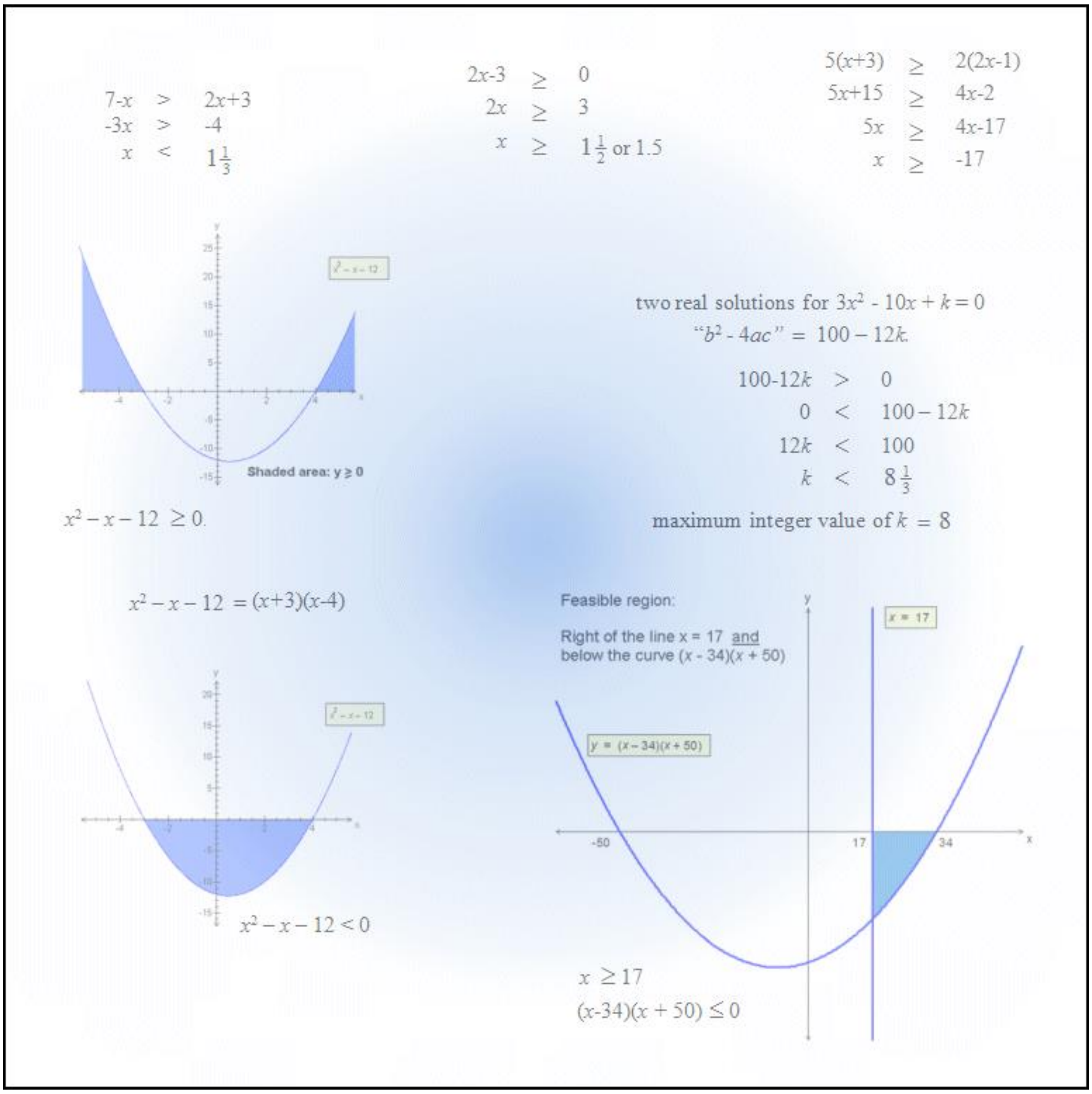
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 1 / AS

LINEAR AND QUADRATIC INEQUALITIES



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INEQUALITIES

Inequalities are solved in the same way as equations, but with one important difference.

The **sign must be reversed** either when multiplying or dividing by a negative number, or when reversing the inequality.

Linear inequalities.

Example (1): Solve the inequality $2x-3 \geq 0$.

$2x-3$	$\geq$	0	
$2x$	$\geq$	3	Add 3 to both sides
x	$\geq$	$1\frac{1}{2}$ or 1.5	Divide both sides by 2

Example (2): Solve the inequality $5(x+3) \geq 2(2x-1)$.

$5(x+3)$	$\geq$	$2(2x-1)$	
$5x+15$	$\geq$	$4x-2$	Expand both sides
$5x$	$\geq$	$4x-17$	Subtract 15 from each side
x	$\geq$	-17	Subtract $4x$ from each side

We could simplify the process by subtracting $4x + 15$ from each side in one step:

$5(x+3)$	$\geq$	$2(2x-1)$	
$5x+15$	$\geq$	$4x-2$	Expand both sides
x	$\geq$	-17	Subtract $4x + 15$ from each side

Example (3): Solve the inequality $7-x > 2x+3$.
The final step here involves dividing both sides of the inequality by a negative number.

$7-x$	$>$	$2x+3$	
$-3x$	$>$	-4	Subtract $2x + 7$ from each side
x	$<$	$1\frac{1}{3}$	Divide both sides by -3 and **reverse the inequality sign**

Alternatively, we could have solved the inequality like this:

$7-x$	$>$	$2x+3$	
4	$>$	$3x$	Add x - 3 to each side
$1\frac{1}{3}$	$>$	x	Divide both sides by 3
x	$<$	$1\frac{1}{3}$	The last result had x on the RHS. We must therefore **reverse the inequality and the sign.**

Example (4): A quadratic function has the equation $3x^2 - 10x + k = 0$, where k is an integer. What is the maximum value of k which guarantees two real solutions for the equation ?

The discriminant of the quadratic, "$b^2 - 4ac$", is $100 – 12k$.
We therefore must solve the inequality $100 – 12k > 0$.

$100-12k$	$>$	0	
0	$<$	$100 – 12k$	Reverse inequality and sign
$12k$	$<$	100	Add $12k$ to both sides
k	$<$	$8\frac{1}{3}$	Divide both sides by 12

$\therefore$ the maximum integer value of k which guarantees two solutions for the quadratic equation $3x^2 - 10x + k = 0$ is $\boldsymbol{k = 8}$.

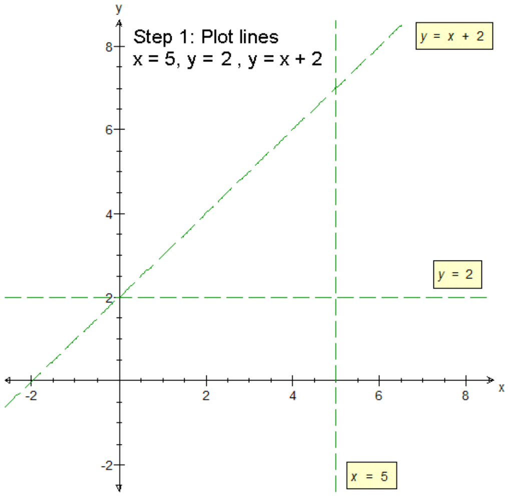
Graphs of linear inequalities. (Revision from GCSE).

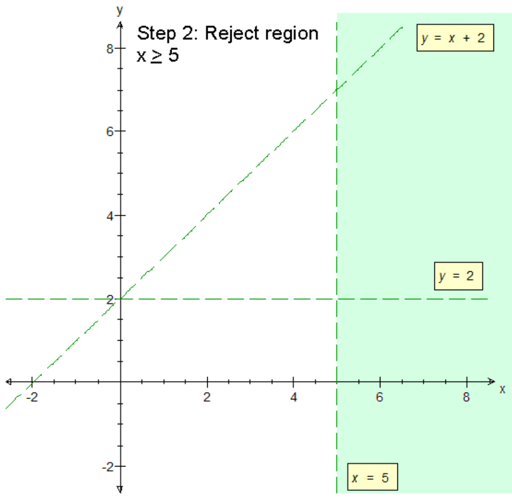
Examples such as the one below seldom turn up in core mathematics, but the topic of linear programming (of which this is an example) is featured in decision mathematics.

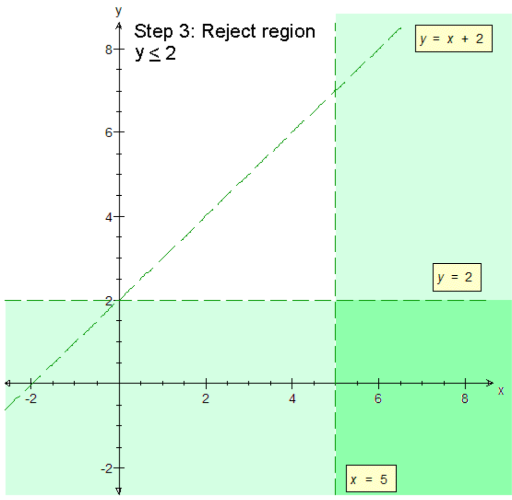
To show the region for a group of given inequalities, first draw the boundary lines as defined by the corresponding equations. Use dotted boundary lines for strict inequalities, and solid ones in other cases.

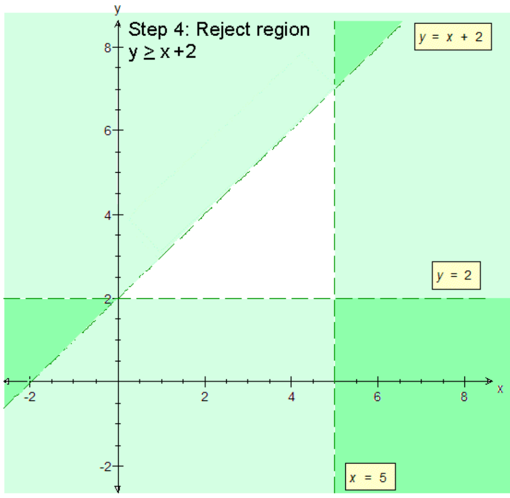
It is generally easier to shade out the **unwanted** regions, so that the solution is shown unshaded.

Example (5): Show graphically the region bounded by the following three inequalities:
$x < 5$; $y > 2$; $y < x + 2$.
In addition, give the coordinates of all points (p, q) inside this feasible region where p and q are integers.









Since we are dealing with strict inequalities, the lines $x = 5$, $y = 2$ and $y = x + 2$ are shown dotted rather than solid.

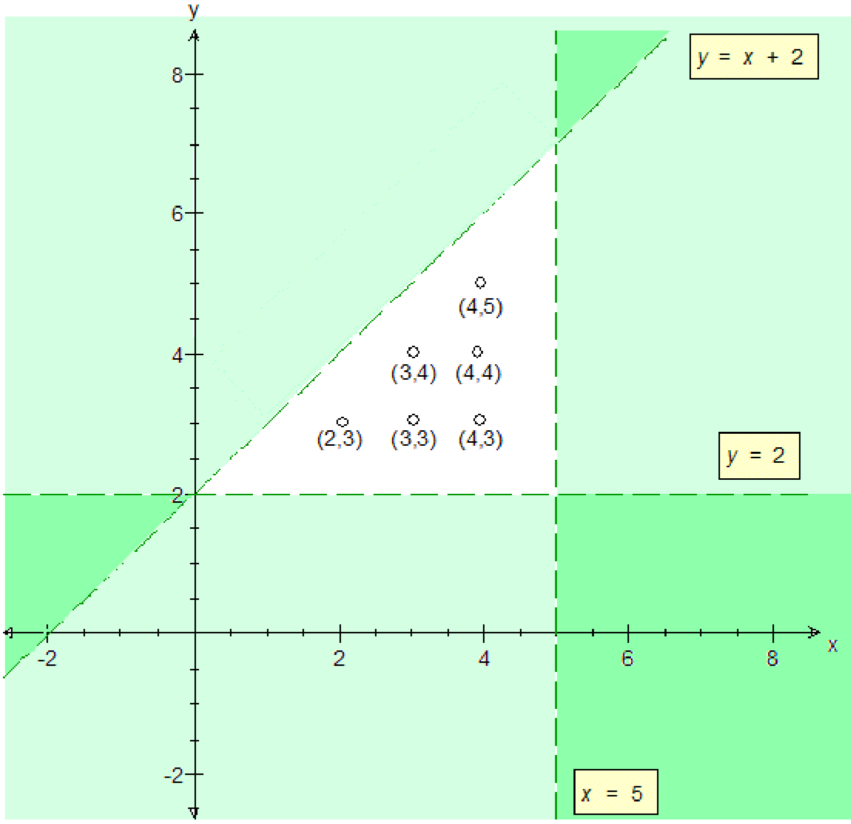
It is easy enough to determine which regions to reject in Steps 2 and 3.

For Step 4, we can choose the origin as a test point for the inequality $y < x + 2$.

When $(x, y) = (0, 0)$, then we have $0 < 2$, which is a true statement.
We therefore keep the region on that side of the line $y = x + 2$, (i.e. below and right of the line) unshaded, and reject and shade the opposite one (above and left).

It now only remains to give the coordinates of the points (p, q) inside the triangular region remaining , and where p and q are integers.

The edges of the triangular region do not count (strict inequalities !), and so we have only the six points highlighted in the right-hand diagram.



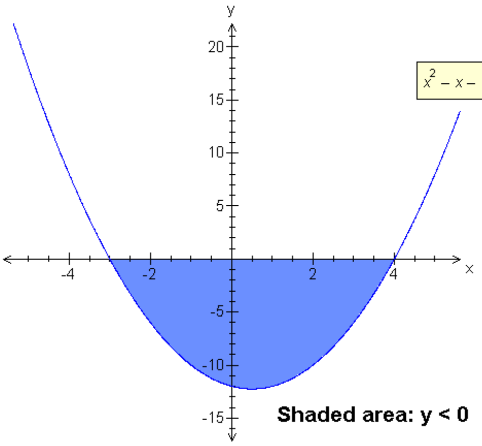
Quadratic inequalities.

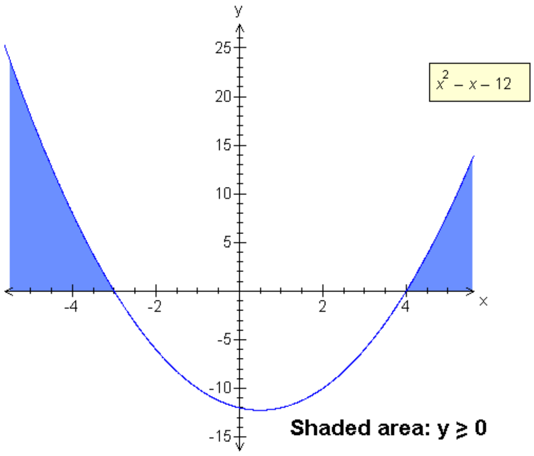
To solve a quadratic inequality, the procedure is to first solve the corresponding equation, followed by either sketching the corresponding graph or producing a table of values.

Examples (6) : Solve the quadratic inequalities a) $x^2 - x - 12 < 0$; b) $x^2 - x - 12 \geq 0$.

The corresponding equation factorises into $(x+3)(x-4) = 0$, giving roots of –3 and 4.

Sketches of the graphs show that there are two different types of solution.





The solution to a) is shown as a single region in the graph on the left, i.e. where the graph passes below the x-axis. All values of x between –3 and 4 satisfy the inequality $x^2 - x - 12 < 0$. The end values of –3 and 4 do not, since this is a strict inequality.

$\therefore$ The solution of $x^2 - x - 12 < 0$ is therefore $-3 < x < 4$, or in set notation, $\{x: -3 < x < 4\}$.

The solution to b) occurs in two regions as shown in the graph on the right, i.e. where the graph lies above the x-axis. This time all values of x less than or equal to –3 satisfy the inequality $x^2 - x - 12 \geq 0$, as do all values of x greater than or equal to 4. (Note that this is not a strict inequality).

$\therefore$ The solution of $x^2 - x - 12 \geq 0$ is therefore $x \leq -3$ or $x \geq 4$, or in set notation,
$\{x: x \leq -3\} \cup \{x: x \geq 4\}$, i.e. the union of the number sets $x \leq -3$ and $x \geq 4$.

This result cannot be combined into a single inequality: an expression such as $-3 \geq x \geq 4$ is nonsense.

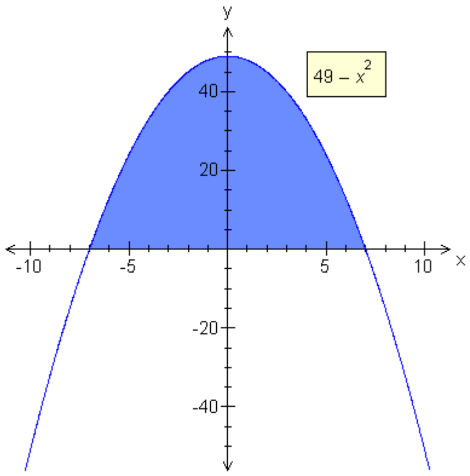
An alternative way of finding the solutions would be to plot a table of the signs of the values of y for values of x within each relevant range of the inequality.

	$x < -3$	$x = -3$	$-3 < x < 4$ (e.g. $x = 0$)	$x = 4$	$x > 4$
$(x + 3)$	(-)	0	(+)	(+)	(+)
$(x - 4)$	(-)	(-)	(-)	0	(+)
Multiply:					
$(x + 3)(x - 4)$	(+)	0	(-)	0	(+)

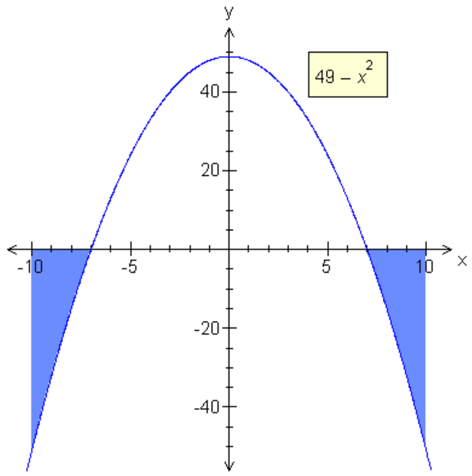
Examples (7) : Solve the quadratic inequalities a) $49 - x^2 > 0$; b) $49 - x^2 \leq 0$.

The corresponding equation is a difference of two squares, factorising into $(7 + x)(7 - x) = 0$, giving roots of –7 and +7.

(This time, the sketch will show an upturned graph since the term in x^2 is negative.)



Shaded area: y > 0



Shaded area: y ≤ 0

The solution to a) is shown on the left. All values of x between –7 and 7 (excluding –7 and 7 themselves) satisfy the strict inequality $49 - x^2 > 0$.

$\therefore$ The solution of $49 - x^2 > 0$ is therefore $-7 < x < 7$, or in set notation, $\{x: -7 < x < 7\}$.

The solution to b) is shown on the right. This time all values of x less than or equal to -7 satisfy the inequality $49 - x^2 \leq 0$, as do all values of x greater than or equal to 7.

$\therefore$ The solution of $49 - x^2 \leq 0$ is therefore $x \leq -7$ or $x \geq 7$,
or in set notation, $\{x: x \leq -7\} \cup \{x: x \geq 7\}$.

Plotting a table of the signs of the values of y for values of x within each relevant range of the inequality gives the following:

	$x < -7$	$x = -7$	$-7 < x < 7$ (e.g. $x = 0$)	$x = 7$	$x > 7$
$(x + 7)$	(-)	0	(+)	(+)	(+)
$(x - 7)$	(-)	(-)	(-)	0	(+)
Multiply:					
$(x + 7)(x - 7)$	(+)	0	(-)	0	(+)

Example (8): The length of a children's football pitch is 16m more than its width. The width of the pitch is x metres.

i) The perimeter of the pitch is no less than 100 metres. Write a linear inequality in x.

ii) The area of the pitch is no greater than 1700 square metres. Show that $(x-34)(x + 50) \le 0$.

iii) By solving the inequalities in i) and ii), determine the set of possible values for x.

i) Using the given data, the perimeter of the pitch is $2((x) + (x + 16))$ metres, or $4x + 32$ m.
The corresponding inequality is therefore $4x + 32 \ge 100 \Rightarrow 4x \ge 68 \Rightarrow x \ge 17$.

ii) The area of the pitch is given by $x(x + 16)$ or $x^2 + 16x$, and we are also told that this area is less than or equal to 1700 square metres.

Hence $x^2 + 16x \le 1700 \Rightarrow x^2 + 16x - 1700 \le 0 \Rightarrow (x-34)(x + 50) \le 0$.

iii) By sketching the graphs of the corresponding equations, we can determine the feasible region satisfying both inequalities.

The region must be to the right of the line $x = 17$ to satisfy $x \ge 17$, and must also be below the curve $(x-34)(x + 50)$ to satisfy $(x-34)(x + 50) \le 0$.

The possible values for x occur in the range $17 \le x \le 34$, so the pitch's width can take any value between 17m and 34m. The values are inclusive as neither inequality is a strict one.