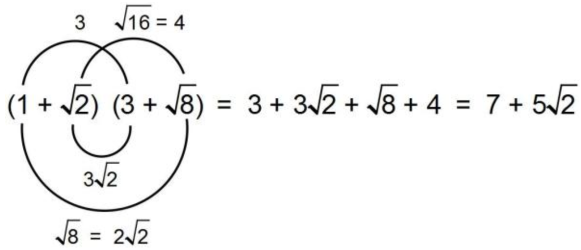
ii) $(1+\sqrt{5})(2-\sqrt{3}) = 2+2\sqrt{5}-\sqrt{3}-\sqrt{15}$.

Notice that the sum of the two middle terms could not be simplified by collecting.



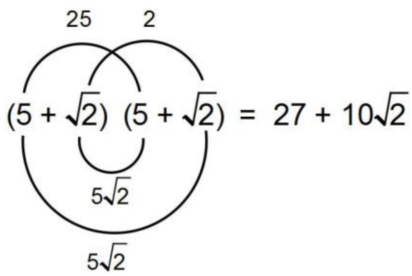
iii) $(1+\sqrt{2})(3+\sqrt{8}) = 7+5\sqrt{2}$.

This time, it is possible to simplify the final result by various manipulations, as shown in the diagram.



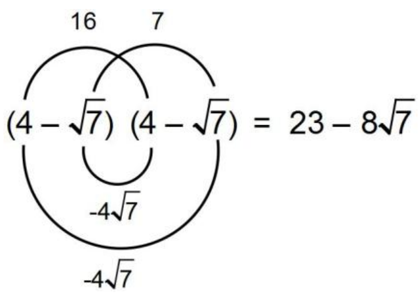
iv) $(5+\sqrt{2})^2 = 27+10\sqrt{2}$.

This perfect square working is analogous to the quadratic result $(a+b)^2 = a^2 + 2ab + b^2$.
Substituting $\sqrt{b}$ for b,
$(a+\sqrt{b})^2 = a^2 + 2a\sqrt{b} + b$



v) $(4-\sqrt{7})^2 = 23-8\sqrt{7}$.

This is related to the quadratic result $(a-b)^2 = a^2 - 2ab + b^2$.

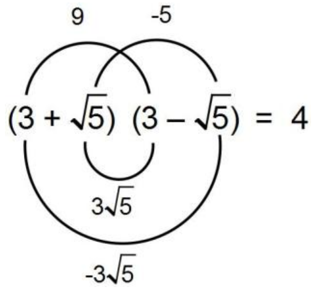


Substituting $\sqrt{b}$ for *b*,

$$\left(a-\sqrt{b}\right)^2 = a^2 - 2a\sqrt{b} + b$$

Example (20): Expand and simplify the expression $(3+\sqrt{5})(3-\sqrt{5})$.

$$(3+\sqrt{5})(3-\sqrt{5}) = 4$$



This surprisingly simple result is connected to the quadratic "difference of two squares" result

$(a+b)(a-b) = a^2 - b^2$

Substituting $\sqrt{b}$ for *b*,

$$\left(a+\sqrt{b}\right)\left(a-\sqrt{b}\right) = a^2 - b$$

Notice how this manipulation eliminates the surd from the expression, by using the difference of squares. This is of particular importance in Examples (22) below.

It is often desirable to manipulate a fractional surd expression so as to make the denominator a rational number. This is termed **rationalising the denominator**.

Examples (21): Rationalise the denominator in the expressions:

i) $\dfrac{1}{\sqrt{2}}$; ii) $\dfrac{2+\sqrt{2}}{3\sqrt{3}}$

i) Multiplying the top and bottom by $\sqrt{2}$ gives the result $\dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$.

ii) Here we multiply the top and bottom by $\sqrt{3}$: $\dfrac{2+\sqrt{2}}{3\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{2\sqrt{3}+\sqrt{6}}{9}$.

The previous examples showed how to rationalise a simple surd denominator.

When the denominator is of the form $a \pm \sqrt{b}$, we can use the "difference of two squares" method $\left(a+\sqrt{b}\right)\left(a-\sqrt{b}\right) = a^2 - b$ to rationalise it.

Examples (22): Rationalise the denominator in the expressions: i) $\dfrac{1}{2+\sqrt{3}}$; ii) $\dfrac{3-\sqrt{7}}{4-\sqrt{7}}$

i) The denominator is $2+\sqrt{3}$, so we rationalise it by multiplying top and bottom by $2-\sqrt{3}$.

(Recall Example (20)).

$$\frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{2-\sqrt{3}}{4-3} = \frac{2-\sqrt{3}}{1} = 2-\sqrt{3}.$$

ii) We need to multiply top and bottom by $4+\sqrt{7}$ to rationalise the denominator

$$\frac{3-\sqrt{7}}{4-\sqrt{7}}\times\frac{4+\sqrt{7}}{4+\sqrt{7}}=\frac{(3-\sqrt{7})(4+\sqrt{7})}{9} = \frac{12-4\sqrt{7}+3\sqrt{7}-7}{9} = \frac{5-\sqrt{7}}{9}$$

Examples (23): If $\phi = \frac{\sqrt{5}+1}{2}$, show (using surd manipulation) that i) $\phi^2 = \phi + 1$; ii) $1/\phi = \phi - 1$.

i) $\frac{\sqrt{5}+1}{2}\times\frac{\sqrt{5}+1}{2} = \frac{5+2\sqrt{5}+1}{4} = \frac{2\sqrt{5}+6}{4} = \frac{\sqrt{5}+3}{2}$

$= \frac{\sqrt{5}+1}{2}+\frac{2}{2} = \boldsymbol{\phi + 1}$ **.**

ii) $\frac{2}{\sqrt{5}+1}\times\frac{\sqrt{5}-1}{\sqrt{5}-1}=\frac{2\sqrt{5}-2}{5-1}=\frac{\sqrt{5}-1}{2}=\frac{\sqrt{5}+1}{2}-\frac{2}{2} = \boldsymbol{\phi - 1}$ **.**

Other surd arithmetic.

Sums and differences of surds **cannot** be simplified like products or quotients:

$\sqrt{a+b} \neq \sqrt{a}+\sqrt{b}$, as the counterexample below will show.

$\sqrt{9+16}=\sqrt{25}=5$; $\sqrt{9}+\sqrt{16}=3+4=7$.

It can still be possible to simplify certain expressions by using the surd laws, however.

The trick here (see Examples (16)) is to look for the largest perfect square factor in the surd expressions, so as to factor out the resulting square root.

Examples (24): Express the following in the form $a + b\sqrt{c}$ where a, b and c are integers

i) $\sqrt{2}+\sqrt{32}$; ii) $\sqrt{75}-\sqrt{12}$

i) $\sqrt{2}+\sqrt{32} = \sqrt{2}+\sqrt{16}\sqrt{2} = \sqrt{2}+4\sqrt{2} = 5\sqrt{2}$.

Because 16 is the largest square factor of 32,, we can say $\sqrt{32}=\sqrt{16}\sqrt{2}$ and hence $\sqrt{32}=4\sqrt{2}$.
We finish by adding the two multiples of √2 using normal algebra.
(Such a case had already cropped up in Example 19(iii)).

ii) $\sqrt{75}-\sqrt{12} = \sqrt{25}\sqrt{3}-\sqrt{4}\sqrt{3} = 5\sqrt{3}-2\sqrt{3} = 3\sqrt{3}$

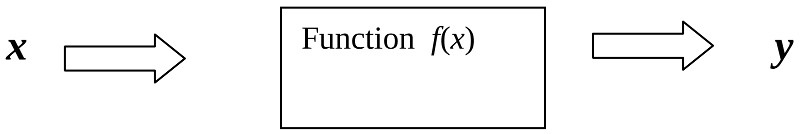
Again, we spot that 25 and 4 are the highest square factors of 75 and 12, respectively. We finish by subtracting multiples of √3.

FUNCTIONS – an introduction.

Much of the coursework in AS and A-Level mathematics involves understanding **functions**.

A function can be thought of as being a rule which takes each member x of a set of values, performs a mathematical operation on it, and then assigns it to a value y.

In short, a function **maps** each value of x to an **image** value of y.



Functions are usually denoted by the letters f, g and h.

A function which squares a number x and then adds 5 to the result can therefore be expressed as

$y = x^2 + 5$ (if the result is assigned the variable y)

$f(x) = x^2 + 5$

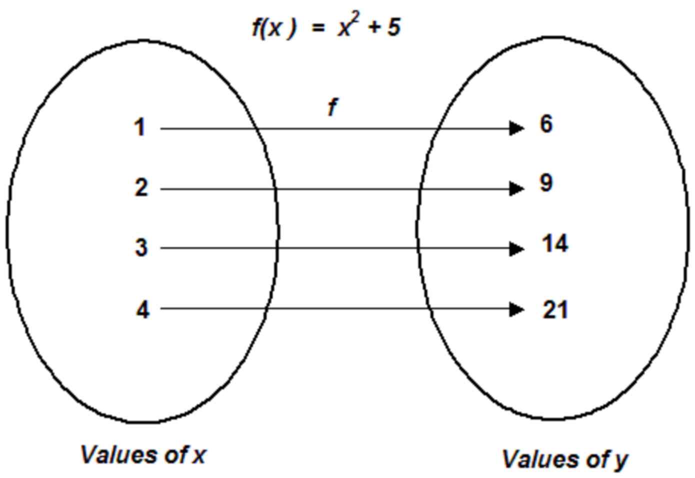
$f: x \mapsto x^2 + 5$

A function can also be illustrated by means of a **mapping diagram**.

The diagram shows how a few values of x are passed through the "function machine" and the results assigned to y.

Here, for example $f(4)= 21$.

The drawback of mapping diagrams is that they can only show a limited number of values, whereas the function could be applicable to *all* values of x.



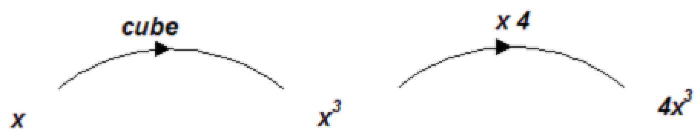
Our choices of (x,y) values in the diagram was purely arbitrary. We could have chosen $f(0) = 5$, $f(-10) = 105$, $f(\sqrt{2}) = 7$, or any others from an infinite number of choices.

Functions can either be single, such as $f(x) = x^2$ ("squaring ") and $g(x) = 2x$ ("doubling"), or they can be composite, such as $h(x) = 3x^2 - 4$.

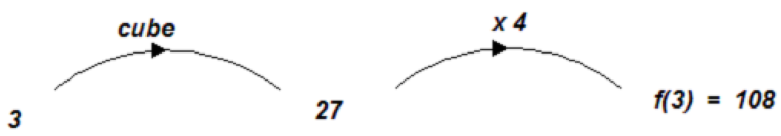
There will be more on the algebra of functions at A2 Level - this overview is just a familiarisation. The form most commonly used in the course documents is (for example) $f(x) = x^2 - 3$.

Example (25) : Describe the function $f(x) = 4x^3$ in terms of mathematical operations. What is $f(3)$?

The function can be seen as a two-step process; first cube a number x and then multiply the result by 4.

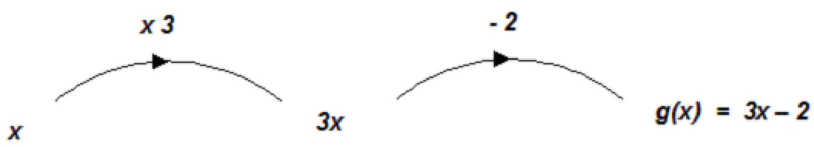


From this, we can deduce that $f(3) = 4 \times 3^3 = 108$.



Example (26) : A function g maps x to y by the following rule; multiply x by 3 and subtract 2 from the result. Express this in function notation, and write out $g(3)$.

The function can be denoted either by
$g(x) = 3x - 2$ or $g: x \mapsto 3x - 2$



Substituting $x = 3$ will give $g(3) = 7$ (multiply 3 by 3 and subtract 2).