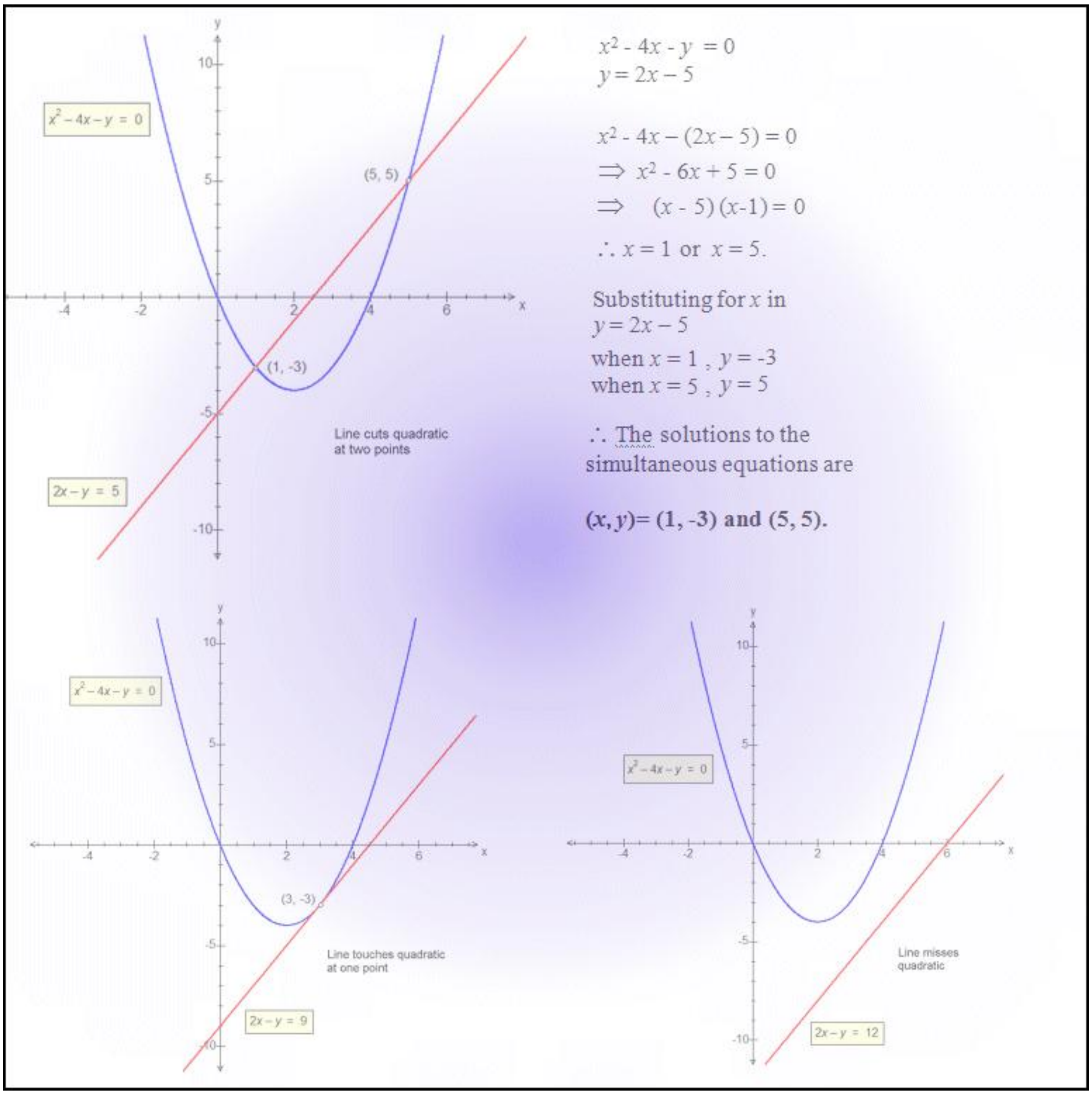
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 1 / AS

SIMULTANEOUS EQUATIONS



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SIMULTANEOUS EQUATIONS

Linear Simultaneous Equations.

To recall, there are three methods of solving linear simultaneous equations – elimination, substitution and graphical methods.

Elimination method.

Example (1): Use the elimination method to solve the simultaneous equations
$x + 4y = 2$; $x - 3y = -5$.

$x + 4y = 2$ A
$x - 3y = -5$ B

It is possible to eliminate x by subtracting equation B from equation A.

$7y = 7$ A-B

This gives $y = 1$, and so the value could be substituted into either of the original equations.

Substituting into equation A gives $x + 4 = 2$, therefore $x = -2$.

The solution to these equations is therefore $\mathbf{x = -2, y = 1}$.

Example (2): Use the elimination method to solve the simultaneous equations
$3x - 4y = 17$; $2x + 5y = -4$

$3x - 4y = 17$ A
$2x + 5y = -4$ B

We need to multiply both equations by suitable factors , so that one of the variables can be eliminated.

(We have chosen to eliminate x here.)

$3x - 4y = 17$ A
$2x + 5y = -4$ B

$6x - 8y = 34$ $2A$
$6x + 15y = -12$ $3B$

$23y = -46$ $3B - 2A$

This makes $y = -2$. Substituting into equation A gives $3x - (-8) = 17$, and thus $x = 3$.
The solution is x = **3,** $\mathbf{y = -2}$.

We could eliminate y instead:

$3x - 4y = 17$ A
$2x + 5y = -4$ B

$15x - 20y = 85$ $5A$
$8x + 20y = -16$ $4B$

$23x = 69$ $5A + 4B$

Because the signs of y are different, we eliminate y by adding rather than subtracting.
This makes $x = 3$. Substituting into equation A gives $9 – 4y = 17$, and thus $y = -2$.
The solution is x = **3,** $\mathbf{y = -2}$.

Substitution method.

In this method, take one of the equations and express one variable in terms of the other.

Example (3): Use the substitution method to solve the simultaneous equations
$x + 4y = 2$; $x - 3y = -5$. (This is identical to Example 1).

This time, we see that the second equation can be rewritten to make x the subject:

$x - 3y = -5$ can be rearranged to $x = 3y - 5$.

Equation A can thus be rewritten by substituting $3y - 5$ for x:

$(3y - 5) + 4y = 2$, or $7y - 5 = 2$, or $7y = 7$.

This gives $y = 1$ as in the elimination method, and substitution into equation A gives $x = -2$.

Example (4): Use the substitution method to solve the simultaneous equations
$x + 2y = 7$; $3x + y = 1$.

The second equation can be rewritten as $y = 1 - 3x$ and the result substituted into the first equation, as
$x + 2(1 - 3x) = 7$,
$\Rightarrow$ $2 - 5x = 7$
$\Rightarrow$ $-5x = 5$
$\Rightarrow$ $x = -1$

Substituting $x = -1$ into the first equation gives $y = 4$.

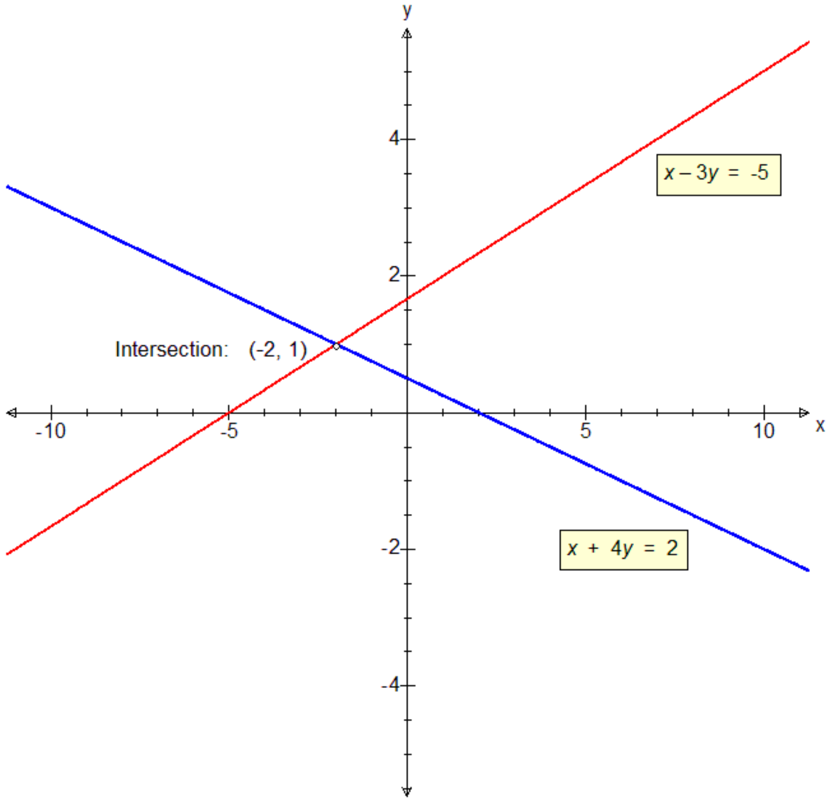
The solution is $\boldsymbol{x = -1, y = 4}$.

Graphical interpretation.

Plot the graphs of the two functions corresponding to the equations. The coordinates of the point of intersection give the solution.

The graph below shows the solution of the equation-pair

$x + 4y = 2$; $x - 3y = -5$.



The point of intersection is (-2, 1), corresponding to $x = -2$, $y = 1$.

Graphical methods are rarely used beyond GCSE, as students would normally be expected to solve such equations algebraically.

Linear / Quadratic Simultaneous Equations

Here we will look at cases where one equation of a simultaneous pair is linear but the other is a quadratic.

Linear / Quadratic Simultaneous Equations.

The examples so far dealt with cases where both equations were linear.
Here we will look at what happens where one equation is linear but the other is a quadratic.

Example (5): Solve the simultaneous equations $y = x^2 - 3x + 4$ and $y = x + 1$.

This example is relatively simple because both equations have y solely on the left-hand side. We can use methods learnt when solving quadratic equations, as per the graphical examples of 'solving many quadratic equations from one'.

The original equations are

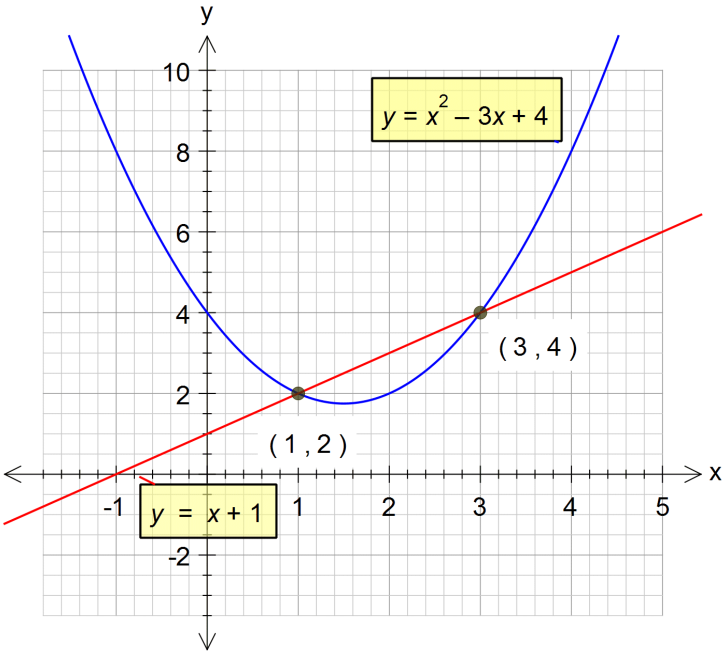
$y = x^2 - 3x + 4$
$y = x + 1$.

They therefore have a solution when $x^2 - 3x + 4 = x + 1$.
This rearranges to give a new quadratic, $x^2 - 4x + 3 = 0$.

Factorising, we have $(x - 3)(x - 1) = 0$
$\therefore\ x = 3$ or $x = 1$.

To find the values of y corresponding to each value of x, we simply substitute into the linear equation $y = x + 1$. Hence, when $x = 1$, $y = 2$ and when $x = 3$, $y = 4$.

The graphical solution is shown below; the quadratic curve and the line intersect at (1, 2) and (3, 4) – the solutions in (x, y) to the simultaneous equations..



Example (6): Solve the simultaneous equations $x^2 + y^2 = 25$ and $x - y = 1$.

The first thing to notice is that it is impossible to use the elimination method here. You cannot add or subtract multiples of x from x^2 and eliminate x from the equations, neither can you do the same with y.

This leaves only the substitution method as a viable option. It is generally easier to manipulate the linear equation, as in the example below.

$x^2 + y^2 = 25$ *A*
$x - y = 1$ *B*

Rearrange equation *B* to give

$x^2 + y^2 = 25$ *A*
$-y = 1 - x$ *B*
$\Rightarrow y = x - 1$ *B*

Now substitute (x-1) for y in equation *A* :

$x^2 + (x - 1)^2 = 25$ *A*
$\Rightarrow y = x - 1$ *B*

Equation *A* can be simplified:

$x^2 + (x - 1)^2 = 25$

$\Rightarrow x^2 + x^2 - 2x + 1 = 25$ (expand)
$\Rightarrow 2x^2 - 2x - 24 = 0$ (collect terms)
$\Rightarrow x^2 - x - 12 = 0$ (take out common factor of 2)

$\Rightarrow (x + 3)(x - 4) = 0$ (factorise)

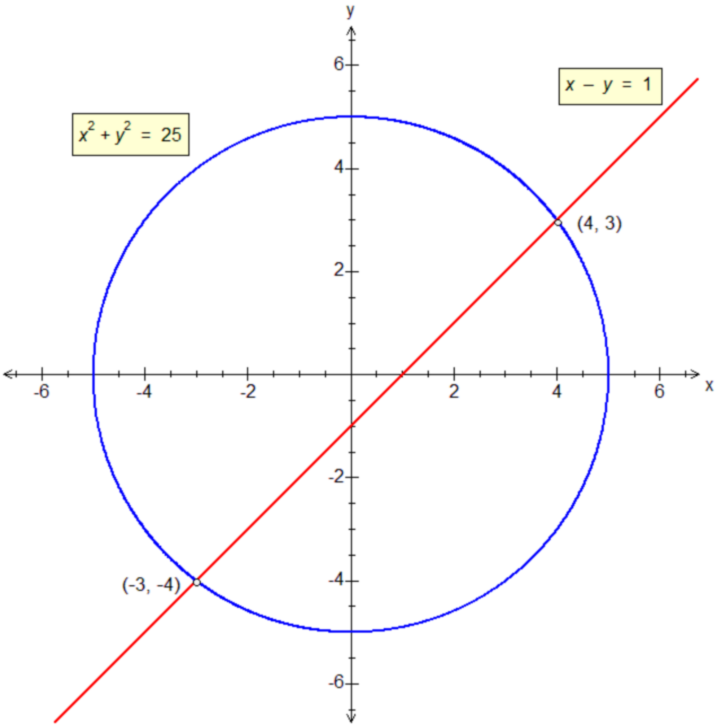
$\therefore x = -3$ or 4.

Having obtained x, the next step is to substitute the x-values back into equation *B* to find the y-coordinates.

This gives $y = -4$ when $x = -3$, and $y = 3$ when $x = 4$.

The solutions in (x, y) are thus (4 ,3) and (-3, -4).

The graphical interpretation is shown on the right.



(N.B. $x^2 + y^2 = 25$ is the equation of a circle centred at the origin and a radius of 5 units.)

Example (7): Solve the simultaneous equations

$x^2 - 4x - y = 0$; $2x - y = 5$

First we express *y* in terms of *x* in the second (linear) equation

$x^2 - 4x - y = 0$ *A*
$y = 2x - 5$ *B*

Substitute $(2x - 5)$ for *y* in equation *A*:

$x^2 - 4x - (2x - 5) = 0$ *A*
$\Rightarrow x^2 - 6x + 5 = 0$ *A*
$y = 2x - 5$ *B*

Factorise the quadratic equation *A*:
$x^2 - 6x + 5 = 0 \Rightarrow (x - 5)(x - 1) = 0$
$\therefore x = 1$ or $x = 5$.

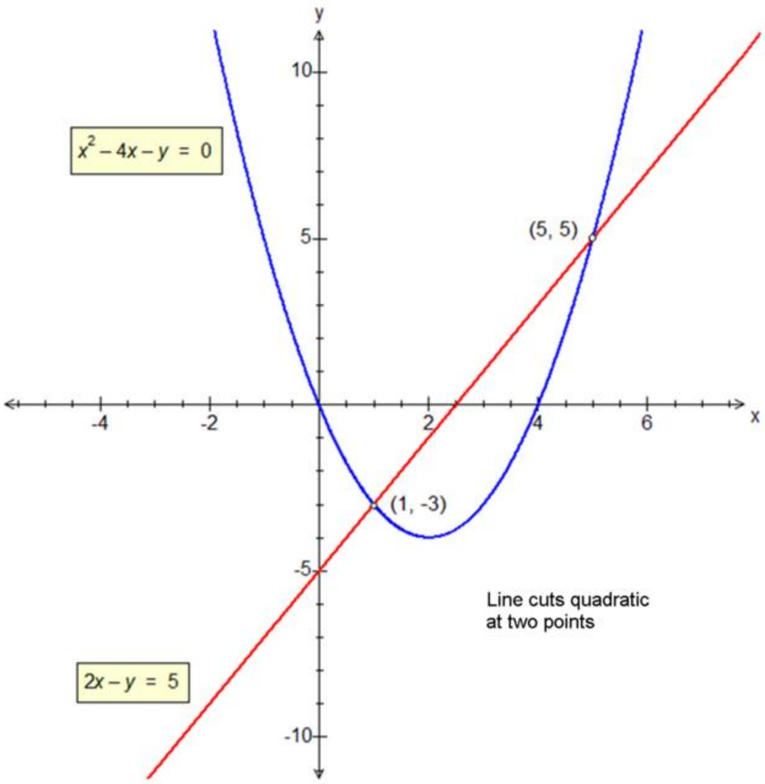
This equation *A* was solved by factorising, but had it not been easy to do so, the discriminant, $b^2 - 4ac$, would have given information about the number of solutions.

Here, $ax^2 + bx + c = x^2 - 6x + 5$, so $b^2 = 36$ and $4ac = 20$.

Because $b^2 - 4ac$ is positive (i.e. $b^2 > 4ac$), the quadratic has two roots and hence the graphs of the equations (a parabola and a line) intersect at two points.

Substituting for *x* in equation *B* , $y = -3$ when $x = 1$, and $y = 5$ when $x = 5$.
$\therefore$ The solutions to the simultaneous equations are

$(x, y) = (1, -3)$ and $(5, 5)$.



The following example will highlight two other possible scenarios.

Example (8): Solve the simultaneous equations
$x^2 - 4x - y = 0$; $2x - y = 9$.

The working is similar to Example (6), but now $y = 2x - 9$.

$x^2 - 4x - (2x - 9) = 0$ *A*
$\Rightarrow x^2 - 6x + 9 = 0$ *A*
$y = 2x - 9$ *B*

The resulting quadratic can also be factorised:

$x^2 - 6x + 9 = 0$
$\Rightarrow (x - 3)^2 = 0 \therefore x = 3.$

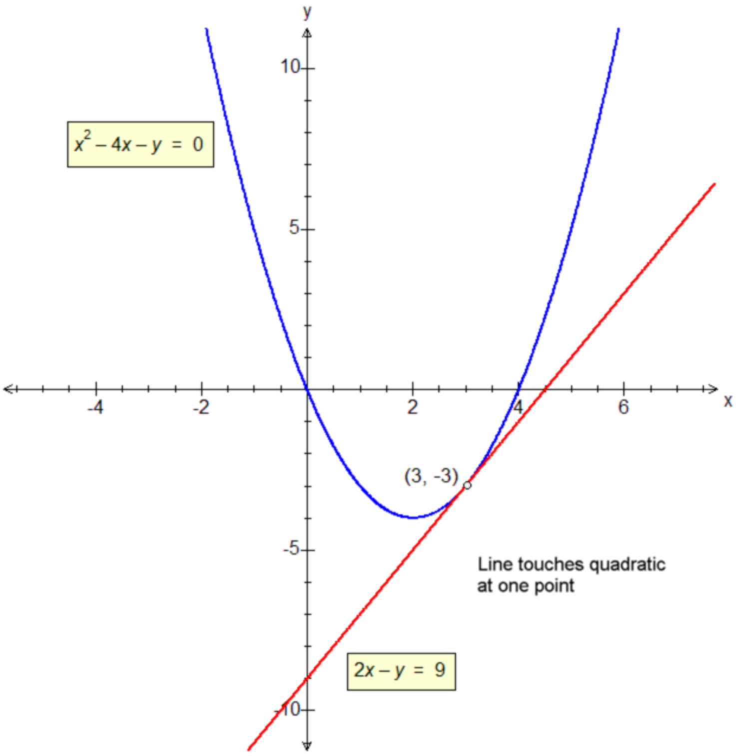
Here, $ax^2 + bx + c = x^2 - 6x + 9$,
so $b^2 = 36$ and $4ac = 36$.

This time $b^2 - 4ac = 0$
(i.e. $b^2 = 4ac$), so there is only one solution.

Substituting for x in equation *B* , $y = -3$ when $x = 3$.

$\therefore$ The single solution to the simultaneous equations is $(x, y) = (3, -3)$.

Graphically, the line is now a tangent to the quadratic.



Example (9): Solve the simultaneous equations $x^2 - 4x - y = 0$; $2x - y = 12$, using the general quadratic formula. What happens here?

The working is again similar to Example (6), but now $y = 2x – 12$.

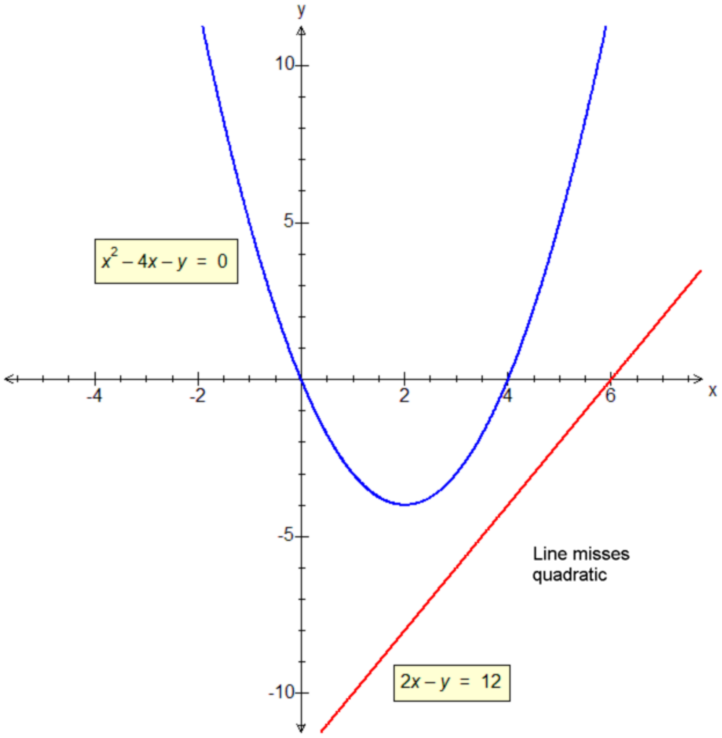
$x^2 - 4x – (2x – 12) = 0$ *A*
$\Rightarrow x^2 - 6x + 12 = 0$ *A*
$y = 2x – 12$ *B*

The clue in this question is that the quadratic cannot be factorised.

Substituting $a = 1$, $b = -6$ and $c = 12$ into the general formula gives $b^2 = 36$ and $4ac = 60$.

Crucially, $b^2 - 4ac < 0$ (i.e. $b^2 < 4ac$), and therefore the quadratic has no real roots and the simultaneous equations have no solution.

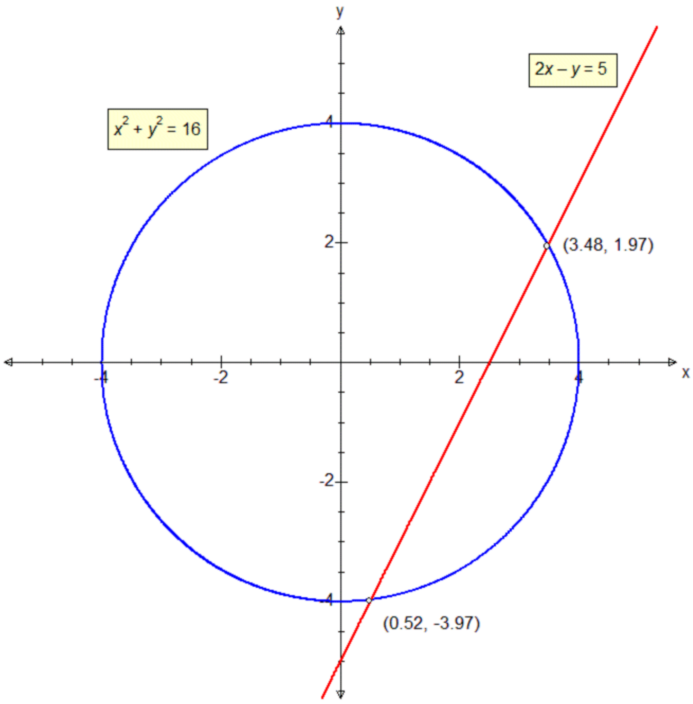
Graphically, the line misses the quadratic.



Example (10): Solve the simultaneous equations

$x^2 + y^2 = 16$; $2x - y = 5$, giving results correct to two decimal places.

There is a clue in the question; it states "2 decimal places". This suggests that there are no easy factors here, and therefore the general formula must be used instead.



This is another example of a line intersecting a circle.

Expressing x in terms of y in the second (linear) equation we have $2x = 5 + y$.

To avoid awkward fractions, it is best to multiply the quadratic by 4 throughout, i.e. re-express it as

$4x^2 + 4y^2 = 64$.

N.B. $4x^2$ is equivalent to $(2x)^2$.

$4x^2 + 4y^2 = 64$ *A*
$2x = 5+y$ *B*

Substitute for y in the first equation

$(5+y)^2 + 4y^2 = 64$ *A*
$2x = 5+y$ *B*

The resulting quadratic in *A* becomes

$25 + 10y + y^2 + 4y^2 = 64$
$\Rightarrow 5y^2 + 10y + 25 = 64$
$\Rightarrow 5y^2 + 10y - 39 = 0$

The solution(s) for y will therefore be

$y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, here $y = \dfrac{-10 \pm \sqrt{100 + 780}}{10}$

or $y = -1 \pm 2.9665$ (keep the extra decimals in the working).

$\therefore y = 1.9665, -3.9665$.

Substituting into the equation *B* gives $x = 3.483$ when $y = 1.9665$ and $x = 0.517$ when $y = -3.9665$

$\therefore$ The solutions to the simultaneous equations (to 2 decimal places) are:

(x, y) = (3.48, 1.97) and (0.52, -3.97).

Example (11): A farmer wishes to use 160m of fencing to build three equal square enclosures of side x metres for his sheep, together with one larger enclosure of side y metres for his cows.

Given that the area to be enclosed is 508 square metres, determine the dimensions of the enclosures.
Hint: find one linear and one quadratic equation linking x and y.

(Copyright Letts Educational, *AS Mathematics Exam Secrets* (2004) ISBN 1-84315-409-9 , Exercise 1.1, Q.6)

The three equal square enclosures have a combined perimeter of $3 \times 4x$ metres and the fourth one has a perimeter of $4y$ metres.

$\therefore$ The perimeters are linked by the equation $12x + 4y = 160$,
or $\mathbf{3x + y = 40}$ (taking out factor of 4).

The combined area of the three equal enclosures is $3x^2$ m^2 and the area of the fourth one is y^2 m^2.

$\therefore$ The areas are linked by the equation $\mathbf{3x^2 + y^2 = 508}$.

We now have a linear / quadratic simultaneous equation pair, to be solved by substitution.

$3x + y = 40$ *A*
$3x^2 + y^2 = 508$ *B*

Substitute $(40-3x)$ for y in equation *B*:

$3x^2 + (40-3x)^2 = 508$ *B*

Expansion of equation *B* gives $3x^2 + (1600 - 240x + 9x^2) = 508$
$\Rightarrow 12x^2 - 240x + 1600 = 508$
$\Rightarrow 12x^2 - 240x + 1092 = 0$
$\Rightarrow x^2 - 20x + 91 = 0$ (dividing throughout by 12)
$\Rightarrow (x - 13)(x - 7) = 0$

$\therefore x = 7$ or 13.

Substituting in $y = 40 - 3x$ gives possible values of $x = 7$, $y = 19$ or $x = 13$, $y = 1$.

The question specifies that $y > x$, therefore the three smaller enclosures have a side of 7m and the larger one has a side of 19m.