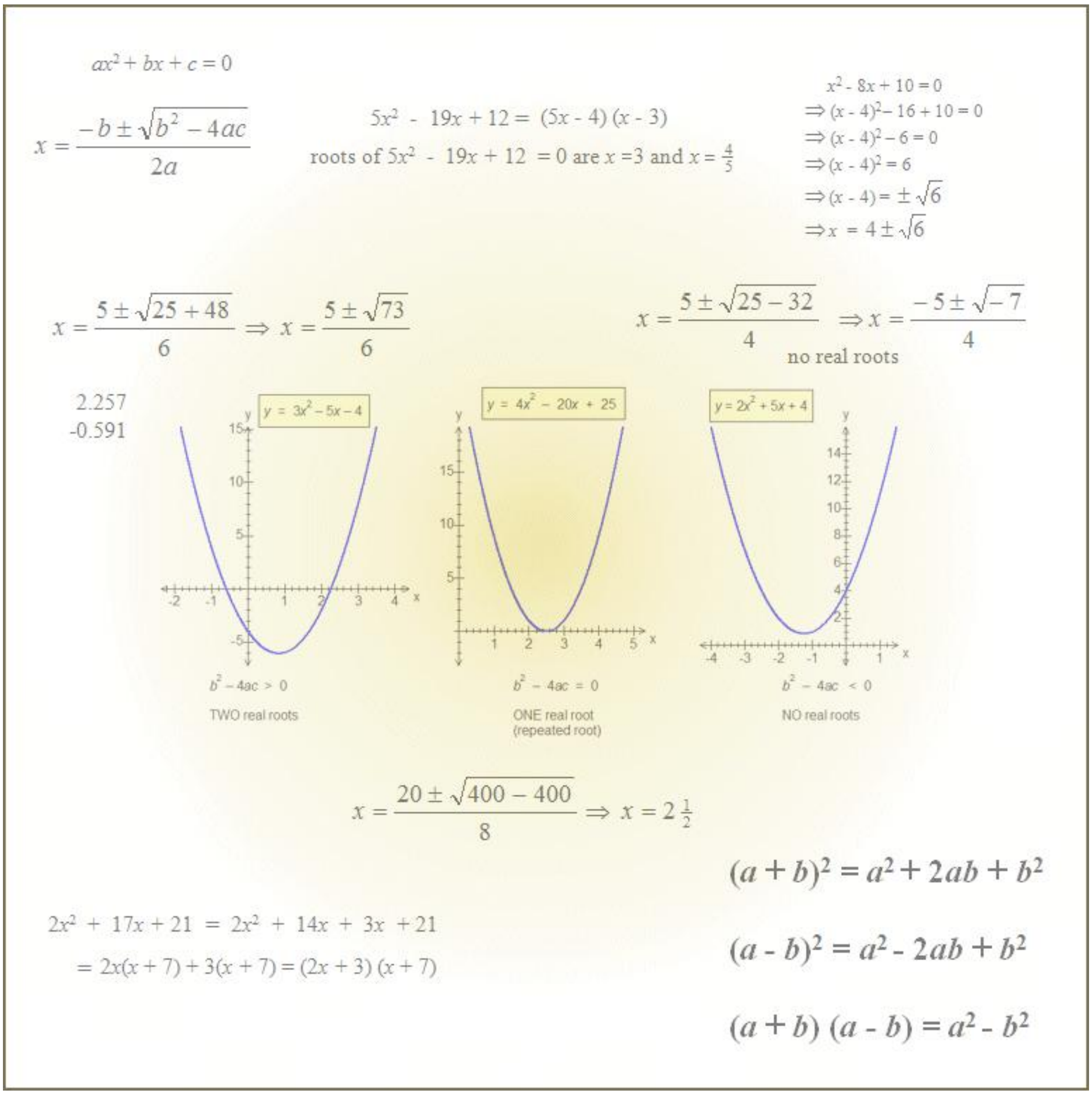
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 1 / AS

QUADRATIC EQUATIONS



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QUADRATIC EQUATIONS

A **quadratic** expression is one of the form $ax^2 + bx + c$ where a, b and c are constants, and a is not zero. The highest power of x is 2 (the square of x). The values a and b are the **coefficients** of x^2 and x, and c is the constant term.

Remember the technique of expansion of brackets. Each term in the first bracket is multiplied by each term in the second bracket, and like terms collected.

Example (1)

$\mathbf{(x + 2)(x + 4)} = x(x + 4) + 2(x + 4)$
$= x^2 + 4x + 2x + 8$ (expand)
$= x^2 + 6x + 8$ (collect)

Also recollect the following special useful results:

$$\boldsymbol{(a + b)^2 = a^2 + 2ab + b^2}$$

$$\boldsymbol{(a - b)^2 = a^2 - 2ab + b^2}$$

$$\boldsymbol{(a + b)(a - b) = a^2 - b^2}$$ This is the 'difference of two squares' result.

Quadratic Equations.

Equations of the form $ax^2 + bx + c = 0$, where a is non-zero, are called quadratic equations.
Such equations can be solved in various ways: factorisation, completing the square, or using the general formula.

Solving Quadratics by Factorisation.

We shall begin with equations of the form $x^2 + bx + c = 0$, where the coefficient of x is unity, i.e. 1.

Example (2). Solve the equation $x^2 = 7x$.

We might be tempted to divide both sides by x to give $x = 7$, but that does not give the complete result. True, $7^2 = 7 \times 7 = 49$, but $x = 0$ is also a solution, since $0^2 = 0 \times 0 = 0$. By dividing by x, we have 'lost' a possible solution of the equation.

The correct procedure is to re-express the equation as $x^2 - 7x = 0$.

Example (2a): Factorise $x^2 - 7x$ and hence solve $x^2 - 7x = 0$.

$x^2 - 7x = 0$ can be rewritten as $x(x - 7) = 0$.
$x^2 - 7x$ is equal to 0 if and only if $x = 0$, or $x - 7 = 0$.
The solutions of the quadratic are thus $x = 0$ and $x = 7$.
They are also known as the **roots** of the equation.

The last case was easy enough, as we were able to factor out x.

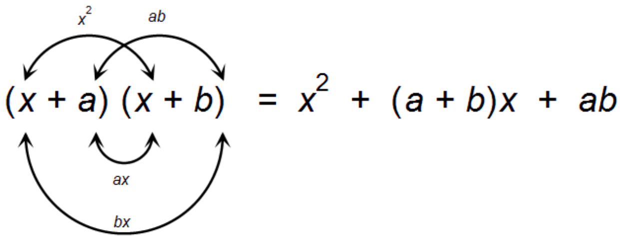
In general, any quadratic equation of the form $x^2 + bx = 0$ (i.e. having no constant term) can be solved by factoring out x to give $x(x + b) = 0$, and hence giving us the solutions of $x = 0$ and $x = -b$.

Example (3a): Case where all the coefficients are positive

Factorise $x^2 + 7x + 10$ and hence find the roots of $x^2 + 7x + 10 = 0$.

To begin to factorise such an expression, let’s look at the general case:

When factorised, the expression would be of the form $(x+a)(x+b)$ where a and b must be found.



$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

Looking at the expansion, we can see that the coefficient of the term in x is the sum of a and b, and the constant term is the product of a and b.

Therefore to factorise $x^2 + 7x + 10$, we must find two numbers a and b that add up to the coefficient of x (here 7) and multiply to give the constant term (here 10). Such a pair of numbers is $a = 5$, $b = 2$.

$x^2 + 7x + 10 = (x+5)(x+2)$.

The solutions of the equation are the values of x which make one of the bracketed terms equal to zero. In this case they are $x = -5$ and $x = -2$.

Example (3b): Factorise $x^2 + 9x + 20$ and hence find the solutions of $x^2 + 9x + 20 = 0$.
Two numbers with a product of 20 and a sum of 9 are 5 and 4, so

$x^2 + 9x + 20 = (x+5)(x+4)$.

The corresponding solutions of $x^2 + 9x + 20 = 0$ are hence $x = -5$ and $x = -4$.

The above method can be used to try and factorise any expression of the form $x^2 + bx + c = 0$, namely **where the coefficient of x^2 is unity.** It also works when b and c are negative.

Example (4a): **Case where the coefficient of x is negative, but the constant term is positive**

Factorise $x^2 - 13x + 40$ and hence solve $x^2 - 13x + 40 = 0$.

Two negative numbers have a positive product but a negative sum, and therefore the factorised form of this expression will be of the form $(x - a)(x - b)$ where a and b are to be determined.

We still look for two numbers whose sum is 13 and whose product is 40.
Such a pair of numbers is $a = 5$ and $b = 8$, but the factorised form of $x^2 - 13x + 40$ is not now $(x+5)(x+8)$, but $(x-5)(x-8)$.

(Note that the sum of -5 and -8 is -13, and their product is 40).

So, $x^2 - 13x + 40 = (x-5)(x-8)$, and hence the roots of $x^2 - 13x + 40 = 0$ are $x = 5$ and $x = 8$.

Example (4b): Factorise $x^2 - 7x + 12$ and hence solve $x^2 - 7x + 12 = 0$.

Two numbers with a product of 12 and a sum of 7 are 3 and 4, so

$x^2 - 7x + 12 = (x - 3)(x - 4)$.

The corresponding solutions of $x^2 - 7x + 12 = 0$ are hence $x = 3$ and $x = 4$.

Example (5a): **Case where the constant term is negative**

Factorise $x^2 - 2x - 15$ and hence solve $x^2 - 2x - 8 = 7$.

Note how the constant term in the expression $x^2 - 2x - 15$ is now negative. The factorised version will therefore be of the form $(x + a)(x – b)$ with one positive solution and one negative one. This is because two numbers of different signs give a negative product.

When looking for the number-pair here, we are now looking for a **difference** of 2 and a product of 15. Two numbers with a difference of 2 are 5 and 3, but we also need to know which one of them goes in the $(x + a)$ term, and which one in the $(x – b)$ term.

If we were to expand $(x + 5)(x - 3)$, then the expansion would be $x^2 + 2x – 15$, not $x^2 - 2x – 15$.
We look at the coefficient of x in the expression $x^2 - 2x – 15$ and see that it is negative.
Therefore, the larger number (5) goes with the term with the (-) sign and the smaller number (3) goes with the term with the (+) sign.

Hence $x^2 - 2x - 15 = (x + 3)(x - 5)$

The equation $x^2 - 2x - 8 = 7$ might look unrelated to the earlier part of the question at first.

What we must not do is to try and factorise the LHS as $(x + 2)(x – 4)$ and solve $x + 2 = 7$ or $x - 4 = 7$.
This is completely wrong !

The correct thing to do is to subtract 7 from both sides, to obtain the equation $x^2 - 2x – 15 = 0$, with the important zero on the RHS.

From the earlier factorisation, the solutions of $x^2 - 2x - 15 = 0$ are $x = 5$ and $x = -3$.

Example (5b): Factorise $x^2 + 3x - 28$ and hence solve $x^2 + 3x - 28 = 0$.

Two numbers with a product of 28 and a difference of 3 are 7 and 4.
The x-coefficient is positive, so the expression factorises to $(x + 7)(x - 4)$
(The larger number of 7 goes with the + sign in the brackets)

The corresponding solutions of $x^2 + 3x - 28 = 0$ are $x = -7$ and $x = 4$.

Example (6): Factorise $x^2 + 12x + 36$ and hence solve $x^2 + 12x + 36 = 0$.

Two numbers with a product of 36 and a sum of 12 are 6 and 6, so
$x^2 + 12x + 36 = (x+6)(x+6) = (x+6)^2$.

This time, there is only one solution – that of $x = -6$, sometimes termed a repeated root or coincident root.

The expression $x^2 + 12x + 36$ is in fact a perfect square of the form $a^2 + 2ab + b^2$
where $a = x$ and $b = 6$. The square of 6 (36) in the constant term and twice 6 (12) in the x-term can be seen by inspection.

Example (7): Factorise $x^2 - 18x + 81$ and hence solve $x^2 - 18x + 81 = 0$.

We have another perfect square here, this time of the form $a^2 - 2ab + b^2$ where $a = x$ and $b = 9$.
The square of 9 (81) and its multiple of 2 (18) can be seen by inspection.

$x^2 - 18x + 81$ is therefore the same as $(x - 9)^2$, and so the root of $x^2 - 18x + 81 = 0$ is $x = 9$; again, this is a unique solution.

Example (8): Factorise $x^2 – 25$ and hence solve $x^2 - 25 = 0$.

Here we can recognise the ‘difference of two squares’ form :

$a^2 - b^2 = (a + b)(a - b)$ where $a = x$ and $b = 5$.
Factorising gives $(x + 5)(x - 5) = 0$, with solutions of $x = 5$ and $x = -5$.
For brevity, we could say the solutions are $x = \pm 5$.

Factorisation of Quadratics where the coefficient of x^2 is not unity.

The previous examples of quadratic equations to be factorised were all of the type $x^2 + bx + c = 0$, i.e. where the coefficient of x^2 was equal to 1.

When the equation takes the form $ax^2 + bx + c = 0$, the coefficient of x^2 is not necessarily equal to 1. Here, factorisation is slightly harder, involving a little more trial and error, but the general principles still hold.

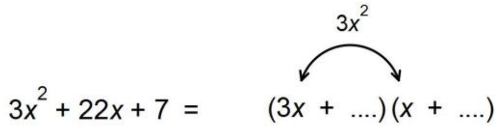
Example (9): Factorise $2x^2 - 5x$ and hence solve $2x^2 - 5x = 0$.

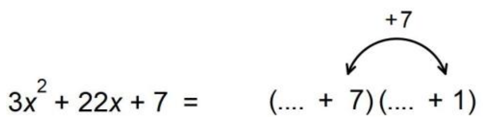
$2x^2 - 5x = 0$ can be rewritten as $x(2x - 5) = 0$.
This is true if and only if $x = 0$, or $2x - 5 = 0$.
The roots of the quadratic are thus $x = 0$ and $x = 2.5$.

Example (10): Factorise $3x^2 + 22x + 7$ and hence solve $3x^2 + 22x + 7 = 0$.

By inspecting the expression, it is clear that by looking at the coefficient of x^2, the factorised expression would take the form (3x + 'thing') (x + 'thing'), as only $3x$ and x can give a product of $3x^2$.

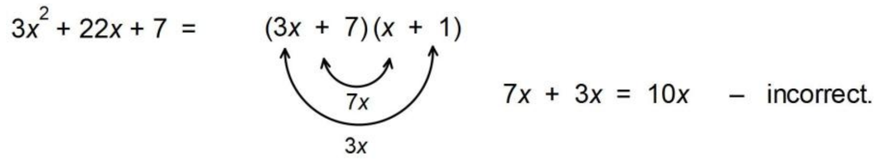
Similarly, by looking at the constant coefficient, the factorised expression would take the form ('thing' + 7) ('thing' + 1) as only 7 and 1 can give a product of 7.

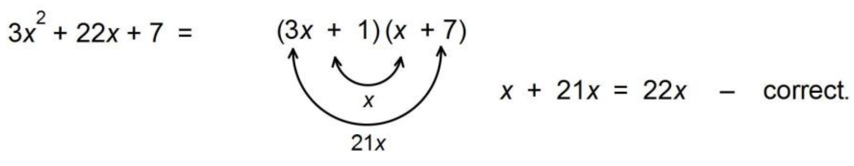




The only possible factorised forms are therefore $(3x + 7)(x + 1)$ and $(3x + 1)(x + 7)$.

To find the correct one, we multiply out the terms which contribute to the x-coefficient.





With $(3x + 7)(x + 1)$, we obtain an x-coefficient of 3 + 7 or 10, so this is incorrect.
With $(3x + 1)(x + 7)$, we obtain an x-coefficient of 21 + 1 or 22, which is the required one.

Therefore, $3x^2 + 22x + 7 = (3x + 1)(x + 7)$

The roots of $3x^2 + 22x + 7 = 0$ are $x = -\frac{1}{3}$ and $x = -7$.

Example (11): Factorise $2x^2 + 17x + 21$ and hence solve $2x^2 + 17x + 21 = 0$.

The factorised expression would take the form (2x + 'thing') (x + 'thing'), as only 2x and x can give a product of $2x^2$.

When it comes to the constant coefficient, we find that 21 is not a prime number, and so we must 'try' factor pairs of that number, namely 1 and 21, or 3 and 7 .

In other words, the factorised expression could take either the form ('thing' + 1) ('thing' + 21), or it could take the form ('thing' + 3) ('thing' + 7) .

We have four possible combinations to test here:

$(2x + 1)(x + 21)$; the coefficient of x adds up to 42 + 1 = 43 and not 17 - reject.

$(2x + 21)(x + 1)$; again the coefficient of x adds up to 2 + 21 = 23 – also reject.

$(2x + 7)(x + 3)$; the resulting x-coefficient is 6 + 7 = 13 - reject.

$(2x + 3)(x + 7)$; this time the x-coefficient works out as 14 + 3 = 17, so this is the required pairing.

Therefore, $2x^2 + 17x + 21 = (2x + 3)(x + 7)$

The roots of $2x^2 + 17x + 21 = 0$ are $x = -1\frac{1}{2}$ and $x = -7$.

Examples (10) and (11) had all-positive coefficients, but the same method can be used when the coefficient of x is negative and the constant term is positive.

Example (12): Factorise $5x^2 - 19x + 12$ and hence solve $5x^2 - 19x + 12 = 0$

Inspection of the x^2 coefficient leads us to a factorised expression which takes the form (5x - 'thing') (x - 'thing'), but note the subtraction signs here (see Example 4).

The constant coefficient is not prime, and so we must 'try' factor pairs of 12 , and test expressions of the form ('thing' - 1) ('thing' - 12), ('thing' - 2) ('thing' - 6) and ('thing' - 3) ('thing' – 4).

The number of combinations to test here has now increased to six:
$(5x - 1)(x - 12)$; $(5x - 12)(x - 1)$; $(5x - 2)(x - 6)$; $(5x - 6)(x - 2)$; $(5x - 3)(x - 4)$; $(5x - 4)(x - 3)$.

Trial and error gives the correct factorised form of $(5x - 4)(x - 3)$.

$\therefore$ the roots of $5x^2 - 19x + 12 = 0$ are $x = 3$ and $x = \frac{4}{5}$.

When the constant term is negative, then we need to 'double up' on the number of test cases.

Example (13): Factorise $2x^2 - 3x - 5$ and hence solve $2x^2 - 3x - 5 = 0$.

This time, the factorised expression will take the form (2x + 'thing') (x - 'thing'),
or it could be (2x - 'thing') (x + 'thing'), by going off the '$2x^2$' part.

By going off the constant term, the expression will take the form ('thing' + 1) ('thing' - 5) or perhaps ('thing' - 1) ('thing' + 5)

The combinations that need testing are $(2x - 1)(x + 5)$; $(2x + 1)(x - 5)$; $(2x - 5)(x + 1)$; $(2x + 5)(x - 1)$.

Inspection and expansion gives $2x^2 - 3x - 5 = (2x - 5)(x + 1)$.

Note: if we had tried $(2x + 5)(x - 1)$, we would have ended up with $2x^2 + 3x - 5$, where only the sign of the x-coefficient was wrong. In that case, the *numbers* would have been correctly matched, but the *signs* would have needed to be exchanged, to give the correct $(2x - 5)(x + 1)$.

Thus the solutions of $2x^2 - 3x - 5 = 0$ are $x = 2\frac{1}{2}$ and $x = -1$.

Sometimes, it is possible to reject certain combinations of 'trial' factors, as the next example shows:

Example (14): Factorise $2x^2 - 7x - 4$ and hence solve $2x^2 - 7x - 4 = 0$.

We can see that $(2x \pm 2)$ or $(2x \pm 4)$ cannot be possible factors, because those terms have a common factor of 2 whereas the original quadratic is in its lowest terms.

We only need to consider the combinations of $(2x - 1)(x + 4)$ and $(2x + 1)(x - 4)$. Inspection and expansion gives $2x^2 - 7x - 4 = (2x + 1)(x - 4)$ and hence the solutions of $x = -\frac{1}{2}$ and $x = 4$.

The roots of any non-zero constant multiple of a quadratic equation are always the same as those of the original equation.

Example (15): Factorise $4x^2 + 12x + 5$ and hence solve $4x^2 + 12x + 5 = 0$.

Inspecting the x^2 coefficient, we have a factorised expression ($4x$ + 'thing') (x + 'thing'), or perhaps ($2x$ + 'thing') ($2x$ + 'thing').

Similarly, by inspecting the constant term. we have a factorised expression ('thing' + 1) ('thing' + 5).

We must therefore test the combinations $(4x + 1)(x + 5)$; $(4x + 5)(x + 1)$; $(2x + 5)(2x + 1)$.

Inspection gives $4x^2 + 12x + 5 = (2x + 5)(2x + 1)$.

Thus the solutions of $4x^2 + 12x + 5 = 0$ are $x = -2\frac{1}{2}$ and $x = -\frac{1}{2}$.

Example (16): Factorise $16x^2 + 40x + 25$ and hence solve $16x^2 + 40x + 25 = 0$.

By inspection, $16x^2$ is the square of $4x$ and 25 is the square of 5. Also, $2 \times 4x \times 5 = 40x$.

This equation is thus of the form $a^2 + 2ab + b^2$ where $a = 4x$ and $b = 5$.

$16x^2 + 40x + 25$ is therefore $(4x+5)^2$, and so the root of $16x^2 + 40x + 25 = 0$ is $x = -1.25$.

Example (17): Solve $2x^2 - 9 = 0$, leaving the answers as surds.

Here we can recognise the form $a^2 - b^2$ or $(a+b)(a-b)$ where $a = \sqrt{2}(x)$ and $b = 3$.
Factorising gives $(\sqrt{2}(x) + 3)(\sqrt{2}(x) - 3) = 0$, thus $\sqrt{2}(x) = \pm 3$, and hence the roots are

$$x = \pm \frac{3}{\sqrt{2}}.$$

Example (18): Factorise $5 + 3x - 2x^2$ and hence solve $5 + 3x - 2x^2 = 0$.

This expression has a negative coefficient of x^2 and is also stated in ascending, rather than descending powers of x.

We could either try factorising *in situ* by trying combinations $(5 + x)(1 - 2x)$, $(5 - x)(1 + 2x)$, $(5 + 2x)(1 - x)$, $(5 - 2x)(1 + x)$. The fourth one is the correct one, so $5 + 3x - 2x^2 + (5 - 2x)(1 + x)$.
The roots of $5 + 3x - 2x^2 = 0$ are $x = -1$ and $x = 2\frac{1}{2}$.

Or we could multiply throughout by -1 to turn it into an equation where the coefficient of x^2 is positive:

$5 + 3x - 2x^2 = -(2x^2 - 3x - 5)$, the quadratic expression in Example (13).

This quadratic factorises to $(2x - 5)(x + 1)$.
We therefore simply multiply either of the factors of $2x^2 - 3x - 5$ by -1 to 'restore' the original:

$5 + 3x - 2x^2 = (5 - 2x)(x + 1)$.

Thus the solutions of $5 + 3x - 2x^2 = 0$ are $x = 2\frac{1}{2}$ and $x = -1$.

Had the question only asked for the roots and not the factorised form, there would have been no need to multiply back by -1. (See the comment at the end of example 14) .

Factorisation of Quadratics where the coefficient of x^2 is not unity – another approach.

The last few examples of factorisation included examples such as finding the solutions of the quadratic equation $2x^2 + 17x + 21 = 0$.

To solve such an equation, we realised that the factorised form would turn out to be of the form (2x + 'thing') (x + 'thing'), and then proceeded to find combinations of the missing numbers which would have a product of 21, until, by trial and error, we found that $2x^2 + 17x + 21 = (2x + 3)(x + 7)$.

There is another method, based on products and sums, but a little more difficult than that used in solving equations of the form $x^2 + bx + c = 0$.

In Example (3), we factorised the expression $x^2 + 7x + 10$ to give the factors of $(x+5)(x+2)$.

Consider the reverse result, i.e. the expansion of $(x+5)(x+2)$.

By the laws of algebra, $(x+5)(x+2) = x(x+2) + 5(x+2) = x^2 + 2x + 5x + 10 = x^2 + 7x + 10$.

When we were given the expression $x^2 + 7x + 10$ to factorise, we looked for a pair of numbers that had a product of 10 and a sum of 7. Such a pair of numbers was 5 and 2.

What we did was effectively 'split' the term in x to give the expression $x^2 + 2x + 5x + 10$.
Then, the first two terms of the expression, $x^2 + 2x$, were factorised to $x(x+2)$.
The last two terms , $5x + 10$, were similarly factorised to $5(x + 2)$.
Combining the two gave $x(x+2) + 5(x+2)$ or $(x+5)(x+2)$.

In the above example, where the coefficient of x^2 was unity, the method seemed long-winded, but with a small modification, it can help solve equations where the coefficient of x^2 is not unity.

Example (11) Recalled: Factorise $2x^2 + 17x + 21$.

This particular example was solved earlier by trial and error, but in this method, we need to find a way of 'splitting' the term in x to factorise the expression.

We must therefore find two numbers that add to 17, and whose product is *not* simply 21, but the product of the two end terms, namely the x^2 coefficient and the constant, i.e. 2×21 or 42.

Such a pair of numbers is 14 and 3, with a product of 17 and a sum of 42.

We can then split the x-coefficient in the expression to $14x + 3x$ to give $2x^2 + 14x + 3x + 21$
The first two terms, $2x^2 + 14x$, can be factorised to $2x(x + 7)$ and the last two, $3x + 21$, to $3(x + 7)$.

Finally, we use the laws of algebra to obtain $2x(x + 7) + 3(x + 7) = (2x + 3)(x + 7)$.

Example (12) Recalled: Factorise $5x^2 - 19x + 12$.

Here we must find two numbers that add to -19, and whose product is that of the outer terms' coefficients, i.e. 5×12 or 60. Such a pair of numbers is -15 and -4.

The quadratic therefore splits into $5x^2 - 15x - 4x + 12$ or $(5x^2 - 15x) - (4x - 12)$
(watch out for the sign reversal with the brackets !).

Factorisation gives $5x(x - 3)$ for the first bracketed term and $4(x - 3)$ for the second.
Hence the full factorised expression is $(5x - 4)(x - 3)$. (Second term subtracted !)

Note that writing the split the other way would still give the same final result:

$5x^2 - 4x - 15x + 12$ or $(5x^2 - 4x) - (15x - 12)$, factorising to $x(5x - 4) - 3(5x - 4)$
and finally $(x - 3)(5x - 4)$.

Example (19): Factorise $6x^2 - 17x - 45$ and hence solve $6x^2 - 17x - 45 = 0$.

This time, the trial and error method of testing combinations of factors would be too long and tedious, as there are rather a lot of them to check. We therefore use the method from the last two examples.

Here we must find two numbers that add to 17 and whose product is that of the outer terms' coefficients, i.e. 6×-45 or -270. Such a pair of numbers is -27 and 10.

The quadratic therefore splits into $6x^2 - 27x + 10x - 45$ or $(6x^2 - 27x) + (10x - 45)$.

Factorisation gives $3x(2x - 9)$ for the first bracketed term and $5(2x - 9)$ for the second.
Hence the full factorised expression is $(3x + 5)(2x - 9)$.

Solving Quadratics by Completing the Square.

An alternative to solving quadratics by factorisation is by a method known as 'completing the square'.

Recalling the general results:

$(x+a)^2 = x^2 + 2ax + a^2$
$(x-a)^2 = x^2 - 2ax + a^2$

This way, any quadratic equation of the form $x^2 + bx + c = 0$ can be re-expressed as $(x+b)^2 = k$ or $(x-b)^2 = k$ where k is a constant to be determined.

The two general results can thus also be written in the form

$x^2 + 2ax = (x+a)^2 - a^2$
$x^2 - 2ax = (x-a)^2 - a^2$

From those two results, and letting $b = 2a$, a general quadratic equation of the form $x^2 \pm bx + c = 0$ can be redefined as $(x \pm \frac{b}{2})^2 - (\frac{b}{2})^2 + c = 0$.

This looks more complicated than it is - let us take one example :

$(x + 7)^2 = x^2 + 14x + 49$ Notice how twice 7 is 14, and the square of 7 is 49.

We can therefore say that $x^2 + 14x = (x + 7)^2 - 49$.

Notice the pattern here; we take half of 14 to obtain 7, and then square the halved value to get 49.

Example (20a): Express $x^2 + 14x + 46$ in the form $(x + 7)^2 + c$, where c is a constant to be determined.

$x^2 + 14x + 46 = (x + 7)^2 - 49 + 46 = (x + 7)^2 - 3$.

We had used the earlier result to replace the non-constant terms $x^2 - 14x$ with the equivalent expression of $(x + 7)^2 - 49$, and then added on the constant term of 46.

Example (20b): Express $x^2 - 12x + 27$ in the form $(x + b)^2 + c$, where b and c are constants.

We start by examining the first two terms of the expression, namely $x^2 - 12x$.
Now $x^2 - 12x = (x - 6)^2 – 36$, because $(x - 6)^2 = x^2 - 12x + 36$.
Again, half of -12 is -6, and the square of -6 is 36.

Therefore:

$x^2 - 12x + 27 = (x - 6)^2 - 36 + 27 = (x - 6)^2 - 9$.

(The highlighted expressions are again equivalent.)

Example (20c): Solve $x^2 + 10x + 25 = 0$ by completing the square.

Here, the coefficient of x, namely $b = 10$, so halving it gives 5 and squaring the halved value gives 25.

By replacing the '$x^2 + 10x$' part with $(x+5)^2 - 25$, the expression becomes $(x+5)^2 - 25 + 25 = 0$ and finally $(x+5)^2 = 0$.

The expression on the LHS of the equation also happens in this case to be a perfect square.

$\therefore$ $x^2 + 10x + 25 = 0$ is the same as $(x+5)^2 = 0$, and hence the solution of $x^2 + 10x + 25 = 0$ is $x = -5$.

Example (21): Solve $x^2 - 8x + 10 = 0$ by completing the square.

Halving the x-coefficient here gives -4, and squaring the halved value gives 16.

$x^2 - 8x + 10 = 0$
$\Rightarrow (x - 4)^2 - 16 + 10 = 0$ (completing square on LHS)
$\Rightarrow (x - 4)^2 - 6 = 0$
$\Rightarrow (x - 4)^2 = 6$ (bringing constant to RHS)
$\Rightarrow (x - 4) = \pm\sqrt{6}$ (taking square roots)
$\Rightarrow x = 4 \pm \sqrt{6}$ (adding 4 to each side)

The roots of the quadratic are therefore $x = 4 + \sqrt{6}$ and $x = 4 - \sqrt{6}$ (leaving as surds).

Example (22): Find the roots of $x^2 - 5x + 3 = 0$ by completing the square.

This is a little more messy, as the coefficient of x is odd and will give rise to fractions, but the process is the same as in the previous cases.
Half the x-coefficient = $\frac{5}{2}$, and *its* square = $\frac{25}{4}$.

$\Rightarrow x^2 - 5x + 3 = 0$
$\Rightarrow (x - \frac{5}{2})^2 - \frac{25}{4} + 3 = 0$ (completing square on LHS)

$\Rightarrow (x - \frac{5}{2})^2 - \frac{13}{4} = 0$
$\Rightarrow (x - \frac{5}{2})^2 = \frac{13}{4}$ (bringing constant to RHS)
$\Rightarrow (x - \frac{5}{2}) = \pm\sqrt{\frac{13}{4}}$ (taking square roots)
$\Rightarrow x = \frac{5}{2} \pm \sqrt{\frac{13}{4}}$ (adding $\frac{5}{2}$ to each side)

The roots of the quadratic are therefore $x = \frac{5}{2} + \sqrt{\frac{13}{4}}$ and $x = \frac{5}{2} - \sqrt{\frac{13}{4}}$ (leaving as surds).

If the coefficient of x^2 is not unity, then part of the equation must be factorised out as in the example below:

Example (23): Solve $5x^2 + 20x + 8 = 0$ by completing the square.

We factorise the first two terms of the equation to obtain $5(x^2 + 4x)$, and then proceed as usual:
(Half of 4 = 2, square of 2 = 4)

$5x^2 + 20x + 8 = 0$
$\Rightarrow 5(x^2 + 4x) + 8 = 0$ (taking out factor of 5)
$\Rightarrow 5((x + 2)^2 - 4) + 8 = 0$ (completing square on LHS)
$\Rightarrow 5(x + 2)^2 - 20 + 8 = 0$
$\Rightarrow 5(x + 2)^2 - 12 = 0$
$\Rightarrow 5(x + 2)^2 = 12$ (bringing constant to RHS)
$\Rightarrow (x + 2)^2 = \frac{12}{5}$ (dividing by 5)
$\Rightarrow (x + 2) = \pm\sqrt{\frac{12}{5}}$ (taking square roots)
$\Rightarrow x = -2 \pm \sqrt{\frac{12}{5}}$ (subtracting 2 from each side)

Some textbooks recommend taking the external factor out of the entire LHS :

$5x^2 + 20x + 8 = 0$
$\Rightarrow 5(x^2 + 4x + \frac{8}{5}) = 0$ (taking out factor of 5)
$\Rightarrow 5((x + 2)^2 – 4 + \frac{8}{5}) = 0$ (completing square on LHS)
$\Rightarrow 5(x + 2)^2 - 20 + 8 = 0$
$\Rightarrow 5(x + 2)^2 - 12 = 0$
$\Rightarrow 5(x + 2)^2 = 12$ (bringing constant to RHS)
$\Rightarrow (x + 2)^2 = \frac{12}{5}$ (dividing by 5)
$\Rightarrow (x + 2) = \pm\sqrt{\frac{12}{5}}$ (taking square roots)
$\Rightarrow x = -2 \pm \sqrt{\frac{12}{5}}$ (subtracting 2 from each side)

Example (24): Solve $x^2 + 4x + 7 = 0$ by completing the square. Does anything go wrong here ?

$x^2 + 4x + 7 = 0$
$\Rightarrow (x + 2)^2 - 4 + 7 = 0$ (completing square on LHS)
$\Rightarrow (x + 2)^2 + 3 = 0$ (completing square on LHS)
$\Rightarrow (x + 2)^2 = -3$ (bringing constant to RHS)

We have run into a problem here, since there is no real number that can give –3 on squaring, and therefore we cannot take square roots.

In fact, no real number has a negative square, and so the quadratic $x^2 + 4x + 7 = 0$ has no real solutions.

With practice, any expression of the form $x^2 + bx + c$ can be expressed as "a perfect square plus a bit" or "a perfect square minus a bit".

Example (24b): Express the following in the form $(x + a)^2 + b$ where a and b are integers :
i) $x^2 + 10x + 21$; ii) $x^2 - 6x + 12$

i) The closest perfect square to $x^2 + 10x + 21$ is $x^2 + 10x + 25$ or $(x + 5)^2$ which is 4 more than it.
Hence $x^2 + 10x + 21 = (x + 5)^2 - 4$.

ii) The closest perfect square to $x^2 - 6x + 12$ is $x^2 - 6x + 9$ or $(x - 3)^2$ which is 3 less than it.
Hence $x^2 - 6x + 12 = (x - 3)^2 + 3$.

Solving Quadratic Equations Using the General Formula.

If a quadratic equation cannot be obviously factorised by inspection, then the general formula can be used as an alternative to 'completing the square'.

To derive the general formula, take the general quadratic equation:

$ax^2 + bx + c$	=	0	
$4a^2x^2 + 4abx + 4ac$	=	0	multiply throughout by $4a$
$4a^2x^2 + 4abx + b^2 + 4ac - b^2$	=	0	add b^2 to complete square and subtract again to equal things out
$(2ax + b)^2 + 4ac - b^2$	=	0	factorise $4a^2x^2 + 4abx + b^2$ as the square of $(2ax + b)$
$(2ax + b)^2$	=	$b^2 - 4ac$	add $b^2 - 4ac$ to both sides
$2ax + b$	=	$\pm\sqrt{b^2 - 4ac}$	take square roots of both sides (allow +ve and –ve)
$2ax$		$-b \pm \sqrt{b^2 - 4ac}$	move b to the right hand side
x	=	$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	divide through by $2a$

Don't worry – you won't be expected to memorise this proof !

The formula for the roots of the general quadratic equation of the form

$ax^2 + bx + c = 0$

is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example (25): Find the roots of $3x^2 - 5x - 4 = 0$ using the formula.

We substitute $a = 3$, $b = -5$ and $c = -4$ into the formula:

$$x = \frac{5 \pm \sqrt{25 + 48}}{6} \Rightarrow x = \frac{5 \pm \sqrt{73}}{6}$$

(Answer given here in surd form - the solutions are 2.257 and -0.591 to 3 decimal places.)

Example (26): find the roots of $4x^2 - 20x + 25 = 0$ using the formula.

Substitute $a = 4$, $b = -20$ and $c = 25$ into the formula.

$$x = \frac{20 \pm \sqrt{400 - 400}}{8} \Rightarrow x = 2\tfrac{1}{2}$$

Notice how the expression inside the square root sign is equal to zero – there is only one root here, namely $\dfrac{-b}{2a}$.

Example (27): find the roots of $2x^2 + 5x + 4 = 0$ using the formula.

Substitute $a = 2$, $b = 5$ and $c = 4$ into the formula.

$$x = \frac{5 \pm \sqrt{25 - 32}}{4} \quad \Rightarrow x = \frac{-5 \pm \sqrt{-7}}{4}$$

The expression inside the square root sign is negative, and because the square of any real number cannot be negative, the equation has no real roots.

Example (28): find the roots of $x^2 - 8x + 1 = 0$ using the formula, and simplify the exact result.

Substituting into the formula ($a = 1$, $b = -8$ and $c = -1$) we have

$$x = \frac{8 \pm \sqrt{64 - 4}}{2} \Rightarrow x = \frac{8 \pm \sqrt{60}}{2}$$

Dividing both sides by 2, this surd expression simplifies into $x = 4 \pm \sqrt{15}$.

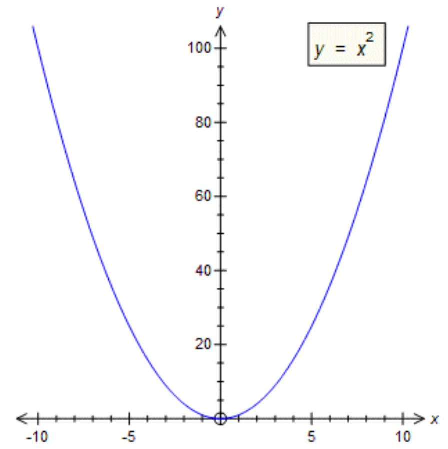
A common error when simplifying surd results of this form is to divide the number in the square root sign by the wrong factor: $x = 4 \pm \sqrt{30}$ is definitely incorrect.

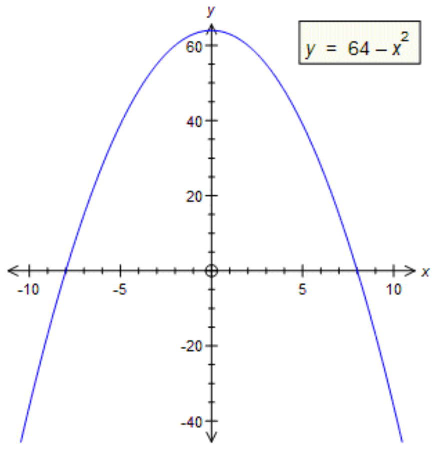
By surd laws, $\frac{\sqrt{a}}{b} = \frac{\sqrt{a}}{\sqrt{(b^2)}} = \sqrt{\frac{a}{b^2}}$.

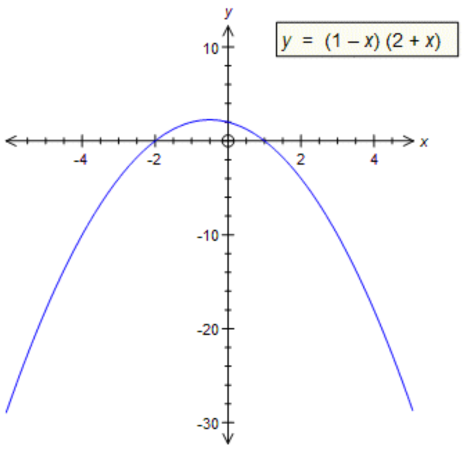
Therefore $\frac{\sqrt{60}}{2} = \frac{\sqrt{60}}{\sqrt{4}} = \sqrt{\frac{60}{4}} = \sqrt{15}$ (not √30), and $\frac{\sqrt{144}}{6} = \frac{\sqrt{144}}{\sqrt{36}} = \sqrt{\frac{144}{36}} = \sqrt{4} = 2$ (not √24).

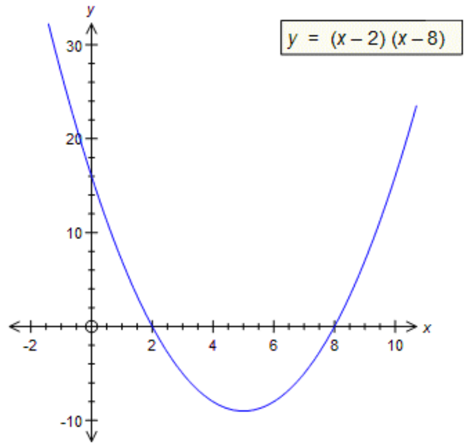
Quadratic graphs.

These graphs are of functions of the form $y = ax^2 + bx + c$ where a, b and c are constants, and a is not zero. The highest power of x is 2 (the square of x). The basic graph of $y = x^2$ is shown upper left.









These graphs are parabolic or 'bucket-shaped'.

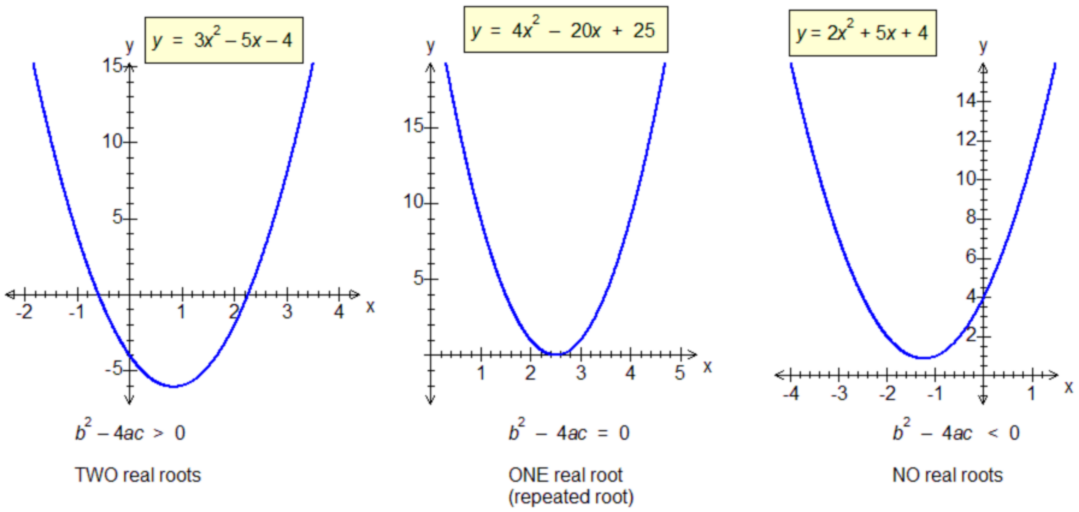
When the x^2 term is positive, the graphs point downwards at a trough and the function takes a minimum value. The expansion of $y = (x – 2)(x – 8)$ is $y = x^2 - 10x + 16$.

On the other hand, they point upwards at a crest and have a maximum value when the x^2 term is negative. The expansion of $y = (1 - x)(2 + x)$ is $y = 2 – x - x^2$.

The ‘depth’ of a parabolic graph can vary, but this is as dependent on the scaling of the graph axes as on the actual function.

The Discriminant of a Quadratic.

The expression $b^2 - 4ac$ within the general quadratic formula can tell us about the roots of the equation $ax^2 + bx + c$ without the need to solve it in full.



The graphs above correspond to examples (25), (26) and (27) above. If there are two real roots, the graph cuts the x-axis at two points; if there is one real root, the graph is a tangent to the x-axis at one point, and if there are no real roots, the graph misses the x-axis entirely.

The conditions can also be quoted as :

$b^2 > 4ac$ for two real roots
$b^2 = 4ac$ for one repeated root
$b^2 < 4ac$ for no real roots

(Although the examples above all have a minimum because the term in x^2 is positive, the same rules apply when the graph points the other way due to a negative x^2 term.)

Example (29): How many real roots do the following quadratic equations have ?
i) $x^2 + 9x + 21 = 0$; ii) $9x^2 + 12x + 4 = 0$; iii) $2x^2 + 11x + 10 = 0$

All the equations below are given in the form $ax^2 + bx + c = 0$, so it is just a matter of comparing the values of b^2 and $4ac$ for each one.

For $x^2 + 9x + 21 = 0$, $b^2 = 81$, $4ac = 84$. We have $b^2 < 4ac$ $\therefore$ no real roots.
For $9x^2 + 12x + 4 = 0$, $b^2 = 144$, $4ac = 144$. We have $b^2 = 4ac$ $\therefore$ one real root.
For $2x^2 + 11x + 10 = 0$, $b^2 = 121$, $4ac = 80$. We have $b^2 > 4ac$ $\therefore$ two real roots.

Example (29a): The quadratic equation $4x^2 + (k + 3)x + 9 = 0$ has coincident roots. Find the possible values of k.

In this equation, $b^2 = (k + 3)^2$ and $4ac = 144$.

The condition for coincident roots is $b^2 = 4ac$, or in this case,
$(k + 3)^2 = 144 \Rightarrow (k + 3) = \pm 12 \Rightarrow$ $\boldsymbol{k = 9}$ or $\boldsymbol{k = -15}$.

Sketching quadratic graphs.

Unlike plotting graphs accurately from a series of points, all that is required of this method is to plot a rough curve but it must go the 'right way up' and have important points included, such as the minimum (or maximum) and the intercept(s). The axes do not even need to be straight for a sketch !

Example (30): Sketch the graph from Example (5) $x^2 - 2x - 15$.

(Factorisation example)

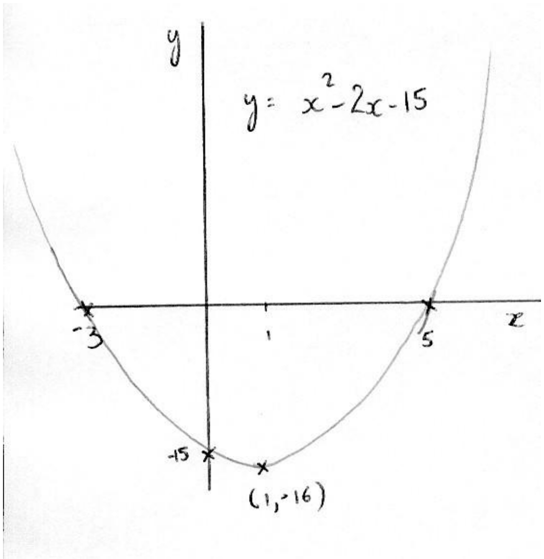
This example factorises into $x^2 - 2x - 15 = (x-5)(x+3)$, so the following can be plotted :

The graph has a 'trough' (i.e. a local minimum) because the term in x^2 is positive.

The roots of the equation are –3 and 5, so the graph crosses the x-axis at (-3,0) and (5,0).

Substituting $x = 0$ means that the graph crosses the y-axis at (0,-15).

As the graph has a line of symmetry, the function takes a minimum value when x is halfway between –3 and 5, i.e. $x = 1$. Substituting $x = 1$ gives the local minimum as (1, -16).



Example(31) : Sketch the graph from Example (24): $x^2 + 4x + 7$.

('Completing the square' example)

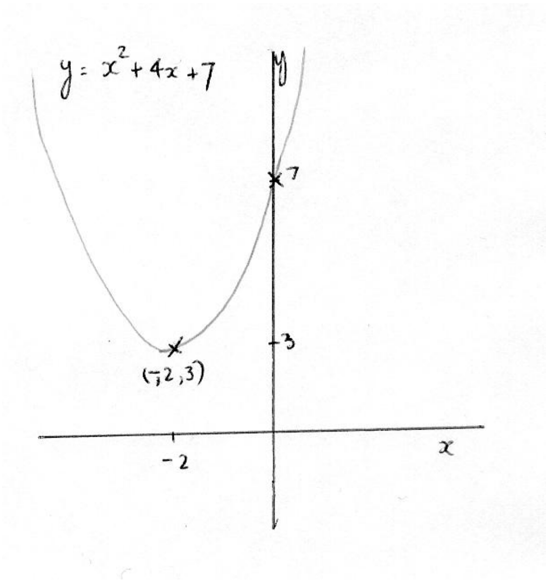
This example did not factorise, but was simplified by 'completing the square'.

$x^2 + 4x + 7 = (x+2)^2 + 3$

The graph again has a minimum since the term in x^2 is positive.

This time the graph does not cut the x-axis at all, but we can deduce the minimum point. The squared term has a minimum of 0, and that occurs when $x = -2$.

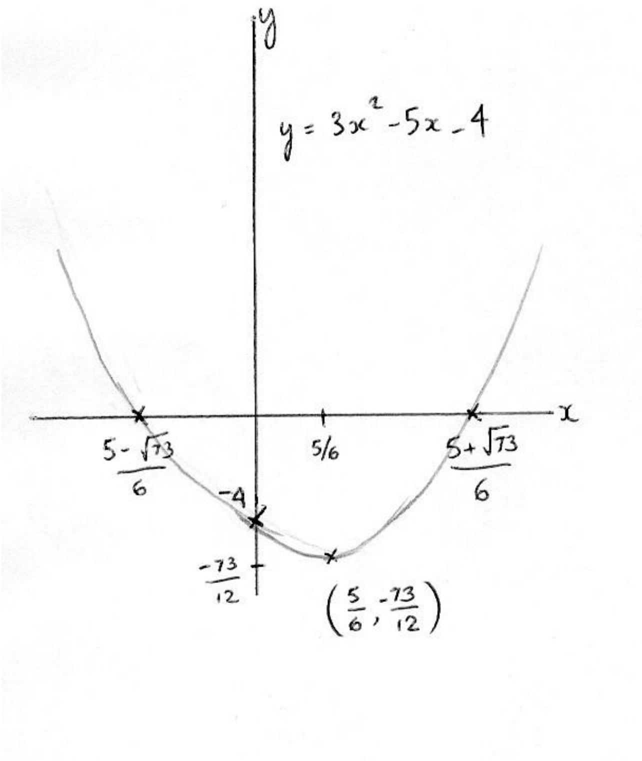
The local minimum is therefore (-2, 3). The only other point that might need plotting here is the y-intercept at (0, 7).



Example(32) : Sketch the graph from Example (25): $3x^2 - 5x - 4$.

(Formula example)

Once again, the graph takes a minimum value and points upwards since the term in x^2 is positive.



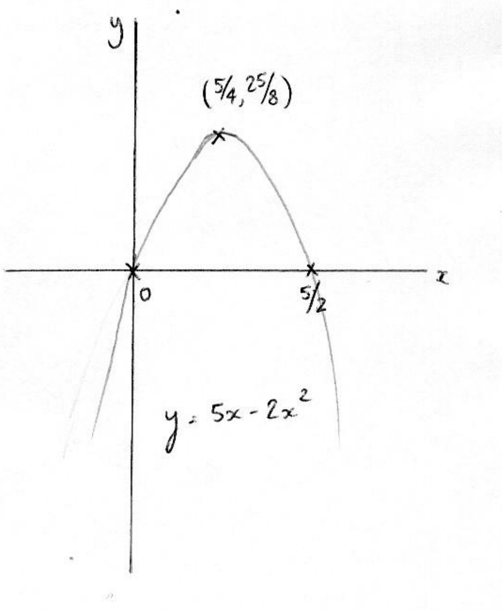
This example was solved using the formula, where the roots were found to be $x = \dfrac{5 \pm \sqrt{73}}{6}$ in surd form, or approximately 2.3 and –0.6.

The graph meets the x-axis for these values of x, and the y-axis at (0,-4).

The local minimum occurs when x is halfway between the roots, or when $x = \frac{5}{6}$. The corresponding value of y is $3(\frac{5}{6})^2 - 5(\frac{5}{6}) - 4$ or $\frac{-73}{12}$.

(For any quadratic $ax^2 + bx + c$, the minimum of maximum occurs when $x = \dfrac{-b}{2a}$.)

Example(33) : Sketch the graph of $5x - 2x^2$.



(Negative term in x^2)
This quadratic factorises into $5x - 2x^2 = x(5 – 2x)$, but differs from the previous examples in that the term in x^2 is negative. The graph has a 'crest' or maximum, rather than a minimum.

The roots of the equation are 0 and $\frac{5}{2}$, so the graph crosses the x-axis at (0,0) and ($\frac{5}{2}$,0).

As the graph has a line of symmetry, the function takes a maximum value when x is halfway between 0 and $\frac{5}{2}$, i.e. $x = \frac{5}{4}$. Substituting $x = \frac{5}{4}$ gives the local maximum as ($\frac{5}{4}$, $\frac{25}{8}$).

Sum and Product of the Roots of a Quadratic Equation.

If the roots of a quadratic equation $ax^2 + bx + c = 0$ are given as

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a},$$

then their sum, $\alpha + \beta = \dfrac{\left(-b+\sqrt{b^2-4ac}\right)+\left(-b-\sqrt{b^2-4ac}\right)}{2a}$

$\Rightarrow \alpha + \beta = \dfrac{-2b}{2a}$ and finally $\alpha + \beta = \dfrac{-b}{a}$.

The product of the roots is $\alpha\beta = \dfrac{\left(-b+\sqrt{b^2-4ac}\right)\left(-b-\sqrt{b^2-4ac}\right)}{(2a)^2}$

$\Rightarrow \alpha\beta = \dfrac{(-b)^2-\left(\sqrt{b^2-4ac}\right)^2}{(2a)^2} \Rightarrow \alpha\beta = \dfrac{b^2-\left(b^2-4ac\right)}{4a^2}$

$\Rightarrow \alpha\beta = \dfrac{4ac}{4a^2}$ and finally $\alpha\beta = \dfrac{c}{a}$.

Example (34): Find the sum and product of the roots α and β of the equation $2x^2 - 11x + 10 = 0$.

Here a, b and c are 2, -11 and 10 respectively, so $\alpha + \beta = \dfrac{-b}{a}$ = 5½ and $\alpha\beta = \dfrac{c}{a}$ = 5.

Conversely, given the sum and product of the roots, we can obtain the original equation:

Example (35): A quadratic equation $ax^2 + bx + c = 0$ has roots
$\alpha = 1 + \sqrt{5}$ and $\beta = 1 - \sqrt{5}$. Find the values of a, b and c.

The sum of the roots is $(1 + \sqrt{5}) + (1 - \sqrt{5})$ $\Rightarrow \alpha + \beta = 2$;
The product is $(1 + \sqrt{5})(1 - \sqrt{5})$ or $1^2 - (\sqrt{5})^2$ $\Rightarrow \alpha\beta = -4$.

$\therefore \dfrac{-b}{a} = 2$ and $\dfrac{c}{a} = -4$.

When $a = 1$, $b = -2$ and $c = -4$, therefore the required quadratic equation is $\mathbf{x^2 - 2x - 4 = 0}$.

Example (36): The roots of a quadratic equation $ax^2 + bx + c = 0$ have a sum of 4
and a product of $\dfrac{5}{3}$. Find the values of a, b and c.

Since the product of the roots = $\dfrac{c}{a}$, we have $a = 3$ and $c = 5$.
The sum of the roots, $\dfrac{-b}{a}$, is -4, $\Rightarrow b = -12$.
$\therefore$ the required quadratic equation is $\mathbf{3x^2 - 12x + 5 = 0}$.

Sometimes an expression can be rewritten as a general quadratic by algebraic manipulation or substitution.

Example (37): The formula $s = ut + \frac{1}{2}at^2$ represents the distance travelled by an object under constant acceleration, where *u* is the initial velocity, *v* is the final velocity, *a* is the acceleration and *t* is the time.

Make *t* the subject of the formula, and hence work out how much time elapses when a particle travels 500 metres given an initial velocity of 5ms^{-1} and acceleration of 2ms^{-2}.

Rewrite the formula as $\frac{1}{2}at^2 + ut - s = 0$ and we have a quadratic in *t*.

Substituting *t* for *x*, $\frac{1}{2}a$ for *a*, *u* for *b* and -*s* for *c* in the general formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we obtain

$$t = \frac{-u \pm \sqrt{u^2 + 2as}}{a}$$

Substitution in the above formula gives

$$t = \frac{-5 \pm \sqrt{25 + 2000}}{2} \text{ or } t = \frac{-5 \pm 45}{2}$$

The only value which makes sense in the context of the question is the positive one, namely $t = 20$

The particle therefore takes 20s to travel 500m, given the starting velocity of 5ms^{-1} and acceleration of 2ms^{-2}.

Sometimes an algebraic question might seem at first to have nothing to do with quadratic equations.

Example (38): Solve the equation $\frac{2x-3}{3x+1}+\frac{x+5}{4(x-1)}=1$.

The first step is to find the L.C.M. of the denominators - $4(3x+1)(x-1)$ - and multiply every term by it.

$$\frac{4(2x-3)(3x+1)(x-1)}{(3x+1)}+\frac{4(x+5)(3x+1)(x-1)}{4(x-1)}=4(3x+1)(x-1)$$

Next, cancel out common factors :

$$4(2x-3)(x-1)+(x+5)(3x+1)=4(3x+1)(x-1)$$

Then, expand and collect:

$4(2x^2 - 5x + 3) + (3x^2 + 16x + 5) = 4(3x^2 - 2x - 1)$

$\Rightarrow$ $8x^2 - 20x + 12 + 3x^2 + 16x + 5 = 12x^2 - 8x - 4$

$\Rightarrow$ $11x^2 - 4x + 17 = 12x^2 - 8x - 4$

$\Rightarrow$ $12x^2 - 8x - 4 - 11x^2 + 4x - 17 = 0$

$\Rightarrow$ $x^2 - 4x - 21 = 0$ $\Rightarrow$ $(x + 3)(x - 7) = 0$

$\therefore$ the solutions are $x = 7$, $x = -3$.

Example (39): Solve the equation $\frac{2x^2 + x - 28}{x^2 - 16} = 3$.

Here we factorise both the top and bottom of the expression and cancel out common factors, before ending with a linear equation.

$$\frac{(x+4)(2x-7)}{(x+4)(x-4)} = 3 \Rightarrow \frac{(2x-7)}{(x-4)} = 3 \Rightarrow 2x - 7 = 3(x-4)$$

$2x + 7 = 3x + 12 \Rightarrow x - 5 = 0 \Rightarrow x = 5.$

$\therefore$ the solution is $x = 5$.

Example (40): The perimeter of a rectangle is 16 cm and its diagonal is 6 cm.

i) Given that its long side is x cm long, find an expression in terms of x for its short side.

ii) Show that $x^2 - 8x + 14 = 0$.

iii) Hence find the lengths of the long and short sides to two decimal places.

i) If the perimeter of the rectangle is 16 cm, half that is 8 cm, so the length of one short side and one long side add up to 8 cm. The short side is therefore $(8 - x)$ cm long.

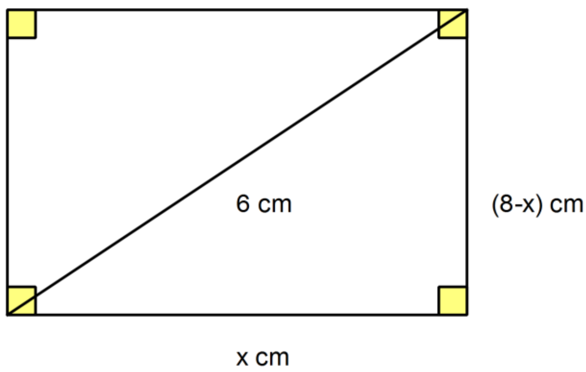
ii) We then apply Pythagoras to form a quadratic equation from the given data :

$x^2 + (8 - x)^2 = 6^2$

$\rightarrow x^2 + 64 - 16x + x^2 = 36$ (expanding both sides)

$\rightarrow 2x^2 - 16x + 28 = 0$ (collecting)

$\rightarrow x^2 - 8x + 14 = 0$ (dividing throughout by 2)



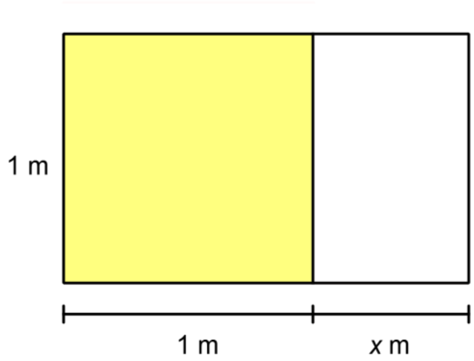
The question states "2 decimal places" so we cannot factorise the quadratic.

Instead we use the general formula (or complete the square).

Substituting into the general formula ($a = 1$, $b = -8$ and $c = 14$) we have

$$x = \frac{8 \pm \sqrt{64 - 56}}{2} \quad \therefore \quad x = \frac{8 \pm \sqrt{8}}{2} \text{ or } 4 \pm \sqrt{2}.$$

The long side x is therefore $4 + \sqrt{2}$ cm, or 5.41 cm, and the short side is $4 - \sqrt{2}$ cm, or 2.59 cm.



Example (41): Damien buys a rectangular piece of canvas whose short side is 1 metre long and whose long side is 1 + *x* metres long.

He then prepares to cut the canvas into two parts, one of which is a square of side 1 metre. The remaining rectangular portion (in white) also has the property of being similar to the original canvas.

Find the length of the long side of the original canvas to the nearest millimetre.

The sides of the original canvas are in the ratio 1 + *x* : 1 and those of the remaining rectangle are 1 : *x*.

The rectangles are similar, so $\frac{1+x}{1} = \frac{1}{x} \rightarrow x(1+x) = 1 \rightarrow x^2 + x - 1 = 0$.

Substituting into the general formula (a = 1, b = 1 and c -1) we have

$x = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$. Since x must be positive here, $x = \frac{\sqrt{5}-1}{2} = 0.618$.

$\therefore$ The long side of the original canvas is 1 + x = **1.618 m** to the nearest millimetre.

The original canvas and the remaining rectangle have their long and short sides in the ratio 1.618 : 1.

As an aside, the ancients considered a rectangle of this shape to be so ideally proportioned, that the ratio of 1.618 : 1 or more exactly, $\frac{\sqrt{5}+1}{2} : 1$ became known as the "golden" ratio.

Example (42) (Harder): Matthew and Nadine take part in a sponsored cycle race from Manchester to Lancaster, a distance of 84 km. Nadine's mean speed is 2 km/h less than Matthew's, and as a result it takes her 12 minutes longer to complete the race.

i) Form an expression for the travel time in hours, *t*, in terms of Matthew's mean speed in km/h, *s*.

ii) Show that the corresponding expression for Nadine's travel time is $t+0.2=\frac{84}{s-2}$.

iii) Show that the expression in part ii) can be rearranged to $t=\frac{84.4-0.2s}{s-2}$

iv) By setting expressions i) and iii) against each other, show that $s^2 - 2s - 840 = 0$, and therefore calculate the outward speeds and resulting times for Matthew and Nadine.

i) Since time is distance divided by speed, Matthew's travel time in hours is given by $t=\frac{84}{s}$.

ii) Nadine takes 12 minutes, or 0.2 hours, longer to complete the race, so her travel time is $t + 0.2$ hours, and her speed is $(s – 2)$ km/h less.

An expression for her travel time is hence $t+0.2=\frac{84}{s-2}$.

iii) We can rearrange the last result; $t+0.2=\frac{84}{s-2} \rightarrow t=\frac{84}{s-2}-0.2$

$$\rightarrow t=\frac{84}{s-2}-\frac{0.2s-0.4}{s-2} \rightarrow t=\frac{84.4-0.2s}{s-2}.$$

iv) We set the expressions from i) and iii) against each other :

$\frac{84}{s}=\frac{84.4-0.2s}{s-2}$, and cross-multiplying gives us $84 (s- 2) = s (84.4 - 0.2s)$.

Therefore $84s - 168 = 84.4s - 0.2s^2 \rightarrow 0.2s^2 – 0.4s - 168 = 0$

$\rightarrow$ $\mathbf{s^2 - 2s - 840 = 0}$ (multiplying both sides by 5)

The last expression factorises to $(s - 30) (s + 28) = 0$.

Hence Matthew's mean speed is **30 km/h** and the time taken for him to complete the race is $\frac{84}{30}=2.8$ hours, or **2 hours and 48 minutes**.

Nadine's mean speed is 2 km/h slower than Matthew's, i.e. **28 km/h**, and as it takes her 12 minutes longer to complete the race, she completes her race in **exactly 3 hours**.

More Disguised Quadratics.

Example (43): Solve the equation $x^3 + 3x^2 - 28x = 0$.

This equation is a cubic rather than a quadratic, but fortunately x can be taken out as a factor and the quadratic factorised again :

$x^3 + 3x^2 - 28x = 0 \Rightarrow x(x^2 + 3x - 28) = 0 \Rightarrow x(x + 7)(x - 4) = 0$

$\therefore$ the solutions are $x = 0$, $x = 4$ and $x = -7$.

Example (44): Solve the equation $x^4 - 13x^2 + 36 = 0$.

This time we have a rather formidable fourth-power equation, but we can turn it into a quadratic one by making the substitution $y = x^2$.

The resulting quadratic is $y^2 - 13y + 36 = 0$, factorising to $(y - 9)(y - 4) = 0$.
The solutions are thus $y = 9$ and $y = 4$, $\Rightarrow$ $x^2 = 9$ and $x^2 = 4$, $\Rightarrow$ $x = \pm 3$ and ± 2.

Note that although the quadratic in y has two solutions, the original equation in x has four.

Example (45): Given that the roots of $x^2 - 13x + 40 = 0$ are 5 and 8,
find the roots of the equation $\frac{1}{x^2} - \frac{13}{x} + 40 = 0$, using the substitution $y = \frac{1}{x}$.

With the substitution $y = \frac{1}{x}$, $\frac{1}{x^2} - \frac{13}{x} + 40 = 0$ transforms to $y^2 - 13y + 40 = 0$

and its solutions are $y = 5$ and $y = 8$.

Taking reciprocals (as $y = \frac{1}{x}$), the solutions of the original equation are $x = \frac{1}{5}$ and $x = \frac{1}{8}$.

(We could still have solved the equation 'the hard way' by multiplying throughout by x^2 and obtaining

$1 - 13x + 40x^2 = 0 \Rightarrow (1 - 5x)(1 - 8x) = 0 \Rightarrow x = \frac{1}{5}$ and $x = \frac{1}{8}$.)