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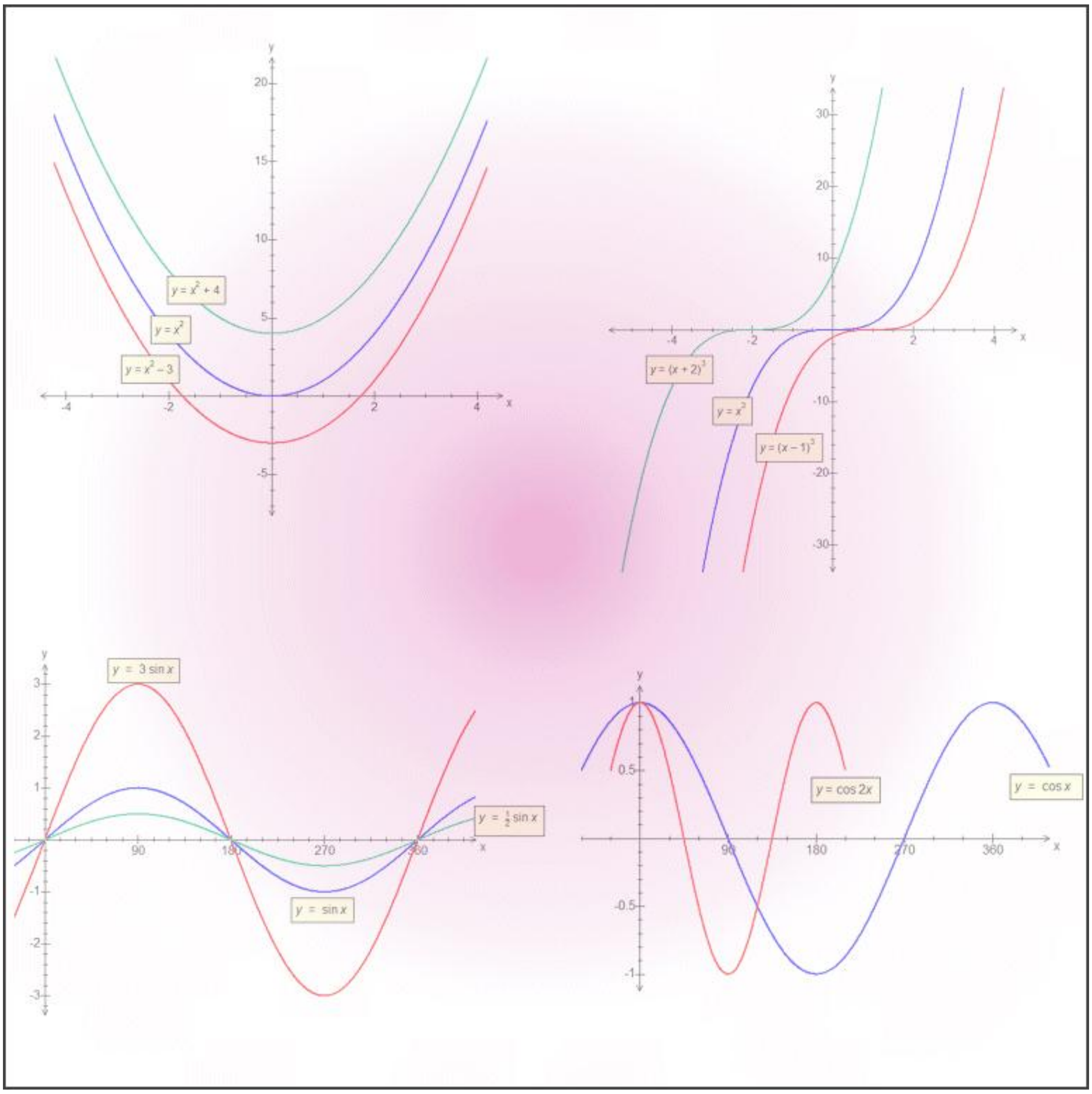
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 1 / AS

TRANSFORMATIONS OF GRAPHS



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TRANSFORMATIONS OF GRAPHS

It is possible to obtain a whole family of graphs from a single one by translating or stretching.

The four transformations to remember are: the *y*-translation, the *x*-translation, the *y*-stretch and the *x*-stretch.

The *y*-translation.

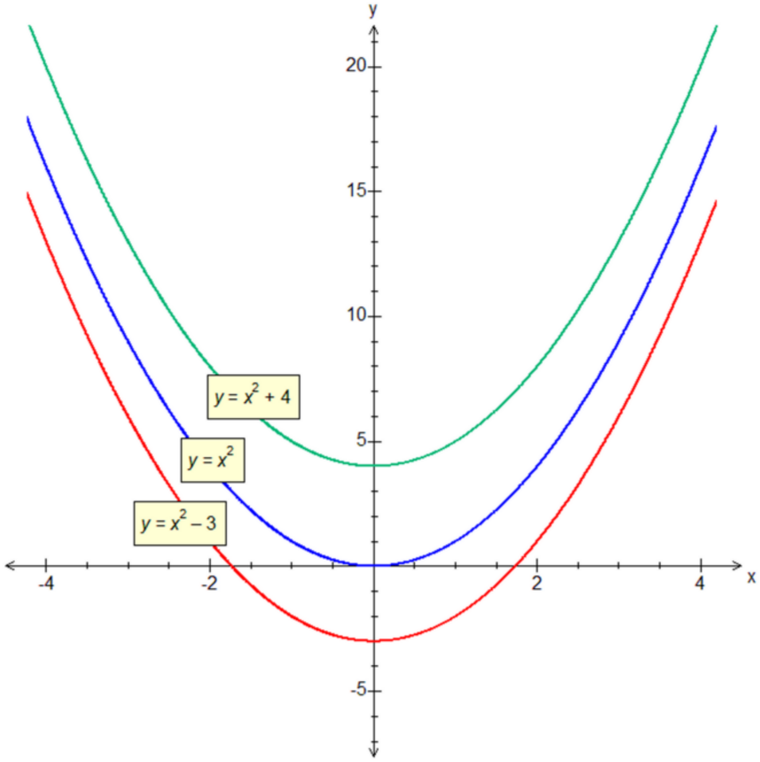
This example will take the function $y = x^2$ and transform it into $y = x^2 + 4$ and $y = x^2 -3$.
The resulting set of graphs is shown below.

The graph of $y = x^2 + 4$ is obtained by translating that of $y = x^2$ upwards by 4 units.

In other words, we apply a translation of +4 units in the *y*-direction.

By contrast, we obtain the graph of $y = x^2 -3$ by translating that of $y = x^2$ downwards by 3 units.

This time, we apply a translation of -3 units in the *y*-direction.



For any function $y = f(x)$, the graph of the function $f(x) + k$ is the same as the graph of $f(x)$, but translated by k units in the *y*-direction. In column vector form, the transformation is $\begin{pmatrix} 0 \\ k \end{pmatrix}$.

In the examples above, the graph of $y = x^2$ is translated to that of $y = x^2 + 4$ by the vector $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$, the and to that of $y = x^2 -3$ by the vector $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$.

The x-translation.

This example will take the function $y = x^3$ and transform it into $y = (x + 2)^3$ and $y = (x - 1)^3$.

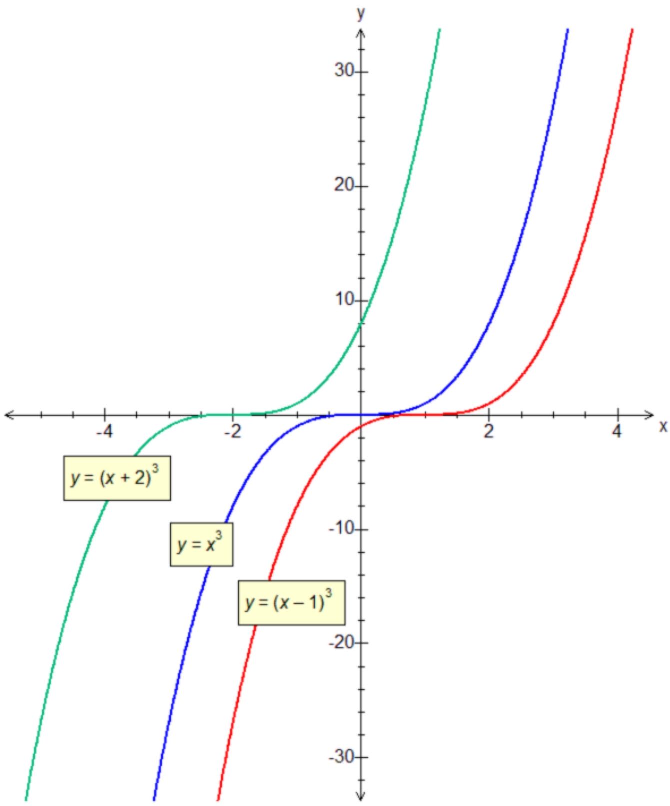
The graph of $y = (x + 2)^3$ is obtained by translating that of $y = x^3$ leftwards by 2 units.

In other words, we apply a translation of -2 units in the x-direction.

Similarly we obtain the graph of $y = (x - 1)^3$ by translating the graph of $y = x^3$ rightwards by one unit.

This time, we apply a translation of +1 unit in the x-direction.

This transformation may seem to work the 'wrong way' at first sight !

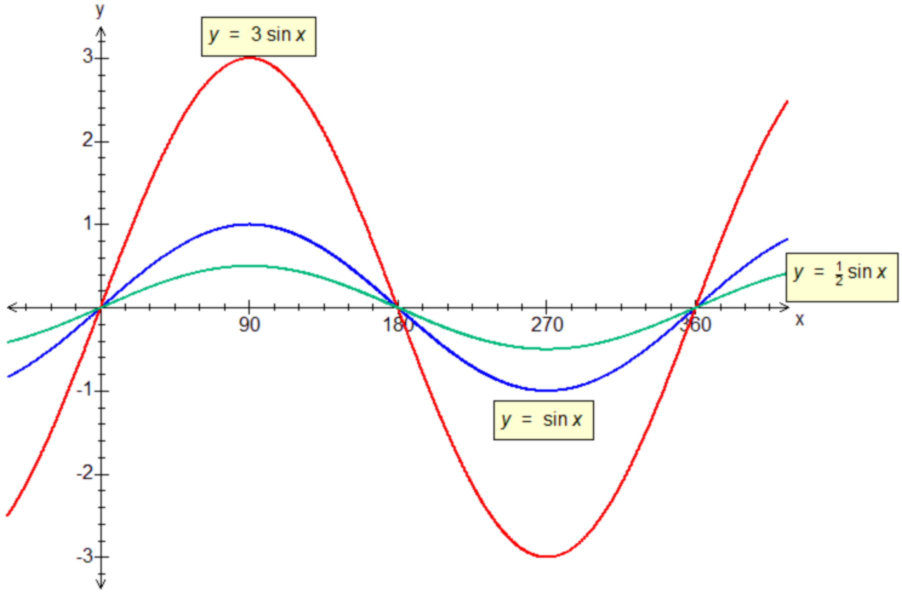


For any function $y = f(x)$, the graph of the function $f(x+k)$ is the same as the graph of $f(x)$, but translated by $-k$ units in the x-direction. In column vector form, the transformation is $\begin{pmatrix} -k \\ 0 \end{pmatrix}$.

In the examples above, the graph of $y=x^3$ is translated to that of $(x + 2)^3$ by the vector $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, and to that of $(x - 1)^3$ by the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The *y*-stretch.

This example will take the function $y = \sin x°$ and transform it into $y = 3 \sin x°$ and $y = ½ \sin x°$.



The graph of $y = 3 \sin x°$ is obtained by stretching that of $y = \sin x°$ with a scale factor of 3 in the *y*-direction.

In other words, the *y*-values are tripled whilst *x*-coordinates are unchanged, leading to a steeper graph.

On the other hand, the graph of $y = ½ \sin x°$ is similarly obtained by stretching that of $y = \sin x°$ with a scale factor of ½ in the *y*-direction.

There, the *y*-coordinates are halved, again leaving *x*-coordinates unchanged, giving a shallower graph.

In general, for any function $y = f(x)$, the graph of the function $kf(x)$ is the same as the graph of $f(x)$, but stretched by a factor of k in the *y*-direction.

In addition, the invariant 'central' line of the stretch is the *x*-axis, which is why the *y*-coordinates are transformed so straightforwardly.

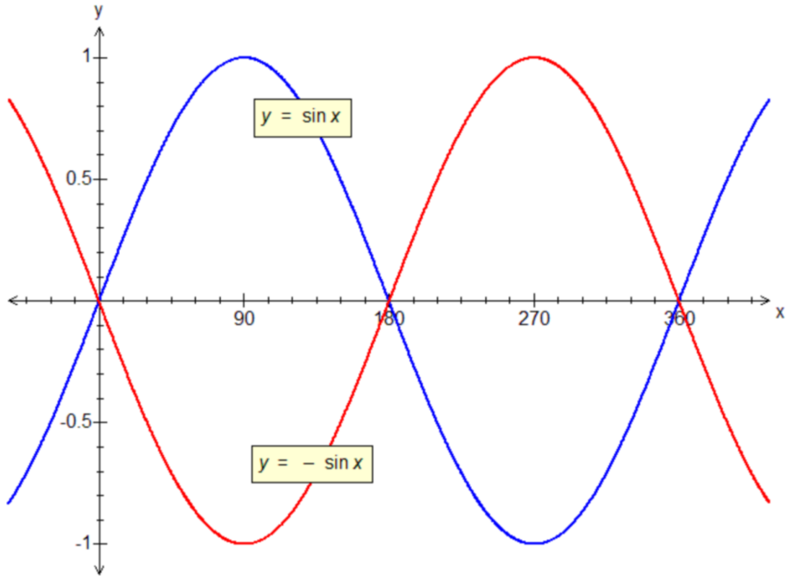
This example used positive values of k - if k is negative, an extra adjustment is needed, as will be shown on the next page.

Special cases of the *y*-stretch.

The special case of $k = -1$, or the graph of $-f(x)$ against $f(x)$, corresponds to a reflection of the graph of $f(x)$ in the x-axis.

The graphs shown here, where $f(x) = \sin x$, illustrate the situation .

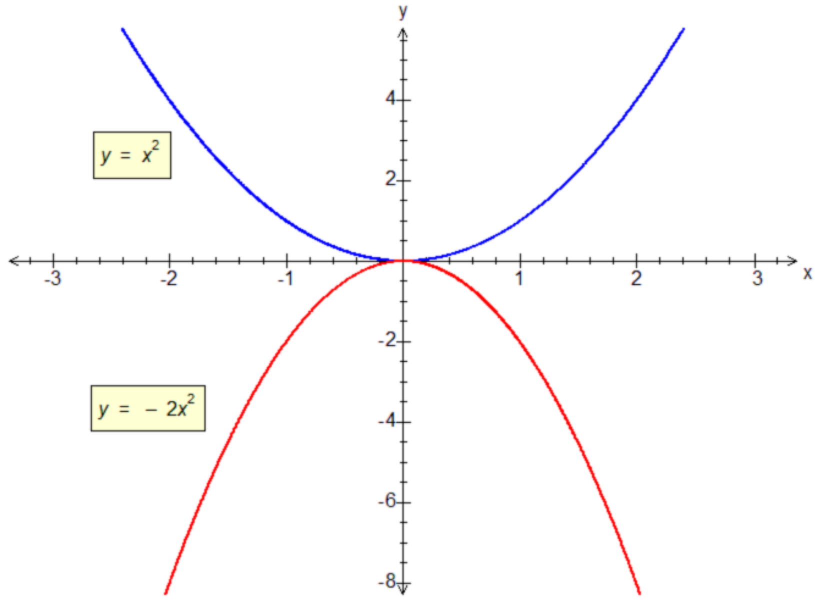
(The y-values have had their sign reversed).



The next graphs show a y-stretch with a scale factor of -2.

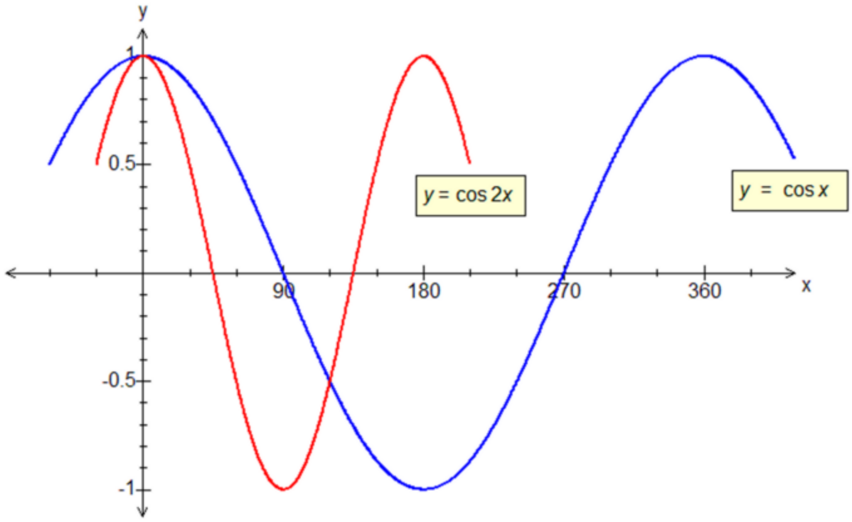
Here, the graph of $y = x^2$ is transformed to $y = -2x^2$.

The transformed graph is obtained by combining a y-stretch of scale factor 2 with a further reflection in the x-axis.



The *x*-stretch.

This example will take the function $y = \cos x°$ and transform it into $y = \cos 2x°$.



The graph of $y = \cos 2x°$ is *compressed* by a factor of 2, or stretched by a factor of ½, in the *x*-direction, relative to that of $y = \cos x°$.

(We have not plotted for $\cos 2x°$ beyond $x = 210°$ in order to show the relationship between the graphs more clearly).

In other words, the *x*-values are halved whilst *y*-values are unchanged.

As in the case of the *x*-translation, the result appears to go the 'wrong way'.

The invariant ‘central’ line of the stretch is the *y*-axis, which is why the *x*-coordinates are transformed so straightforwardly.

In general, for any function $y = f(x)$, the graph of the function $f(kx)$ is the same as the graph of $f(x)$, but stretched by a factor of $(1/k)$ in the *x*-direction.

This example used a positive value for k - if k is negative, an extra adjustment is needed, as will be shown on the next page.

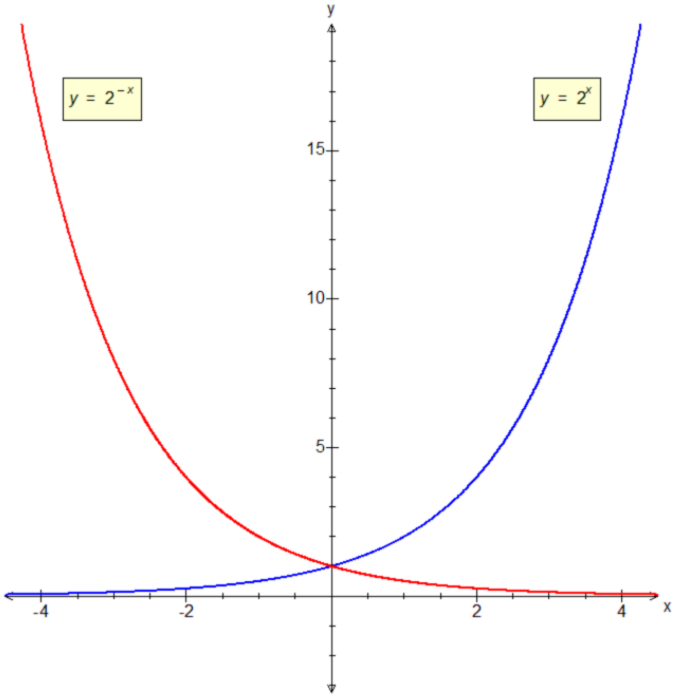
Special case of the *x*-stretch.

The special case of k = -1, or the graph of $f(-x)$ against $f(x)$, corresponds to a reflection of the graph of $f(x)$ in the y-axis.

The graphs shown here, where $f(x) = 2^x$, illustrate the situation .

The transformed graph is that of $y = 2^{-x}$.

(The x-values have had their sign reversed).

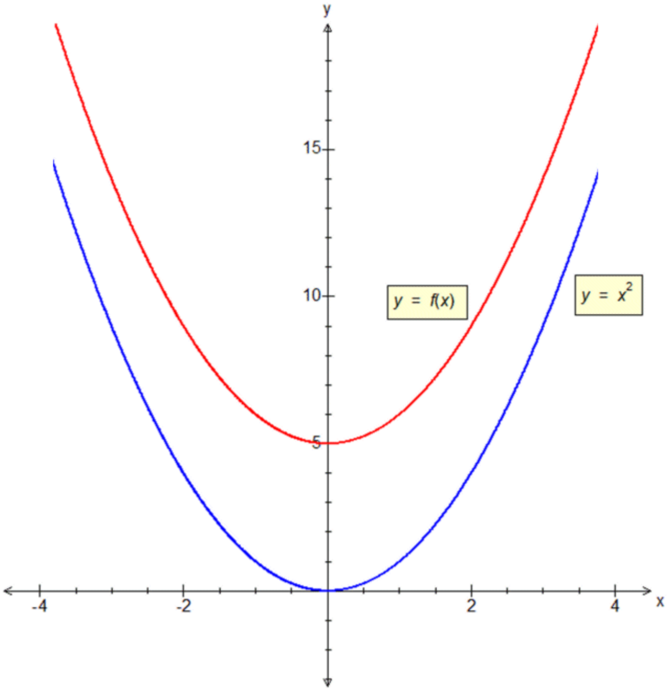


Example (1): Describe the transformation mapping the graph of $y = x^2$ to $y = f(x)$.

Hence find the equation of $f(x)$.

We see that the graph of $y = f(x)$ is a translation of $y = x^2$ by the vector $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$, i.e. a translation in the y-direction by +5 units.

Therefore $f(x) = x^2 + 5$.

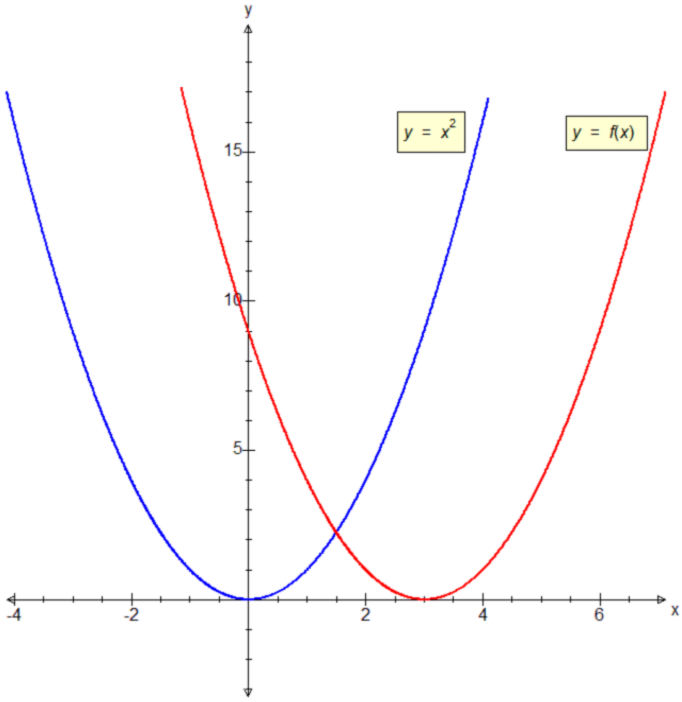


Example (2): Describe the transformation mapping the graph of $y = x^2$ to $y = f(x)$.

Hence find the equation of $f(x)$.

We see that the graph of $y = f(x)$ is a translation of $y = x^2$ by the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, i.e. a translation in the x-direction by +3 units.

Therefore $f(x) = (x - 3)^2$.



Example (3): The graph of $y = x^3$ is transformed to the graph of $y = g(x)$.

The graph of $g(x)$ passes through the point (3, 108).

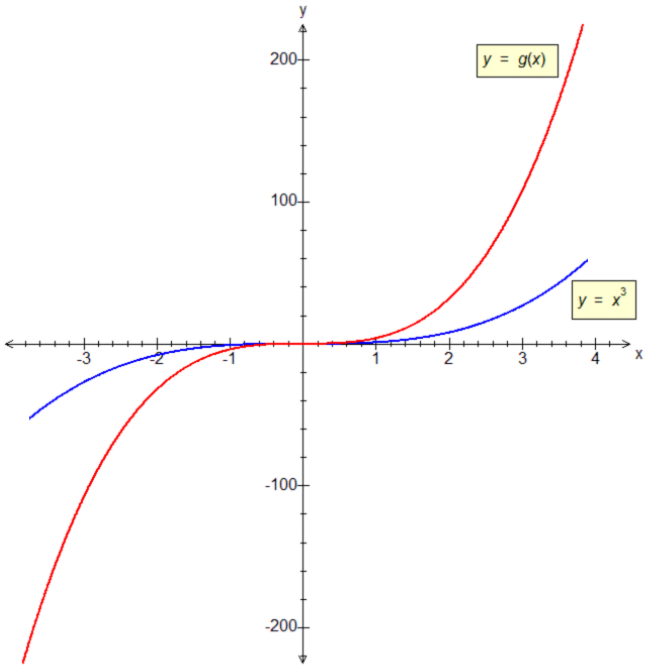
Describe the transformation and hence find the equation of $g(x)$.

The transformation is a stretch in the y-direction, and the scale factor k can be worked out by solving the equation

$kx^3 = 108$ when $x = 3$.

This gives $27k = 108$ and $k = 4$.

The scale factor of the stretch is therefore 4, and the equation of $g(x)$ is $y = 4x^3$.



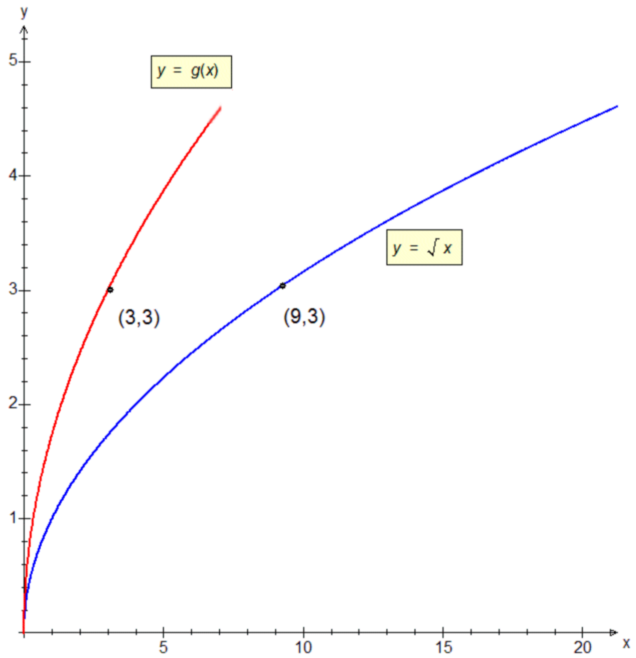
Example (4): The graph of $y = \sqrt{x}$ is transformed to the graph of $y = g(x)$.

We are also told that the point (9,3) on the graph of $y = \sqrt{x}$ is transformed to (3, 3) on the graph of $y = g(x)$.

Describe the transformation and hence find the equation of $g(x)$.

The point (9, 3) has its x-coordinate divided by 3 after the stretch, and so we are dealing with an x-stretch with scale factor of $\frac{1}{3}$, or a compression of scale factor 3 in the x-direction.

The equation of $g(x)$ is therefore $y = \sqrt{(3x)}$.



Examples (5): Describe the transformations required to map:

i) $f(x) = x^2$ to $g(x) = (x+4)^2$

ii) $f(x) = x^3$ to $g(x) = \left(\frac{x}{2}\right)^3$

iii) $f(x) = \sqrt{x}$ to $g(x) = -\sqrt{x}$

iv) $f(x) = \sin x$ to $g(x) = \sin(-x)$

v) $f(x) = x^2 - 3x - 5$ to $g(x) = 2x^2 - 6x - 10$

i) $f(x) = x^2$ is transformed to $g(x) = (x + 4)^2$ by a translation using vector $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$.

ii) $f(x) = x^3$ is transformed to $g(x) = \left(\frac{x}{2}\right)^3$ by an x-stretch with scale factor of 2.

iii) $f(x) = \sqrt{x}$ is transformed to $g(x) = -\sqrt{x}$ by a y-stretch with scale factor -1, which is equivalent to a reflection in the x-axis.

iv) $f(x) = \sin x$ is transformed to $g(x) = \sin(-x)$ by an x-stretch with scale factor -1, which is equivalent to a reflection in the y-axis.

v) We notice here that the expression for $g(x)$ is exactly twice that for $f(x)$. Since all y-values are doubled, the equivalent transformation is a y-stretch with scale factor 2.

Examples (6): The transformations in each case map $f(x)$ to $g(x)$. Find $g(x)$.

i) $f(x) = \sqrt{x}$ by a translation with vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

ii) $f(x) = \frac{1}{x}$ by a y-stretch with scale factor $\frac{1}{4}$

iii) $f(x) = x^3$ by a translation with vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

iv) $f(x) = 2x^2 – 1$ by an x-stretch with scale factor ½

i) Translating $f(x) = \sqrt{x}$ with the vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ gives $g(x) = \sqrt{x} + 2$.

ii) When we transform $f(x) = \frac{1}{x}$ by a y-stretch with scale factor $\frac{1}{4}$, we get $g(x) = \frac{1}{4x}$.

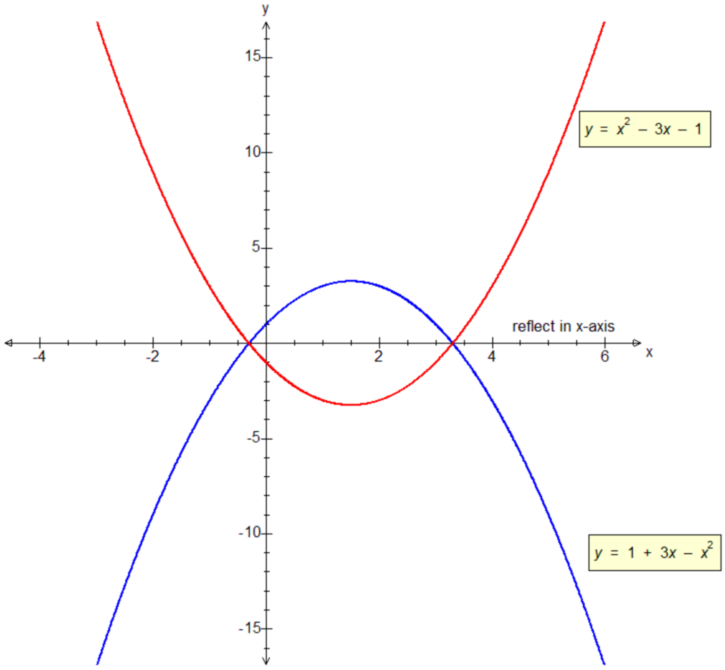
iii) By translating $f(x) = x^3$ with the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, we obtain $g(x) = (x - 3)^3$.

iv) An x-stretch of factor ½ transforms $f(x)$ to $f(2x)$.

Substituting $2x$ for x therefore gives $f(2x) = g(x) = 2(2x)^2 - 1 \rightarrow g(x) = 2(4x^2) - 1 \Rightarrow g(x) = 8x^2 - 1$.

(Beware of this common error: $(2x)^2 = 4x^2$, **not** $2x^2$!)

Example (7): The graph of $y = f(x) = 1 + 3x - x^2$ is reflected in the x-axis to give the graph of $y = g(x)$.
Give the equation of $g(x)$.



Reflection in the x-axis maps $f(x)$ to $-f(x)$, with all y-coordinates multiplied by –1.

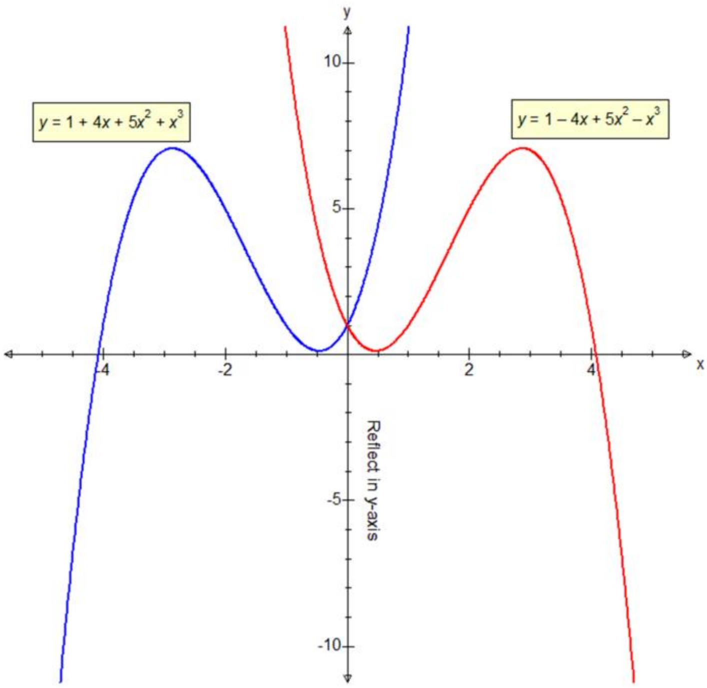
$\therefore$ The coefficients of the terms in the equation of $g(x)$ are exactly those of the terms in $f(x)$ but with the + and - signs reversed.

Hence the equation of $g(x)$ is

$y = -(1 + 3x - x^2) \Rightarrow$
$y = x^2 - 3x - 1$.

Example (8): The graph of $y = f(x) = 1 + 4x + 5x^2 + x^3$ is reflected in the y-axis to give the graph of $g(x)$.

Give the equation of $g(x)$.



Reflection in the y-axis maps $f(x)$ to $f(-x)$, with all references to x replaced by those to $-x$.
$\therefore g(x) = 1 + 4(-x) + 5(-x)^2 + (-x)^3$
$\Rightarrow g(x) = 1 - 4x + 5x^2 - x^3$.

Notice how the coefficients of the even powers of x (and the constant) remain the same, but those of the odd powers of x have their sign reversed.

This is because $(-x)^2 = x^2$, $(-x)^4 = x^4$, and so on for even powers of x.

Conversely $(-x) = -(x)$, $(-x)^3 = -(x^3)$, and so on for odd powers of x.

Summary of single transformations and their effects on graphs of functions.

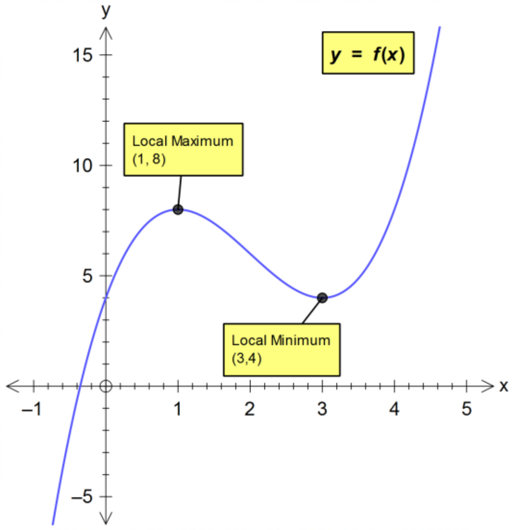
Translations.

We shall look at the effects of translations on the graph of a general function.

The graph of $y = f(x)$ is of an unspecified function.

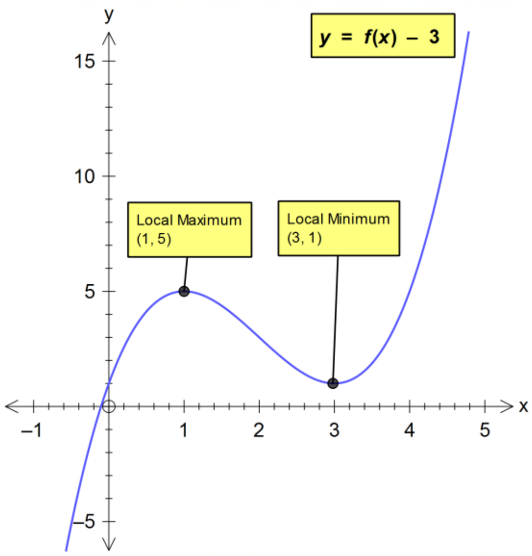
All we are given here is that it has a local minimum at the point (3, 4) and a local maximum at the point (1, 8) .

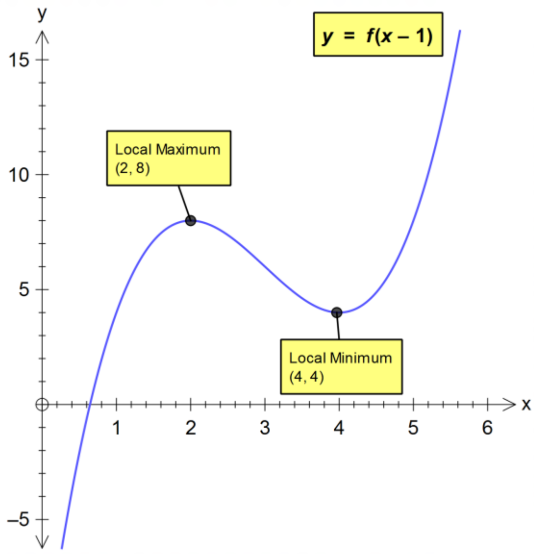
(It has a y-intercept at (0, 4) but we are not concerned about it here).



The left-hand graph below shows the effects of transforming $y = f(x)$ to $y = f(x)$-3.
This is a vertical translation via the vector $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$, so the y-coordinates of the turning points are decreased by 3, with the x-coordinates unchanged.

The right-hand graph below shows the transformation of $y = f(x)$ to $y = f(x$-1).
This is a horizontal translation via the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so the x-coordinates of the turning points are increased by 1, while the y-coordinates stay the same.





Note that points that are roots on the original graph cease to be roots after a y-translation, The graph of another unspecified function, $g(x)$, is shown here.

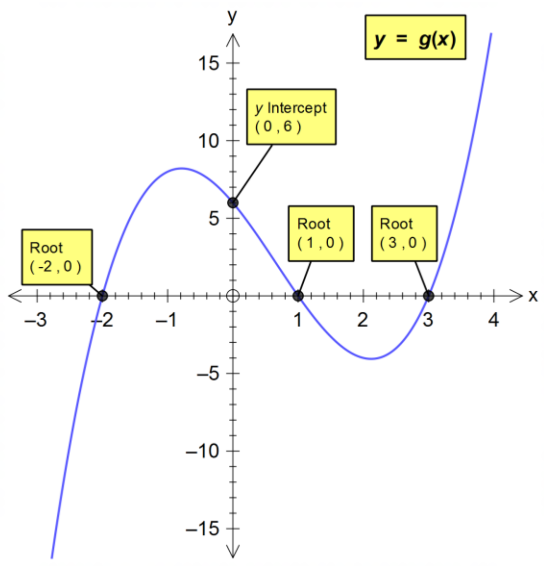
This time, we are given the coordinates of the points where the graph intersects the coordinate axes.

The y-intercept is at (0, 6) and the roots are at (-2, 0), (1, 0) and (3, 0).

Note that points that are roots on the original graph cease to be roots after a y-translation, and the same applies to former y-intercepts after an x-translation.

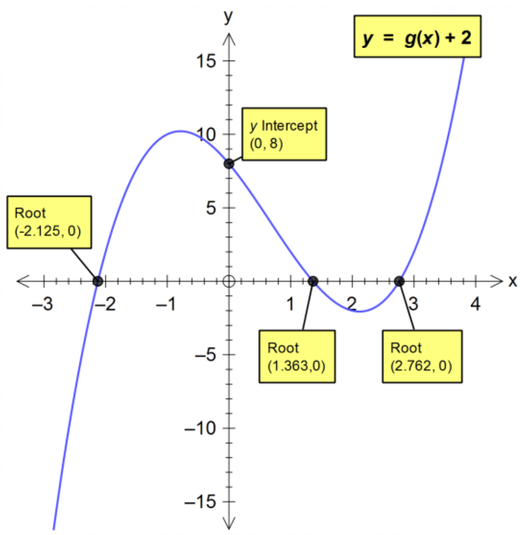
The left-hand graph below shows the effects of transforming $y = g(x)$ to $y = g(x)+2$, which is a vertical translation via the vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

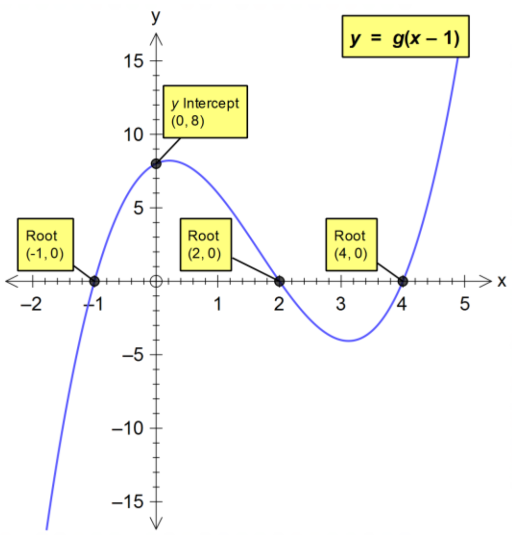
The right-hand graph below shows the transformation of $y = g(x)$ to $y = g(x-1)$, or a horizontal translation via the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.



By looking at the results of the vertical translation, we can see that the y-intercept has been translated up by 2 units to (0, 8) . The roots, however, have had their x-coordinates altered because they do not correspond to those of the original function. This is because the original roots at (-2, 0), (1, 0) and (3, 0) have been translated to (-2, 2), (1, 2) and (3, 2) and are no longer roots of the equation $g(x) + 2 = 0$. In fact, the new roots have to be calculated by using algebraic methods.

With the horizontal translation, the roots have been moved right by 1 unit to (-1, 0), (2, 0) and (5, 0) respectively, and are therefore still roots. The y-intercept of (0, 6) on the original has been transformed to (1, 6), and is thus no longer an intercept; the new one is at (0, 8) and would have to be calculated algebraically.





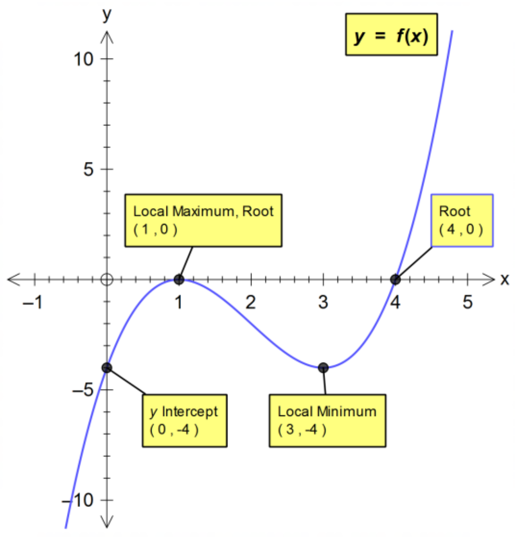
Stretches.

Here we have the graph of another function $f(x)$, but this time we are interested in all the important points of the graph – turning points, y-intercepts and roots.

We have the point (1, 0) which is both a local maximum and a repeated root; then we have another root at (4, 0), a local minimum at (3, -4), and a y-intercept at (0, -4)

The left-hand graph below is a transformation of $y = f(x)$ to $y = 2f(x)$, in other words, a y-stretch with scale factor 2.

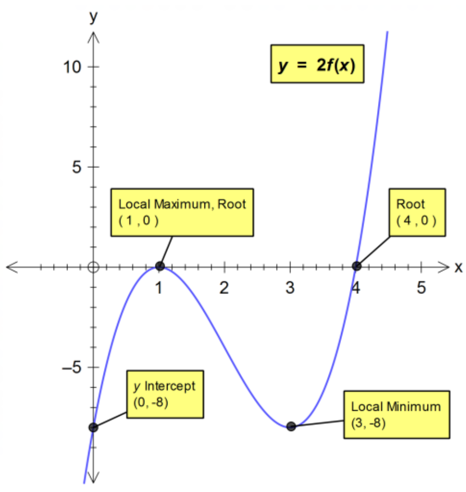
The right-hand graph below is a transformation of $y = f(x)$ to $y = -2f(x)$, in other words, a y-stretch with scale factor -2.

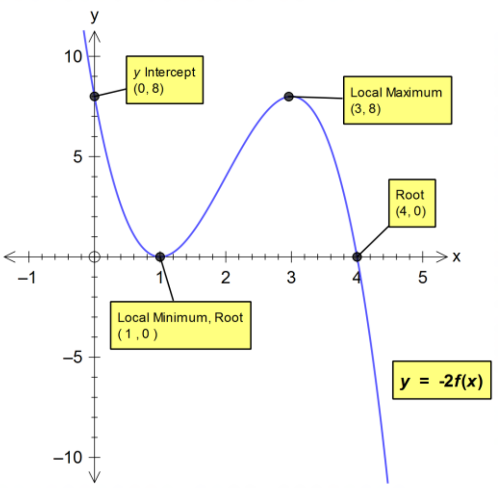


In each graph, the y-coordinates have been multiplied by the respective scale factors and any points on the x-axis, namely the roots, have remained unchanged.

The right-hand graph bears out an important point. If the scale factor is negative, then any local maximum on the original graph is transformed into a local minimum, and vice versa.

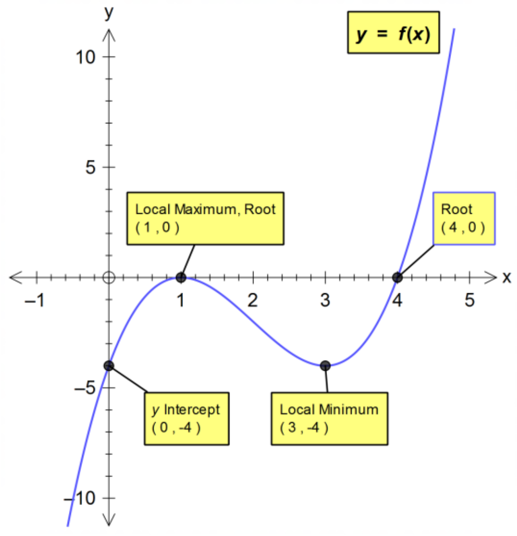
Recall also that a y-stretch with a scale factor of -1 is equivalent to reflection in the x-axis.





The treatment of *x*-stretches is similar.

We take the graph of the same function *f*(*x*) used for the *y*-stretches.

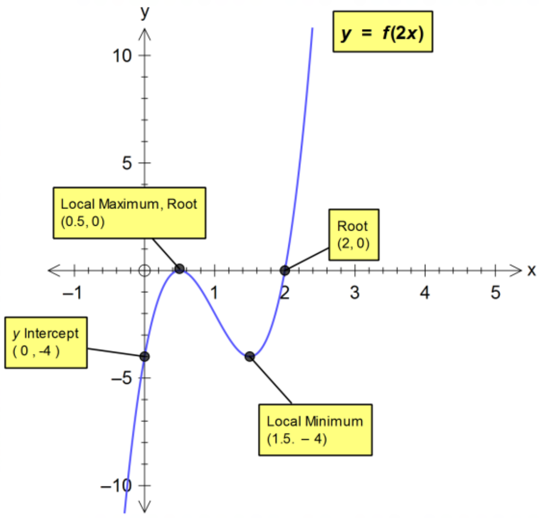


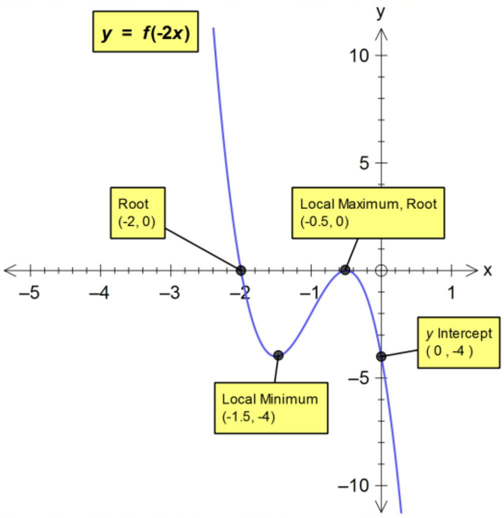
The left-hand graph below is a transformation of $y = f(x)$ to $y = f(2x)$, in other words, an *x*-stretch with scale factor ½.

The right-hand graph below is a transformation of $y = f(x)$ to $y = f(-2x)$, in other words, a *y*-stretch with scale factor -½.

In each graph, the *x*-coordinates have been multiplied by the respective scale factors and any points on the *y*-axis, i.e. the *y*- intercepts, have remained unchanged.

Recall also that an *x*-stretch with a scale factor of -1 is equivalent to reflection in the *y*-axis.



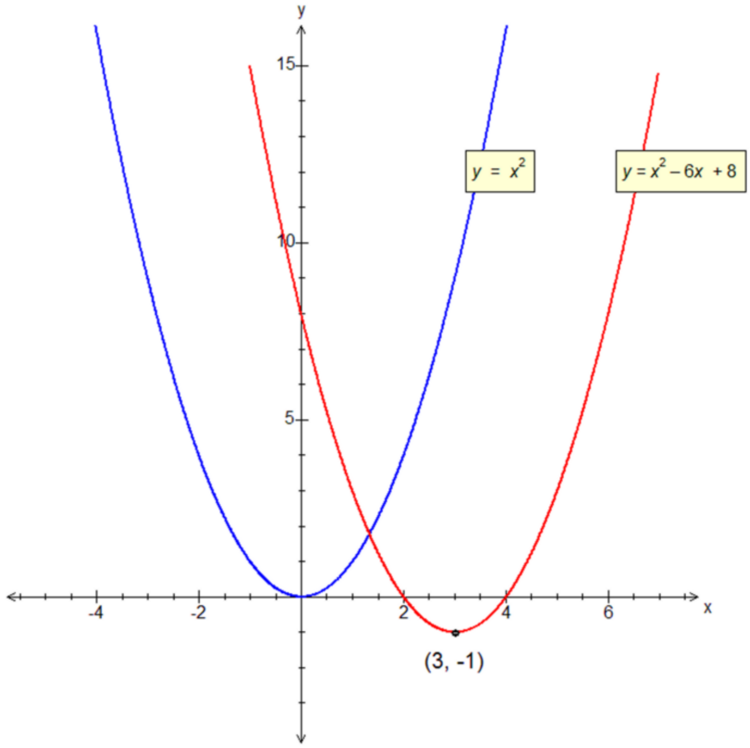


Composite transformations.

It is possible to combine transformations of graphs, as the next examples show.

Example (9): How can we transform the graph of $y = x^2$ to the graph of $y = x^2 - 6x + 8$?

When we look at the two graphs , we can see that they have the same shape, and that the second is a translation of the first.



By inspecting the minimum points of the two graphs, we can see that $y = x^2 - 6x + 8$ is a translation of $y = x^2$ using the vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

In other words, we have translated the graph of $y = x^2$ by 3 units right and 1 unit down.

How could we obtain the vector without plotting the graph ?

The solution lies in completing the square !

$x^2 - 6x + 8 = (x - 3)^2 - 9 + 8$

$= (x - 3)^2 - 1$.

(We are not asked to solve the equation.)

This completed-square expression therefore gives us information about the required transformations.

Starting with $y = x^2$, we begin with an x-translation using the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, thus producing an intermediate result of $y = (x - 3)^2$.

Next, we perform a y-translation using the vector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ to obtain the final result of $y = (x - 3)^2 - 1$.

The two translations can be combined in a single vector, namely $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

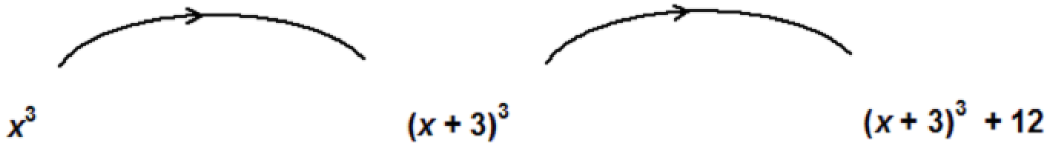
We could therefore have carried out the y-translation first to map $y = x^2$ to $y = x^2 - 1$, followed by the x-translation to map $y = x^2 - 1$ to $y = (x - 3)^2 - 1$.

This is because we are combining two transformations of the same kind, namely vector translations, and these obey the laws of vector addition.

It is however more logical to start within the brackets and perform the x-translation first, which is important when dealing with other types of composite transformations.

Example (10): The graph of $y = x^3$ can be transformed to $y = (x + 3)^3 + 12$ in two separate transformations. Describe them and sketch the graphs.

For compound transformations of this kind, it is often helpful to use function diagrams to break down the process into steps.

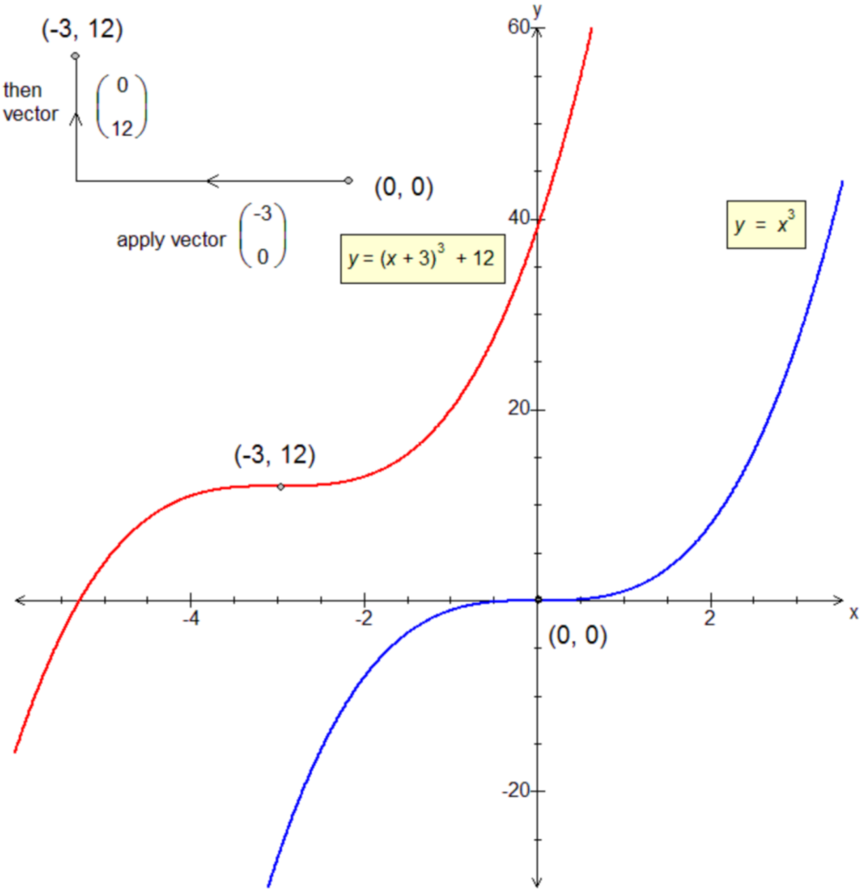


The first step is to get from x^3 to $(x + 3)^3$, and this means a translation using the vector $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$.

The second is to get from $(x + 3)^3$ to $(x + 3)^3 + 12$. This requires a translation using the vector $\begin{pmatrix} 0 \\ 12 \end{pmatrix}$.

The two translations can be combined in one vector, namely $\begin{pmatrix} -3 \\ 12 \end{pmatrix}$.

The graphs are shown below with the point (0,0) on the original for reference. (The intermediate function of $(x + 3)^3$ is not shown.)

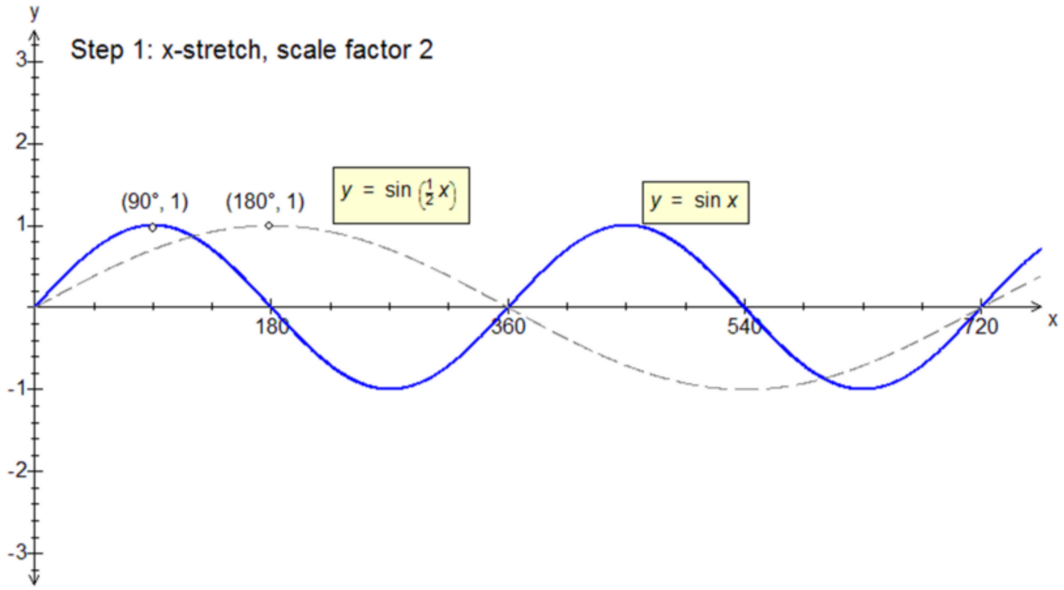


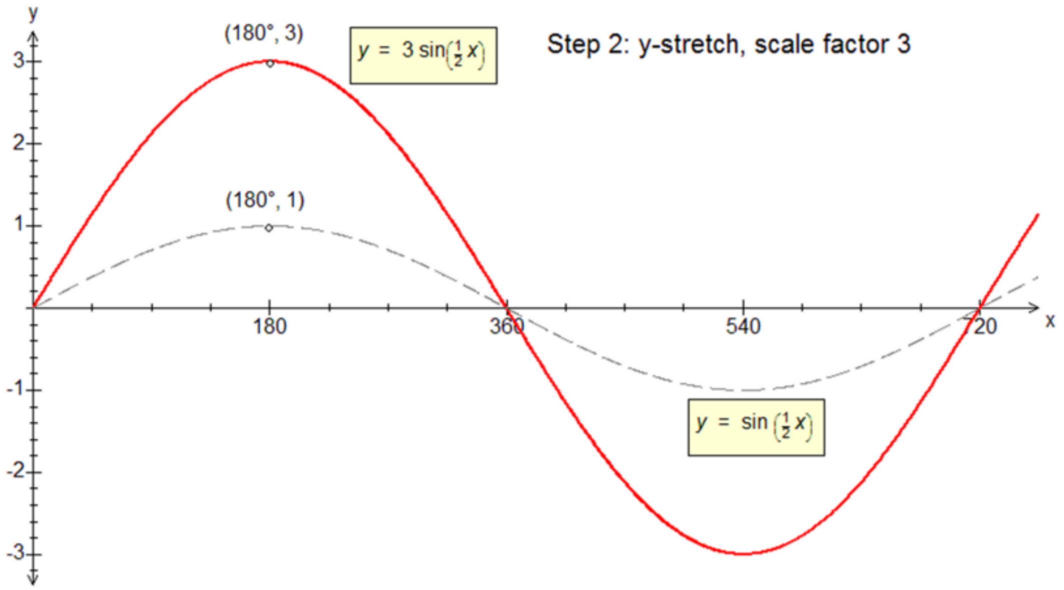
These graphs are drawn accurately here, but a rougher sketch would be acceptable in an examination provided the general shape is recognisable and the vector(s) noted.

Example (11): The graph of $y = \sin x°$ can be mapped to $y = 3 \sin (½x)$ in a two-part transformation. Describe the stages in full, and sketch the graphs for $0° \le x° \le 360°$, showing also the transformations of the starting point (90°, 1).

Firstly, an x-stretch with scale factor 2 transforms $y = \sin x°$ to $y = \sin (½x°)$.
The point (90°, 1) is transformed to (180°, 1).

Secondly, a y-stretch with scale factor 3 transforms $y = \sin (½x°)$ to $y = 3 \sin (½x°)$.
The point (180°, 1) is transformed to (180°, 3).



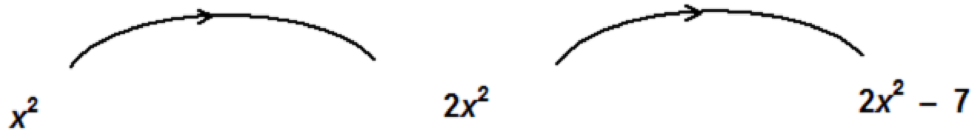


As in example (9), both transformations are of the same kind, i.e. both are stretches.

We could have carried out the y-stretch first and still obtained the same result, but the order shown above is more logical, starting with the 'inner' transformation.

The last examples involved transformations all of one kind.
It is however possible to combine different types of transformation, namely translations and stretches.

Example (12): The graph of $y = x^2$ can be transformed to $y = 2x^2 - 7$ in two separate transformations. Describe them and sketch the graphs.

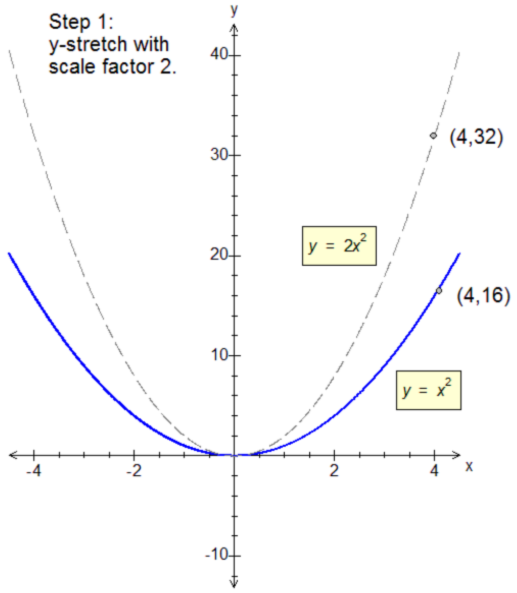


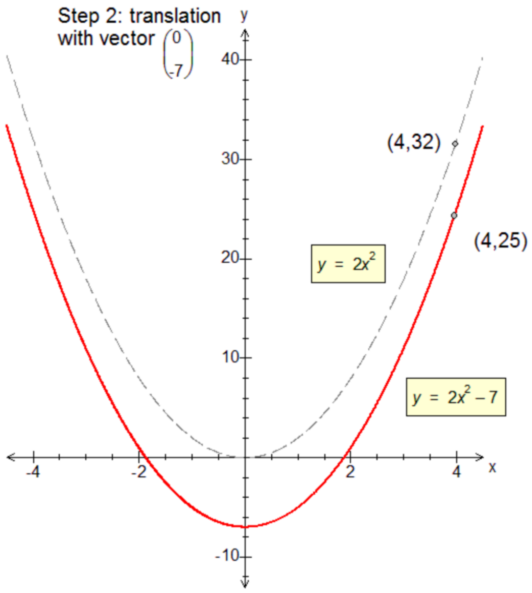
Step 1: transform $y = x^2$ to $y = 2x^2$. This involves a y-stretch with scale factor 2.
Step 2: transform $y = 2x^2$ to $y = 2x^2 - 7$. This involves a translation using the vector $\begin{pmatrix} 0 \\ -7 \end{pmatrix}$.

The graphs below illustrate the processes, and we have chosen the point (4, 16) as a marker.

In step 1, the point (4, 16) to transformed to (4, 32) as the y-coordinate is doubled in the stretch.
In step 2, point (4, 32) is translated down 7 units to (4, 25).

Notice the minimum point of the graph after the combined transformation; it is at (0, -7).





It is important to carry out composite transformations **in the correct order**, as the next example will show . This is especially true when the transformations are of different kinds !

Example (12a): What happens if we attempt to repeat the transformations in Example (12), but in the incorrect order ?

$$x^2 \longrightarrow x^2 - 7 \longrightarrow 2(x^2 - 7)$$

$$= 2x^2 - 14$$

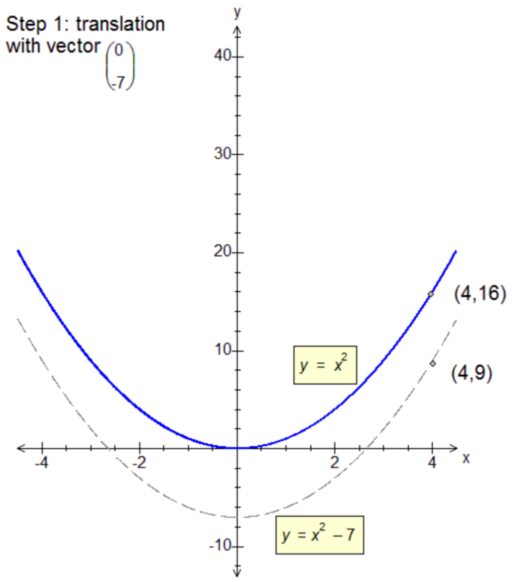
Step 1: we translate with the vector $\begin{pmatrix} 0 \\ -7 \end{pmatrix}$, where $y = x^2$ is transformed to $y = x^2 - 7$. So far, so good.

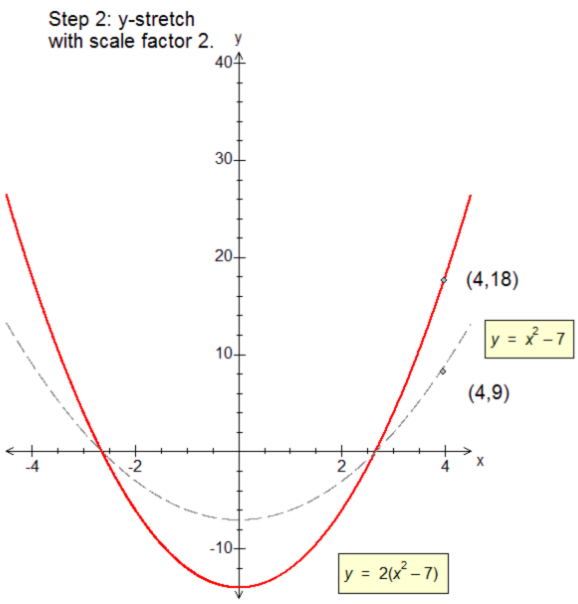
Step 2: we perform the y-stretch with scale factor 2.
We now have a problem, because it is the whole of $y = x^2 - 7$ which is doubled, and not just x^2 itself.

The end result is not $y = 2x^2 - 7$, but $y = 2(x^2 - 7)$ which is $y = 2x^2 - 14$.

Point (4, 16) is now transformed to (4, 9) and then to (4, 18).
In Example (9), the transformations were (4, 16) $\rightarrow$ (4, 32) $\rightarrow$ (4, 25).

The minimum point on the graph is now at (0, -14), not (0, -7).





Example (13): The graph of $y = \sin x$ is transformed to that of $y = 1 - 2 \sin x$.

Describe the transformations in detail, and plot each of the three steps on a separate graph, for $0° \leq x° \leq 360°$.

What range of values can $y = 1 - 2 \sin x$ take, and how is it related to the transformations ?

$$\sin x \rightarrow 2 \sin x \rightarrow -2 \sin x \rightarrow 1 - 2 \sin x$$

Step 1: a y-stretch of scale factor 2 to transform $y = \sin x$ to $y = 2 \sin x$.
Step 2: a reflection in the x-axis to transform $y = 2 \sin x$ to $y = -2 \sin x$.

(Steps 1 and 2 could be said to consist of a single y-stretch of scale factor -2.)

Step 3: a translation with vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to transform $y = -2 \sin x$ to $y = 1 - 2 \sin x$.

The original function $y = \sin x$ can only take values between -1 and 1, i.e. the range is $-1 \leq y \leq 1$.

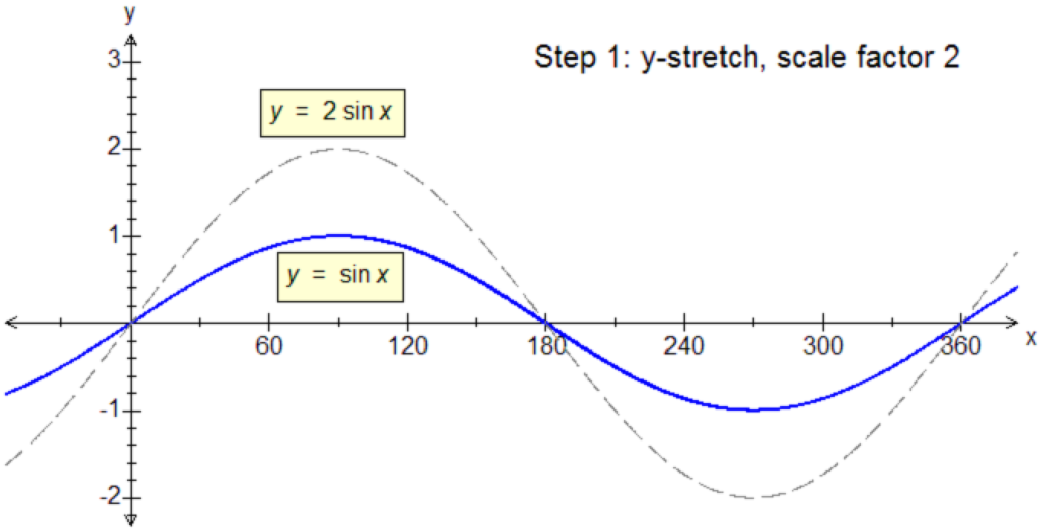
After Step 1, all the y-coordinates are doubled, so the range becomes transformed to $-2 \leq y \leq 2$.

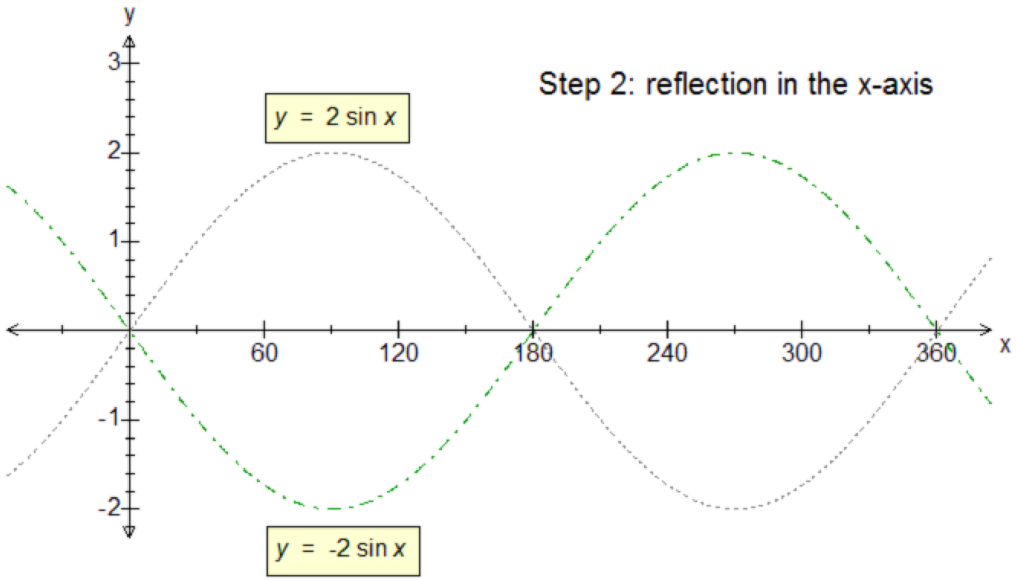
Step 2 has no effect on the range of $y = -2 \sin x$, as it still remains between -2 and 2.

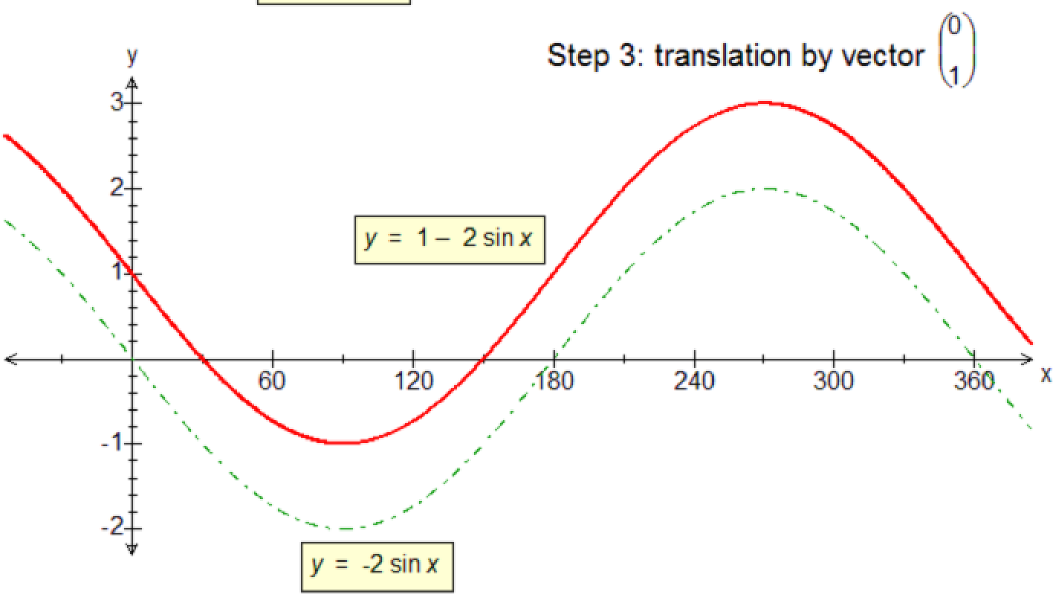
In Step 3, all the y-coordinates are increased by 1, and so the range is transformed again to $-1 \leq y \leq 3$.

$\therefore$ the function $y = 1 - 2 \sin x$ takes a range of values from -1 to 3 inclusive.

See the next page for the graphs.


Step 1: y-stretch, scale factor 2
y = 2 sin x
y = sin x
y
x
60
120
180
240
300
360


Step 2: reflection in the x-axis
y = 2 sin x
y = -2 sin x
y
x
60
120
180
240
300
360


Step 3: translation by vector (0 1)
y = 1 – 2 sin x
y = -2 sin x
y
x
60
120
180
240
300
360

Example (14): (After Example (6) of 'Polynomials' – full working of factorisation process given there).

(Original question copyright OCR 2004, MEI Mathematics Practice Paper C1-A, Q.7)

Given that $(x\text{-}2)$ is a factor of $f(x) = x^3 - x^2 - 4x + 4$, find the equation of the function $g(x)$, obtained by transforming $f(x)$ by firstly an x-translation of $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, followed by a y-translation of $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$.

The first step is to factorise the cubic. Dividing $f(x)$ by $(x\text{-}2)$ gives a quotient of $x^2 + x - 2$, which in turn factorises to $(x + 2)(x - 1)$.

$\therefore$ $\mathbf{x^3 - x^2 - 4x + 4 = (x - 2)(x + 2)(x - 1).}$

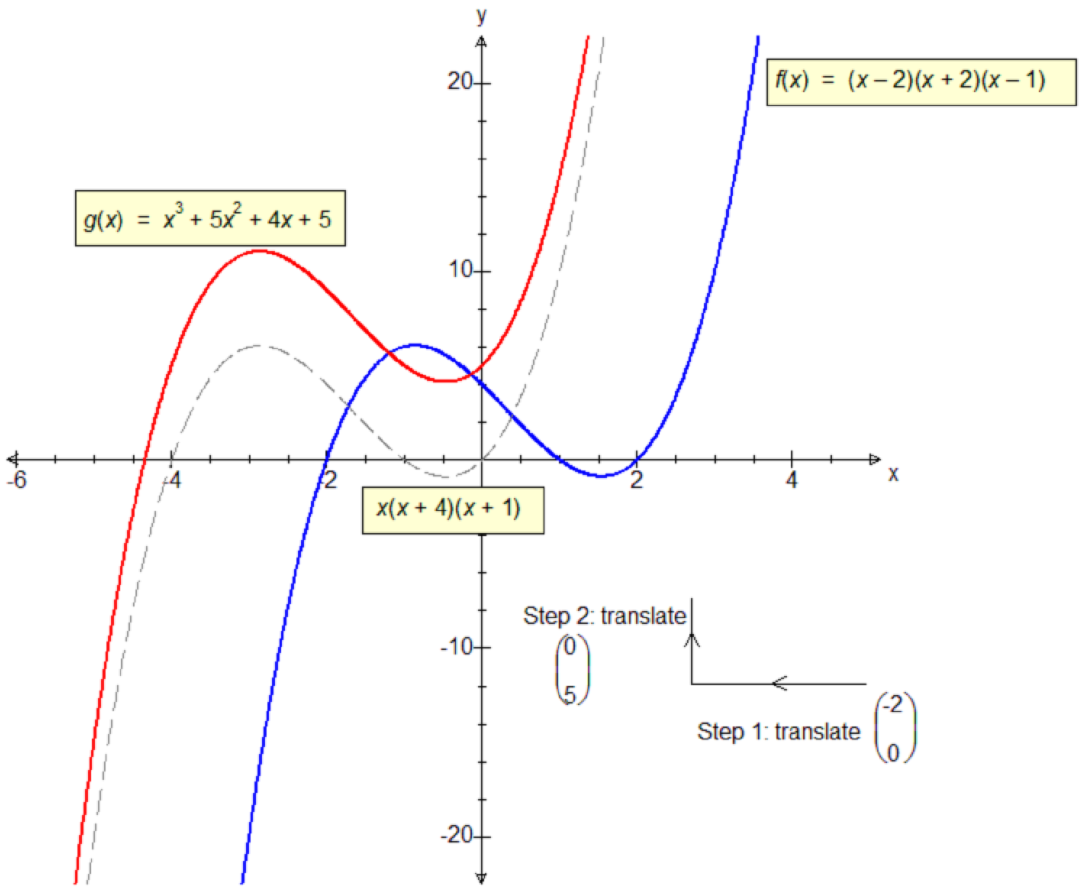
The first part, i.e. the x-translation, transforms $f(x)$ to $f(x+2)$.

Substituting $(x+2)$ for x therefore gives the intermediate function of
$((x - 2) + 2)((x + 2) + 2)((x - 1) + 2)$, simplifying to $\mathbf{x(x + 4)(x + 1).}$

The second part of the transformation, namely the y-translation, gives
$\mathbf{g(x) = x(x + 4)(x + 1) + 5.}$

This result can either be left in quotient-remainder form as above or expanded to give

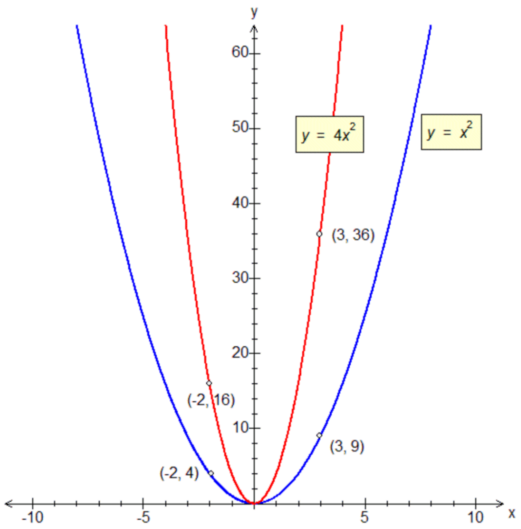
$g(x) = x(x^2 + 5x + 4) + 5 \Rightarrow$ $\mathbf{g(x) = x^3 + 5x^2 + 4x + 5.}$



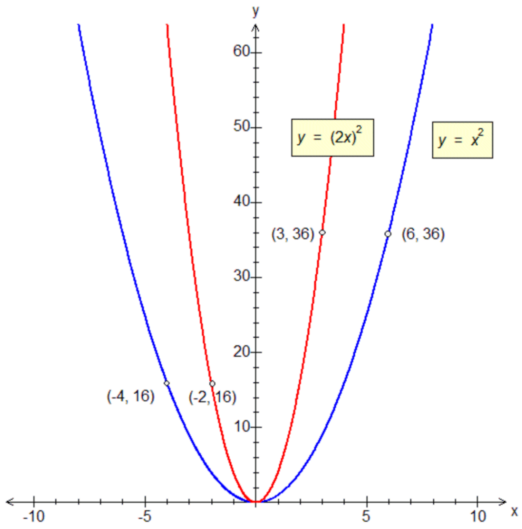
Alternative transformations.

When it comes to certain pairs of functions, it is sometimes possible to express the same transformation in alternative ways.

Example (15): Look at the two transformations of $y = x^2$ below.



y-stretch, scale factor 4



x-stretch, scale factor ½

In the transformation on the left, $y = x^2$ is transformed to $y = 4x^2$.
This is a y-stretch with scale factor 4, so the point (3, 9) is mapped to (3, 36) and (-2, 4) to (-2, 16).

In the transformation on the right, $y = x^2$ is transformed to $y = (2x)^2$.
This is an x-stretch with scale factor ½, so the point (6, 36) is mapped to (3, 36) and (-4, 16) to (-2, 16).

Both transformations represent the same function, since $(2x)^2 = 4x^2$.

Example (16): Show that a y-stretch with scale factor k and an x-stretch with the same scale factor k are equivalent transformations of the graph of $y = \frac{1}{x}$.

A y-stretch with scale factor k transforms the graph of $y = \frac{1}{x}$ to that of $y = \frac{k}{x}$.

An x-stretch with scale factor k transforms the graph of $y = \frac{1}{x}$ to that of $y = \frac{1}{\left(\frac{1}{k}\right)x}$.

This is equivalent to $y = \frac{1}{\left(\frac{x}{k}\right)}$ or $y = \frac{k}{x}$.

$\therefore$ Both stretches represent the same transformation.

Examples (17a): State the sequence of transformations mapping the graph of x^2 to the following graphs. In addition, give the images of the origin and the point (4, 16) at each stage of the composite transformation.

i) x^2 to $2x^2 – 3$; ii) x^2 to $2(x^2 – 3)$; iii) x^2 to $2(x – 3)^2$

i) x^2 to $2x^2 – 3$

Step 1: transform $y = x^2$ to $y = 2x^2$. This involves a *y*-stretch with scale factor 2.
The origin is unchanged, and (4, 16) is mapped to (4,32).

Step 2: transform $y = 2x^2$ to $y = 2x^2 - 3$. This involves a translation using the vector $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$.

The origin is mapped to (0, -3), and (4, 32) is mapped to (4, 29).

ii) x^2 to $2(x^2 – 3)$

Step 1: transform $y = x^2$ to $y = x^2 - 3$. This involves a translation using the vector $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$.

The origin is mapped to (0, -3), and (4, 16) is mapped to (4,13).

Step 2: transform $y = x^2 - 3$ to $y = 2(x^2 – 3)$. This involves a *y*-stretch with scale factor 2.
The point (0, -3) is mapped to (0, -6), and (4, 13) is mapped to (4,26).

iii) x^2 to $2(x – 3)^2$

Step 1: transform $y = x^2$ to $y = (x- 3)^2$. This involves a translation using the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

The origin is mapped to (3, 0), and (4, 16) is mapped to (7,16).

Step 2: transform $y = (x- 3)^2$ to $y = 2(x- 3)^2$. This involves a *y*-stretch with scale factor 2.
The point (3,0) is unchanged at (3, 0), and (7, 16) is mapped to (7,32).

The three composite transformations in the last example were fairly straightforward to work out, but there still remains one tricky one !

Transformations mapping $f(x)$ to $f(ax + b)$.

These transformations combine x-translations and x-stretches. These can be rather treacherous, with plenty of scope for errors.

Example (17b):
i) What is the solution of $(2x - 3)^2 = 0$?
ii) What sequence of transformations maps the graph of x^2 to the graph of $(2x - 3)^2$?

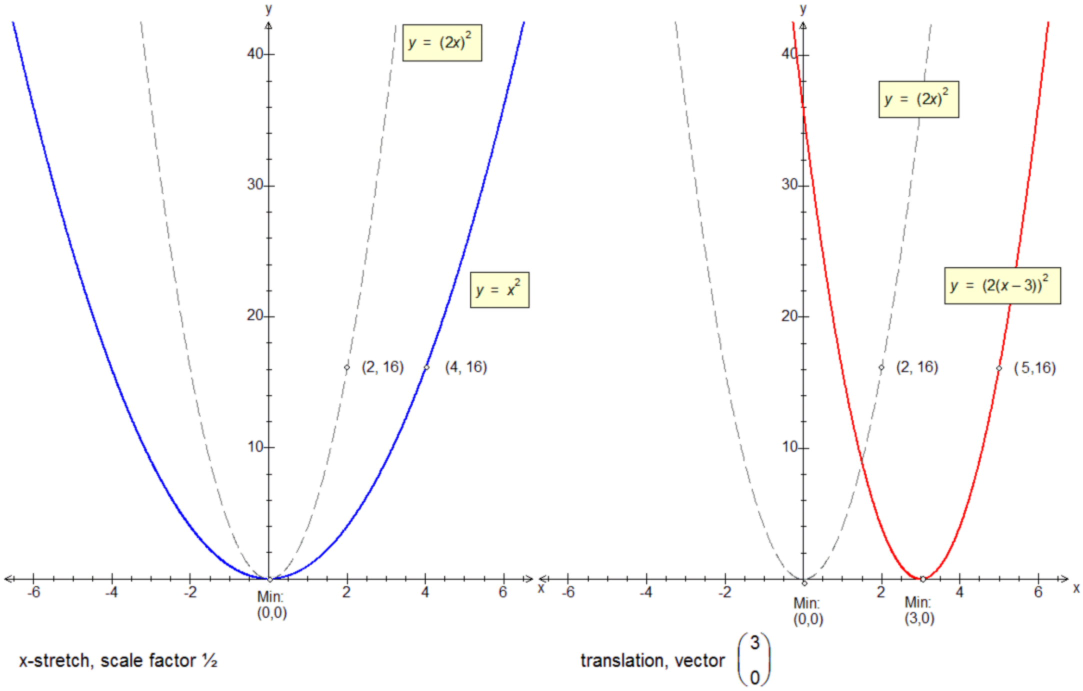
i) The solution of $(2x - 3)^2 = 0$ is $(2x - 3) = 0$ and hence $x = 1\frac{1}{2}$.

ii) We might think that the required sequence of transformations is :

Step 1: an x-stretch with scale factor $\frac{1}{2}$ mapping x^2 to $(2x)^2$

Step 2: a translation by vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ mapping $(2x)^2$ to $(2x - 3)^2$.

This is incorrect, as can be shown below:



The x-stretch mapping x^2 to $(2x)^2$ seems to work correctly – the minimum point is still at the origin, and the point (4, 16) is transformed to (2, 16).

The translation by vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ then maps the minimum point to (3, 0) and (2, 16) to (5, 16) .

However, when we solve $(2x - 3)^2 = 0$, we get $2x = 3$ or $x = 1\frac{1}{2}$, not 3.

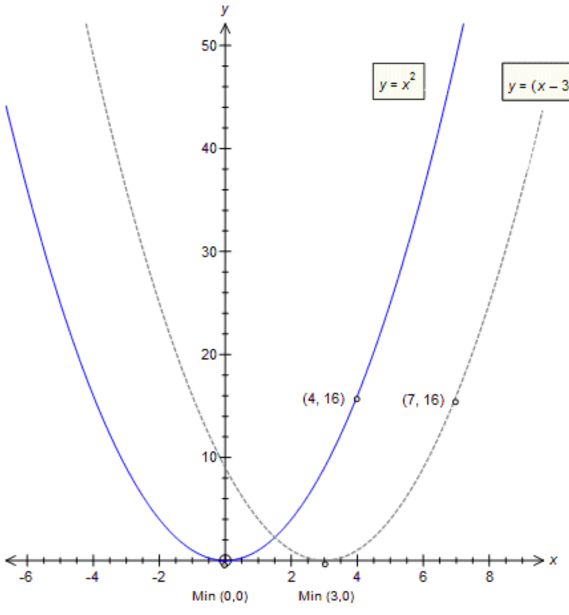
Similarly, solving $(2x - 3)^2 = 16$ gives us $2x - 3 = 4$ or $x = 3\frac{1}{2}$, not 5. (We are not interested in the solution of $2x - 3 = -4$, as this is on the other side of the minimum.)

The translation by vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ seems to have transformed the graph of $(2x)^2$ to that of $(2(x - 3))^2$ or $(2x - 6)^2$ and not $(2x - 3)^2$.

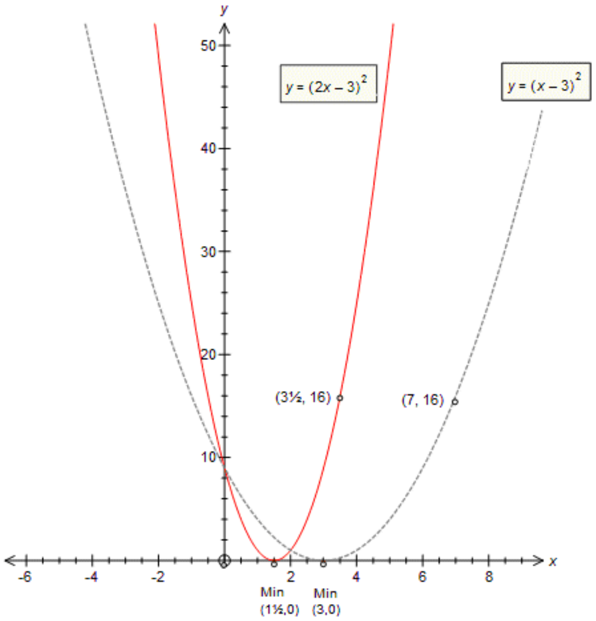
The correct order of transformations is:

Step 1: a translation by vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ mapping x^2 to $(x-3)^2$.

Step 2: an x-stretch with scale factor $\frac{1}{2}$ mapping $(x-3)^2$ to $(2x-3)^2$.



Step 1: Translation, vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$



Step 2: x-stretch, factor ½

The translation with vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, maps the origin to (3, 0) and the point (4, 16) to (7, 16).

Next, the x-stretch with scale factor ½ maps the point (3, 0) to (1½, 0) and the point (7, 16) to the point (3½, 16). (The x-coordinates are halved whilst the y-coordinates are unchanged).

The rules for transforming $f(x)$ to $f(ax+b)$ are :

i) Perform an x-translation by vector $\begin{pmatrix} -b \\ 0 \end{pmatrix}$.

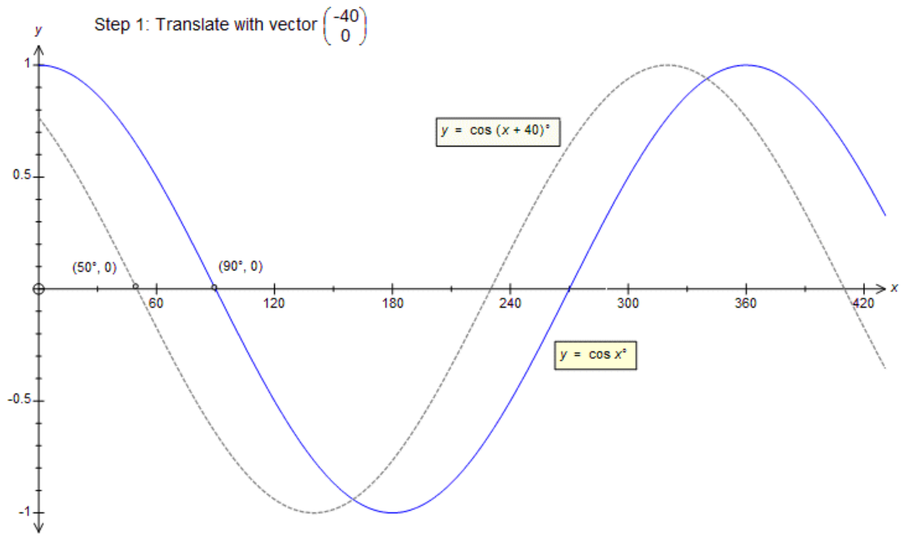
ii) Perform an x-stretch with scale factor $\frac{1}{a}$.

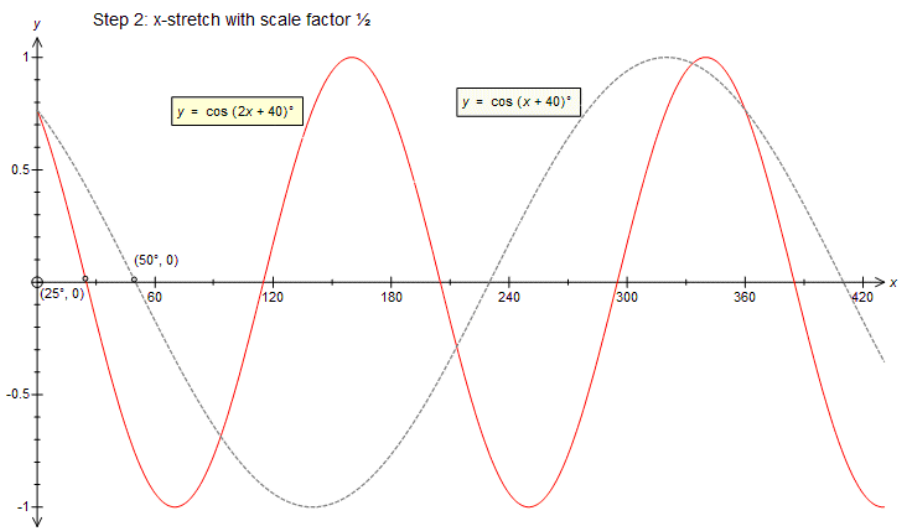
Example (18):

i) What sequence of transformations maps the graph of $y = \cos x°$ to the graph of $y = \cos (2x + 40°)$?
ii) The point (90°, 0) on the graph of $y = \cos x°$ is mapped to (p, q) after the combined transformation. Find the values of p and q.

i) Firstly, we translate in the x-direction by the vector $\begin{pmatrix} -40° \\ 0 \end{pmatrix}$ - this maps $\cos x°$ to $\cos (x + 40°)$.

Secondly, an x-stretch with scale factor ½ maps $\cos (x + 40°)$ to $\cos (2x + 40°)$.





ii) Because the combined transformation only affects x-coordinates of any points, the value of q remains unchanged at 0.

To find the value of p, we solve $(2x + 40°) = 90°$ for x, giving $x = 25°$.
$\therefore$ $(p, q) = (25°, 0)$