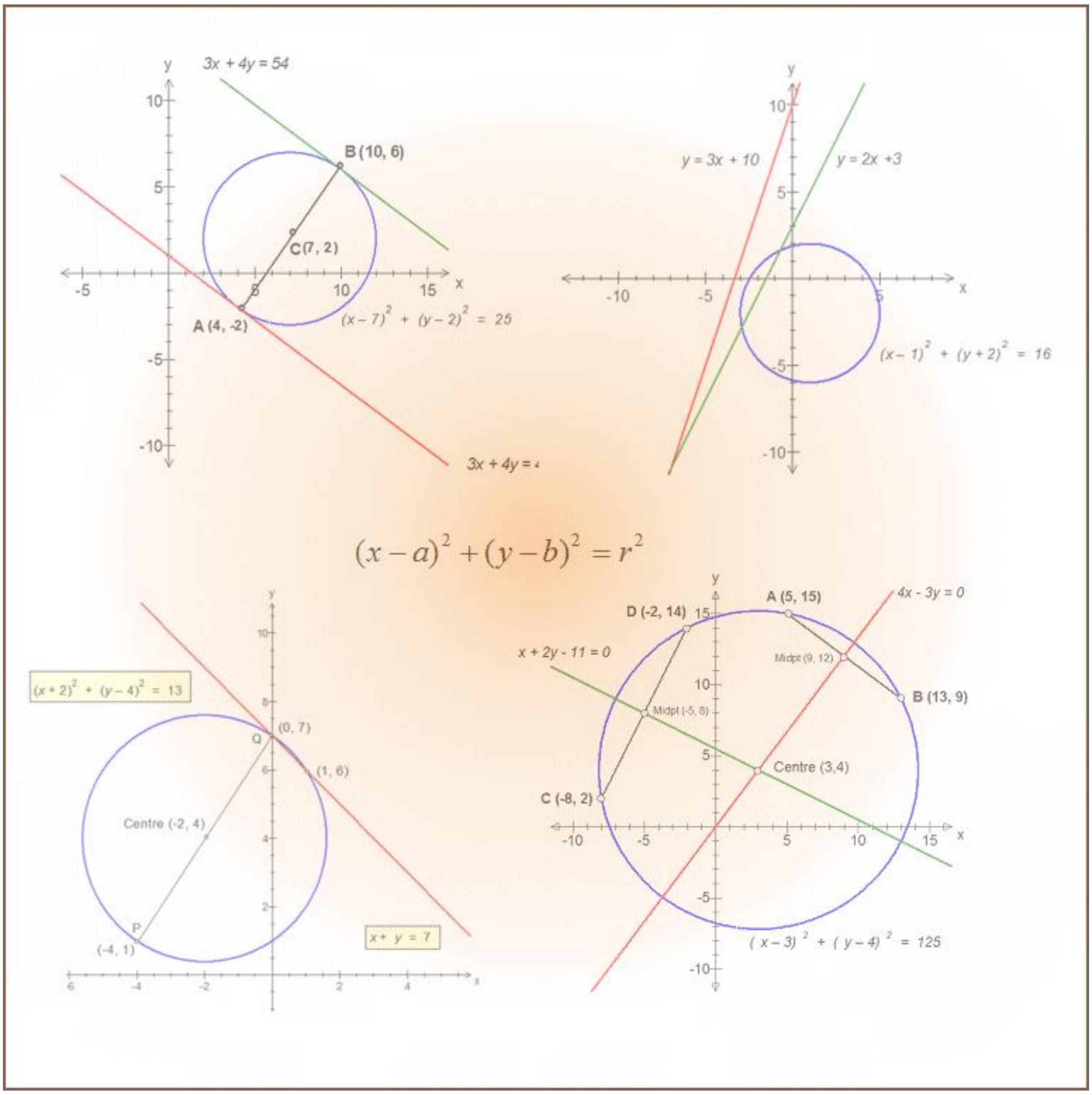
# M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 1 / AS

# COORDINATE GEOMETRY - CIRCLES

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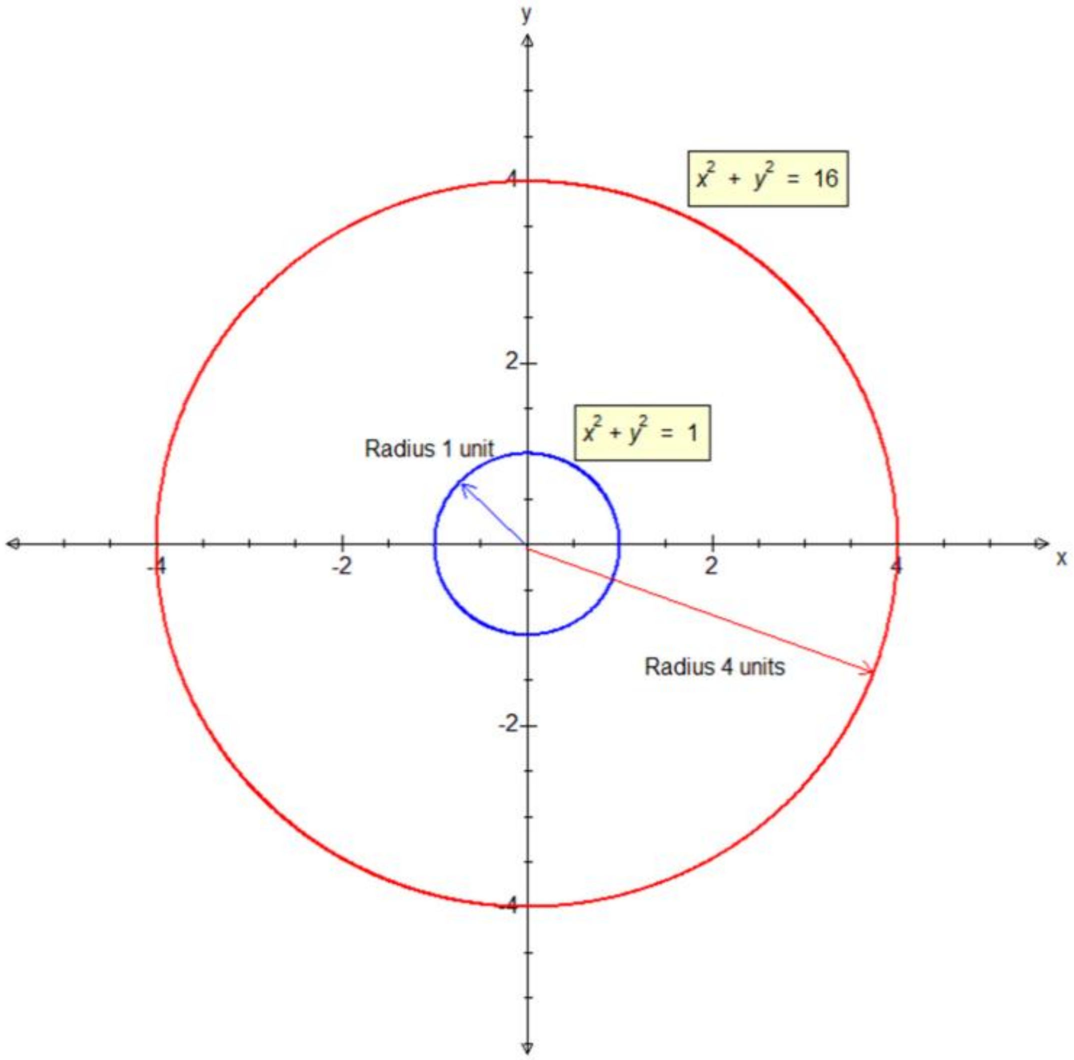
**The equation of a circle.**




Both circles here are centred on the origin; the inner one has a radius of one unit, and the outer one a radius of 4 units.

By using Pythagoras, any point $(x,y)$ on the small inner circle will have coordinates (sin θ, cos θ) for a corresponding angle of θ measured anticlockwise from the positive $x$-axis.

We have expressed $x$ and $y$ in terms of a third variable or **parameter**, θ. This equation, $x = \cos\theta$; $y = \sin\theta$ is the **parametric equation** of a unit circle centred on the origin.

Its Cartesian equation is $x^2 + y^2 = 1$; substituting the parametric values gives the identity $\cos^2\theta + \sin^2\theta = 1$.

(There will be more on parametric equations in Core 3-4).

The larger circle has a radius of 4 units but is still centred on the origin, and its Cartesian equation is $x^2 + y^2 = 16$. Note that the RHS of the equation is equal to the *square* of the radius, not the radius itself.

The example on the right has its centre at the point (3, 4) and a radius of 5 units.

Using Pythagoras, we can show that the origin is on the circle's circumference.

$$\sqrt{(3-0)^2+(4-0)^2}=5$$

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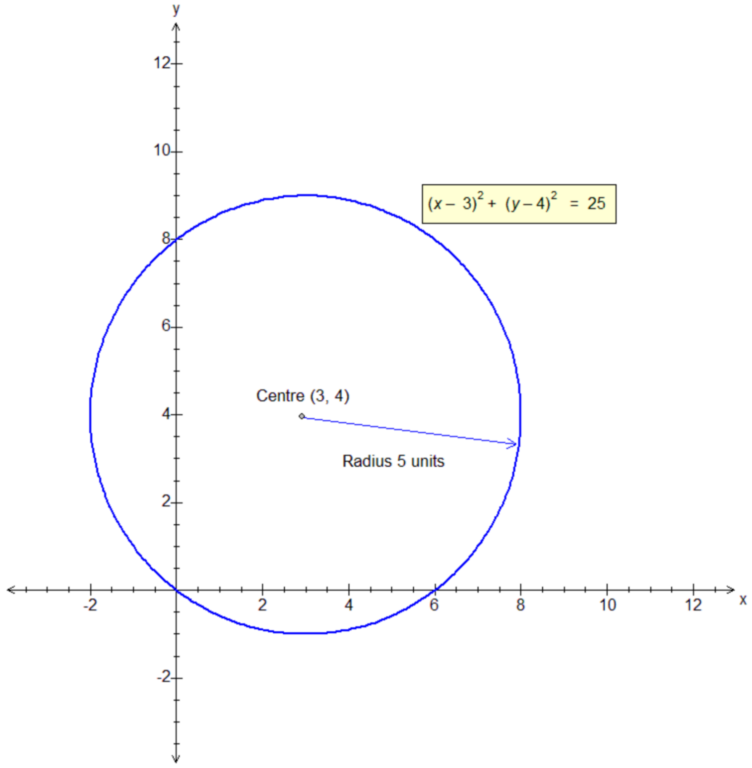
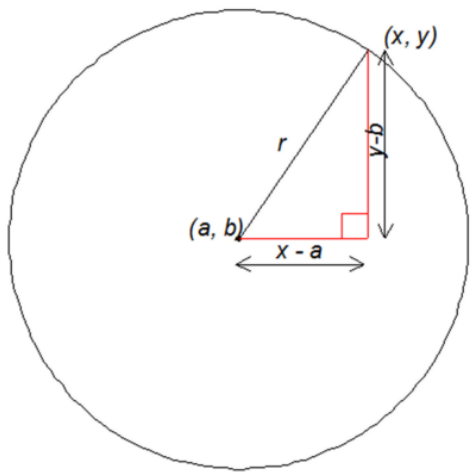



The example above illustrates a general formula:

The equation of a circle with centre ($a$,$b$) and radius $r$ *is*

$$(x-a)^2+(y-b)^2=r^2$$

The diagram on the right shows how the equation is derived from Pythagoras.



**Example (1) :** Find the equation of a circle of radius 6 units centred on (5,0).

Here $r = 6$, $a = 5$ and $b = 0$,
so the equation of the circle is $(x-5)^2 + y^2 = 36$.

**Example (1a):** Find the equation of a circle centred on (2,-3) passing through the point (5, 4).

The equation of the circle is $(x - 2)^2 + (y + 3)^2 = r^2$

We could use Pythagoras to find $r^2 = (5-2)^2 + (4-(-3))^2 = 58$
or we could substitute $x = 5$ and $y = 4$ into the circle equation to obtain $(5 - 2)^2 + (4 + 3)^2 = 58$.

The equation of the circle is $\mathbf{(x - 2)^2 + (y + 3)^2 = 58}$**.**

**Examples (2)** : Find the centre and radius of

i) the circle with equation $x^2 + y^2 + 4x - 10y + 13 = 0$
ii) the circle with equation $x^2 + y^2 + 12x - 14y -15 = 0$

Here we need to collect the terms in $x$ and $y$ and put them into 'completed square' form, followed by bringing the number terms to the right-hand side of the equation.

i) $x^2 + y^2 + 4x - 10y + 13 = 0$

Collecting the terms in $x$ and $y$ we have

$(x^2 + 4x) + (y^2 - 10y) + 13 = 0$.

We then complete the square for both the $x$ and $y$ terms:

$(x^2 + 4x) + (y^2 - 10y) + 13 = 0$
$\Rightarrow (x + 2)^2 - 4 + (y - 5)^2 – 25 + 13 = 0$ (Completing squares)
$\Rightarrow (x + 2)^2 + (y - 5)^2 = 16$ (Bringing numbers to RHS)

Having converted the equation into $(x-a)^2 + (y-b)^2 = r^2$ form, we can now deduce that the circle is centred at (-2, 5) and that it has a radius of 4 units.

ii) $x^2 + y^2 + 12x - 14y -15 = 0$

$\Rightarrow (x^2 + 12x) + (y^2 - 14y) -15 = 0$. (Collecting)
$\Rightarrow (x +6)^2 -36 + (y - 7)^2 – 49 - 15 = 0$ (Completing squares)
$\Rightarrow (x +6)^2 + (y - 7)^2 = 100$ (Bringing numbers to RHS)

$\therefore$ the centre of the circle is (-6, 7) and its radius is 10 units.

**Alternative form of the equation of a circle (Not all syllabuses).**

This form of the equation of a circle is derived from the algebraic processing in the last example.
$x^2 + y^2 + 2gx + 2fy + c = 0.$
The centre of the circle is ($-g$ , $-f$) and the radius is $\sqrt{g^2 + f^2 - c}$ .

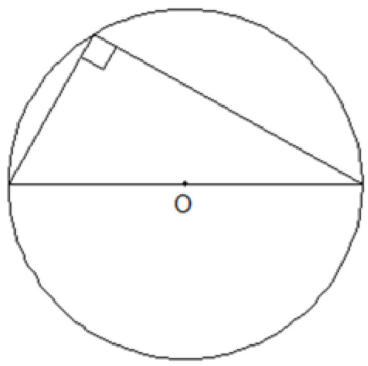
**Examples (3)** : Use the alternative form to find the centre and radius of i) the circle with equation $x^2 + y^2 + 4x - 10y + 13 = 0$ and ii) the circle with equation $x^2 + y^2 + 12x - 14y - 15 = 0$.

i) Here $g$ = 2, $f$ = -5 and $c$ = 13, and thus the centre of the circle is at (-2, 5) and its radius is $\sqrt{(4+25)-13}$ or 4 units.

ii) Here $g$ = 6, $f$ = -7 and $c$ = -15, and thus the centre of the circle is at (-6, 7) and its radius is $\sqrt{(36+49)+15}$ or 10 units.

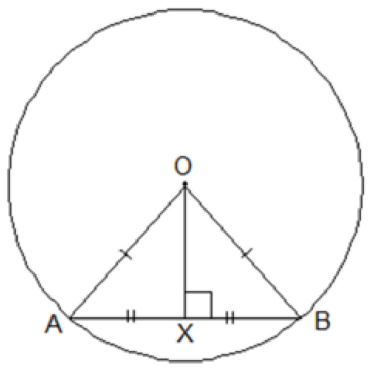
**Revision of circle properties.**

The angle in a semicircle is a right angle.



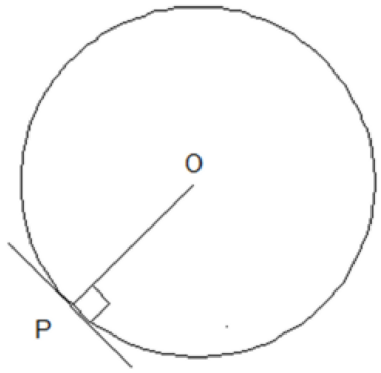

The perpendicular from the centre of a circle to a chord is also a bisector of the chord.

The lengths of *AX* and *XB* are therefore the same. (*OA* = *AB* because they are both radii.)




The tangent to a circle is perpendicular to the radius at the point of contact.

If the gradient of *OP* is known, then the fact that the product of the gradients of perpendicular lines is –1 can be used to find the gradient of the tangent at P.




These properties are often used in examination questions involving circles. Typical questions include

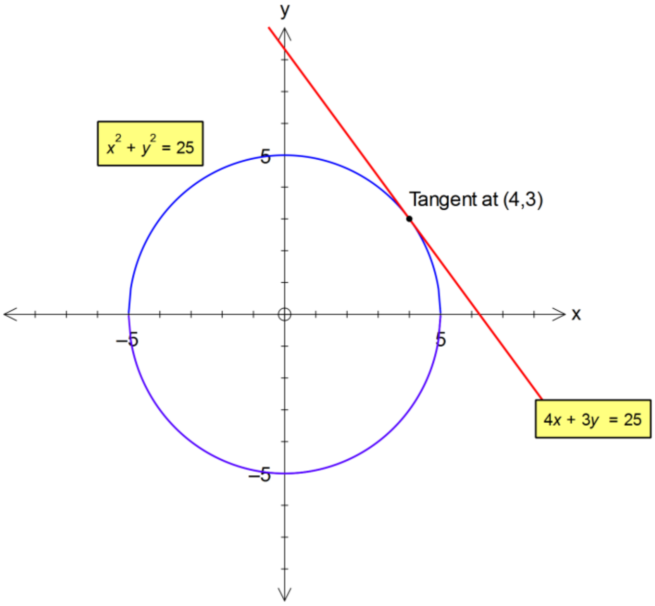
i) finding points of intersection between a circle and a line (see the section on Simultaneous Equations for an example).

ii) finding equations of tangents and normals to a point on a circle's circumference.

iii) finding the equation of a circle given the coordinates of three or more points on its circumference.

**Equation of the tangent to a circle**.

For any point ($a$,$b$) on a circle centred on the origin, and with a radius $r$, the equation of the tangent to the circle at that point is $\boldsymbol{ax + by = r^2}$**.**




In the above example, the equation of the circle is $x^2 + y^2 = 25$, and the selected point on its circumference is (4, 3).

The equation of the tangent is thus $\mathbf{4x + 3y = 25}$.

Another method:

Gradient of radius from (0, 0) to (4, 3) = $\frac{3}{4}$ .

Since the tangent is perpendicular to the radius, its gradient is $-\frac{4}{3}$ , as two perpendicular lines have a gradient product of -1.

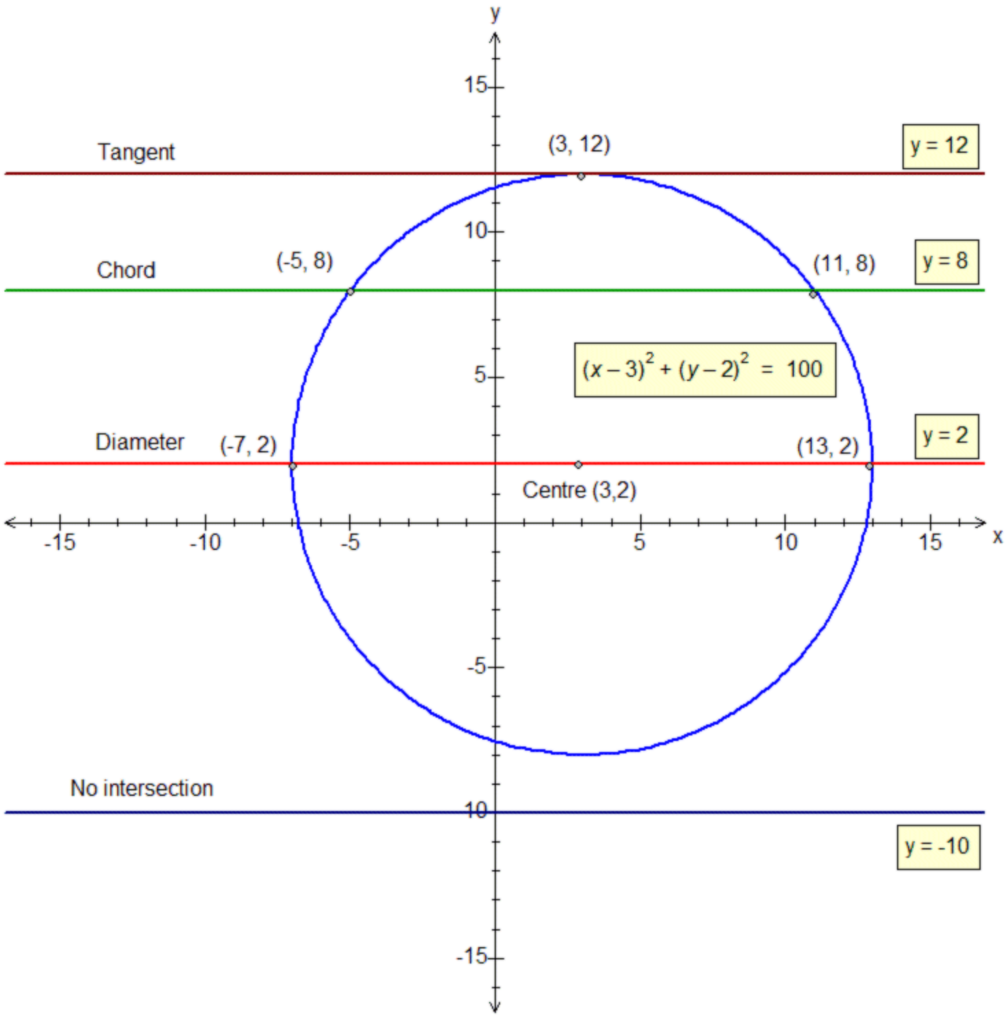
The equation of the tangent is therefore $y = -\frac{4}{3}x + c$.

Substituting $x = 4$, $y = 3$, we have $-\frac{16}{3} + c = 3$.

Hence $c = \frac{25}{3}$ and the equation of the tangent is $\boldsymbol{y = \frac{25}{3} - \frac{4}{3}x}$ .

(This equation is the same as $4x + 3y = 25$, but in $y = mx + c$ form).

**Intersection of a line and a circle.**




The graphs of a line and a circle can meet at two points (a chord) , one point (a tangent) , or not meet at all. The example above shows examples of all cases.

The circle with equation $(x\text{-}3)^2 + (y\text{-}2)^2 = 100$ has a radius of 10 units and a centre of (3, 2).
The four examples of straight lines are all parallel to the $x$-axis for ease of illustration.

When $y = 2$, we can substitute to give $(x\text{-}3)^2 + 0^2 = 100 \Rightarrow x - 3 = \pm 10$.
The line and the circle therefore intersect when $x = 13$ or –7, namely at the points (13, 2) and (-7, 2), i.e. part of the line is a chord. In addition, the midpoint of that line is (3, 2), or coincident with the centre of the circle, and thus the line $y = 2$ includes a diameter.

When $y = 8$, we can substitute to give $(x\text{-}3)^2 + 6^2 = 100 \Rightarrow (x\text{-}3)^2 = 64 \Rightarrow x - 3 = \pm 8$.
The line and the circle therefore intersect when $x = 11$ or -5, namely at the points (11, 8) and (-5,8).
$\therefore$ the line $y = 8$ also includes a chord to the circle.

When $y = 12$, then $(x\text{-}3)^2 + 10^2 = 100 \Rightarrow x\text{-}3 = 0$.
The line and the circle therefore intersect only when $x = 3$ , i.e. at (3, 12)
$\therefore$ the line $y = 12$ is now a tangent to the circle.

When $y = -10$, then $(x\text{-}3)^2 + (-12)^2 = 100 \Rightarrow (x\text{-}3)^2 = -44$.
There is no real solution here, and therefore the line and the circle never intersect.

**Example(4):** A circle has the equation $(x-1)^2 + (y+2)^2 = 16$.
Does it have any points of intersection with either of the lines i) $y = 3x + 10$ ; ii) $y = 2x + 3$ ?
If so, state how many. (Do not calculate the coordinates in full.)

Both equations can be solved using the substitution method for simultaneous equations.
Substituting for $y$ in the first equation

$(x - 1)^2 + (3x + 12)^2 = 16$ *A*
$y = 3x + 10$ *B*

Expanding the equation in *A* gives the quadratic $x^2 - 2x + 1 + 9x^2 + 72x + 144 = 16$
which further simplifies to $10x^2 + 70x + 129 = 0$.

The discriminant of this quadratic, $b^2-4ac$, is 4900 – 5160, or –260 (i.e. < 0), so there are no real roots and the simultaneous equations have no solution , $\therefore$ the circle does not intersect the line $y = 3x + 10$ .

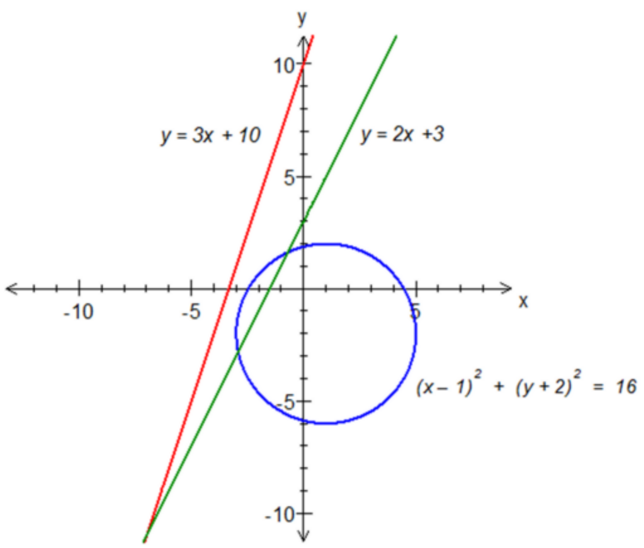
Substituting for $y$ in the second equation

$(x - 1)^2 + (2x + 5)^2 = 16$ *A*
$y = 2x + 3$ *B*

Expanding the equation in *A* gives $x^2 - 2x + 1 + 4x^2 + 20x + 25 = 16$
which simplifies to $5x^2 + 18x + 10 = 0$.

This time $b^2-4ac$ is 324 – 200, or 124 (i.e. > 0) , so there are two real roots and the simultaneous equations have two solutions $\therefore$ the line $y = 2x + 3$ cuts the circle at two points, forming a chord.

See the diagram below.



**Example (5):** A circle has equation $x^2 + y^2 + 4x – 8y + 7 = 0$.

i) Find the centre and radius of the circle.
ii) The circle passes through the point $P$ (-4, $k$) where $k < 4$. Find the value of $k$.
iii) Find the coordinates of the points where the circle meets the line with equation $x + y = 7$.
iv) One point in iii), point $Q$, is an intercept of the line. Show that $PQ$ is a diameter of the circle.



i) Collect the $x$ and $y$ terms first: $(x^2 + 4x) + (y^2 - 8y) + 7 = 0$
$\Rightarrow (x + 2)^2 - 4 + (y - 4)^2 – 16 + 7 = 0$ (Completing squares)
$\Rightarrow (x + 2)^2 + (y - 4)^2 = 13$ (Bringing numbers to RHS)

The circle is centred at (-2, 4) and its radius is $\sqrt{13}$ units.

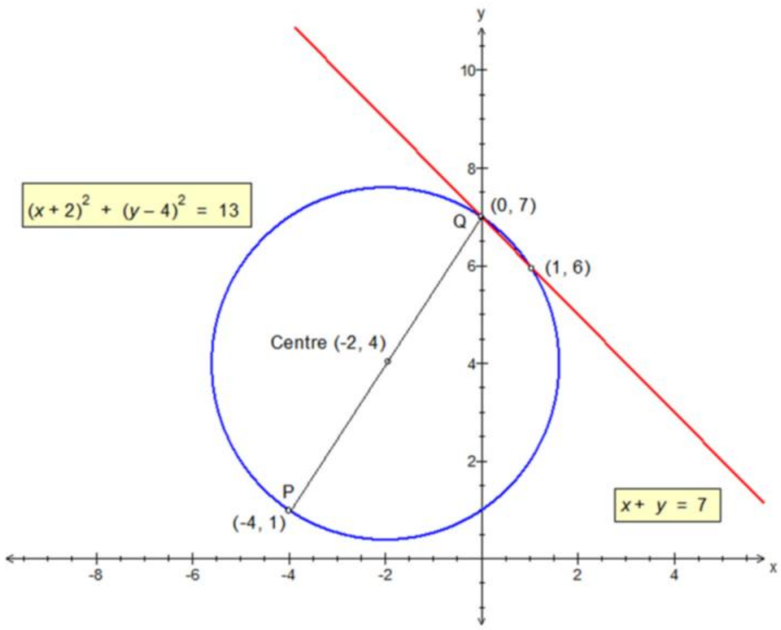
ii) Substitute $x = -4$ into the equation of the circle:
$(-2)^2 + (y - 4)^2 = 13 \Rightarrow (y - 4)^2 = 9 \Rightarrow y – 4 = \pm 3$.
As we are given $k < 4$, $y – 4 = – 3$ gives the required solution of $y = k = 1$.

iii) We solve the equations $(x + 2)^2 + (y - 4)^2 = 13$ and $x + y = 7$ simultaneously.
Substituting $y = 7 – x$ into the original circle equation we have

$x^2 + (7 – x)^2 + 4x – 8(7 – x) + 7 = 0$
$\Rightarrow x^2 + 49 – 14x + x^2 + 4x – 56 + 8x – 7 = 0$
$\Rightarrow 2x^2 – 2x = 0$
$\Rightarrow 2x(x – 1) = 0$

The $x$-coordinates of the intersection points are $x = 0$, $x = 1$.
Substituting into the equation $x + y = 7$, we have the full coordinates of (0, 7) and (1, 6).
The point (0,7) is therefore the $y$-intercept of the line. (See diagram)

iv) The midpoint of $PQ$ is $\left(\frac{-4+0}{2}, \frac{1+7}{2}\right)$ or (-2, 4), which is also the centre of the circle.
$\therefore$ $PQ$ is a diameter of the circle.



**Example(6):** Points *A* and *B* have coordinates (4, -2) and (10, 6) respectively. *C* is the midpoint of *AB*.

Find:
i) the equation of the circle whose diameter is *AB*;
ii) the equation of the tangent to that circle at the point *A*, giving the answer in '$ax + by = c$' form .
iii) the equation of the tangent to that circle at point *B*.



i) We need to find the radius and the centre of the circle here. Since the diameter passes through *A* and *B*, its length is $\sqrt{(10-4)^2+(6-(-2))^2} = \sqrt{36+64} = 10$.
The radius is therefore half that, or 5 units.

The centre is the midpoint of *AB*, or $\left(\frac{4+10}{2}, \frac{(-2)+6}{2}\right)$ or (7, 2).

The equation of the circle is therefore $(x - 7)^2 + (y - 2)^2 = 25$.

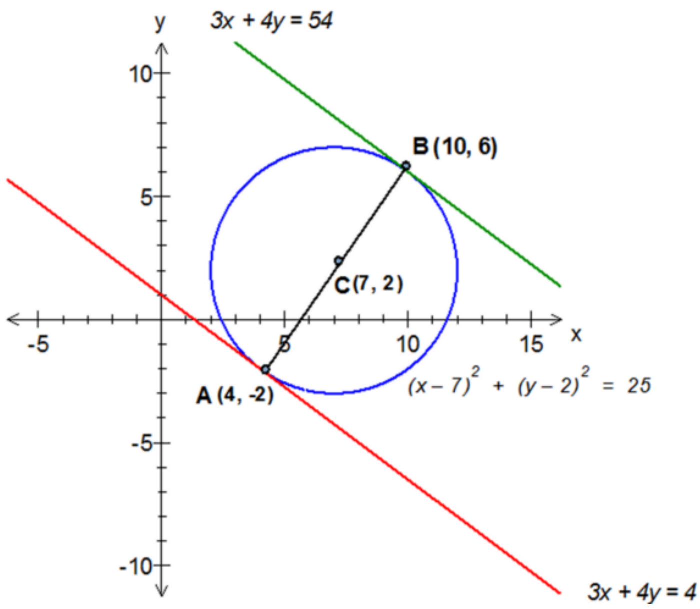
ii) We first need to find the gradient of the radius *CA* ; here it is $\frac{2-(-2)}{7-4} = \frac{4}{3}$.

Since the radius *CA* is normal to the tangent at point *A*, the gradient of the tangent at *P* is equal to $-\frac{3}{4}$, and its equation is $y + 2 = -\frac{3}{4}(x - 4) \Rightarrow 4y + 8 = -3x + 12 \Rightarrow 3x - 12 + 4y + 8 = 0$

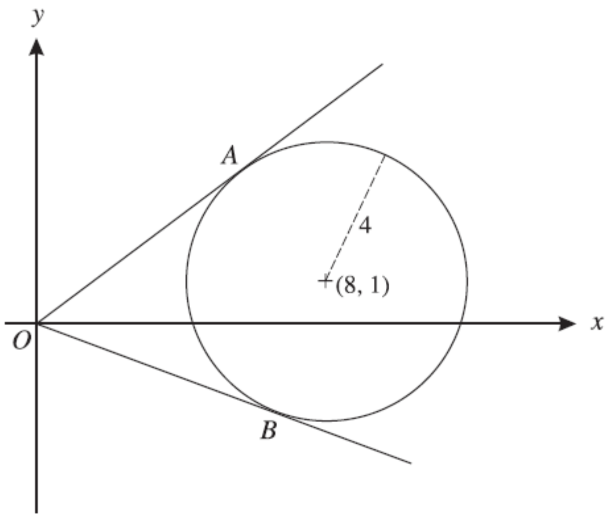
$\Rightarrow$ **$3x + 4y = 4$.**

iii) Since parallel lines have the same gradient, the equation of the tangent to that circle at point *B* is $3x + 4y = k$. We can find *k* by substituting $x = 10$, $y = 6$ to obtain $3x + 4y = 30 + 24$ or **$3x + 4y = 54$.**

See the diagram below.



**Example (7):** A circle has centre (8,1) and radius of 4 units. The points *A* and B on the circle are such that the tangents *OA* and *OB* pass through the origin.




i) State the equation of the circle.

ii) Without finding out the coordinates of *A* or *B*, show that the tangents *OA* and *OB* are 7 units long.

iii) Given that the equation of any line through the origin is $y = mx$, show that the $x$-coordinates of any points of intersection of this line and the circle are $x^2(1 + m^2) - 2x(8 + m) + 49 = 0$.

iv) Find the values of $m$ for which the line is a tangent to the circle, and hence state the equations of lines *OA* and *OB*.

v) Find the coordinates of *A* , giving the results as exact decimals.



i) The equation of the circle is $(x-8)^2 + (y-1)^2 = 16$.

ii) The distance from *O* to the centre of the circle is $\sqrt{8^2+1^2} = \sqrt{65}$ units.
The radius of the circle is 4 units, and since the tangent and radius meet at right angles,

the points *O*, *A* and the centre of the circle form a right-angled triangle, so
$OA = \sqrt{65-4^2} = \sqrt{49} = 7$ units.

Since tangents from the same point are equal in length, *OB* is also 7 units long.

iii) Because the two tangents at *A* and *B* pass through the origin, the equation for each is $y = mx$.
The line and the circle must therefore meet when the equations $y = mx$ and $(x-8)^2 + (y-1)^2 = 16$ have a simultaneous solution.

Substituting $mx$ for $y$ in the equation of the circle gives $(x-8)^2 + (mx-1)^2 = 16$.

Expanding, $(x-8)^2 + (mx-1)^2 = 16 \Rightarrow x^2 - 16x + 64 + m^2x^2 - 2mx + 1 = 16$

$\Rightarrow (1 + m^2)x^2 - (16 + 2m)x + 65 = 16 \Rightarrow (1 + m^2)x^2 - (16 + 2m)x + 49 = 0$

$\Rightarrow x^2(1 + m^2) - 2x(8 + m) + 49 = 0.$

iv) Since the lines passing through *A* and *B* are tangents to the circle, we must find values of *m* such that the quadratic in iii) has coincident roots, i.e. $b^2 = 4ac$.

In this case, $a = 1 + m^2$, $b = -16 - 2m$, $c = 49$ and therefore we must solve the equation $b^2 - 4ac = 0$.

$(-16 - 2m)^2 - 196(1 + m^2) = 0$
$\Rightarrow 256 + 64m + 4m^2 - 196 - 196m^2 = 0$
$\Rightarrow 60 + 64m - 192m^2 = 0$
$\Rightarrow 48m^2 - 16m - 15 = 0$

Factorisation is a little awkward here, so we use the general formula :

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow m = \frac{16 \pm \sqrt{256 + 2880}}{96}$$

$$\Rightarrow m = \frac{16 \pm 56}{96} \Rightarrow m = \frac{2 \pm 7}{12} \quad , \quad \therefore \text{ line meets circle when } m = \frac{3}{4},\ m = -\frac{5}{12}.$$

The equations of the tangents at *A* and *B* are therefore $y = \frac{3}{4}x$ and $y = -\frac{5}{12}x$ respectively.

v) To find the coordinates of *A*, we substitute the equation of the tangent with the positive gradient, namely $y = \frac{3}{4}x$, into the equation for the circle..

$$(x-8)^2 + \left(\frac{3}{4}x - 1\right)^2 = 16 \Rightarrow x^2 - 16x + 64 + \frac{9}{16}x^2 - \frac{3}{2}x + 1 = 16$$

Multiplying throughout by 16 to get rid of the fractions, the last equation becomes

$$16x^2 - 256x + 1024 + 9x^2 - 24x + 16 = 256$$

$$\Rightarrow 25x^2 - 280x + 784 = 0$$

As this is the equation of a tangent, it is also a perfect square: $(5x - 28)^2 = 0$.

Hence $5x = 28$ and $x = 5.6$. Substituting for $y = \frac{3}{4}x$ gives $y = 4.2$.
$\therefore$ the coordinates of *A* are (5.6, 4.2).

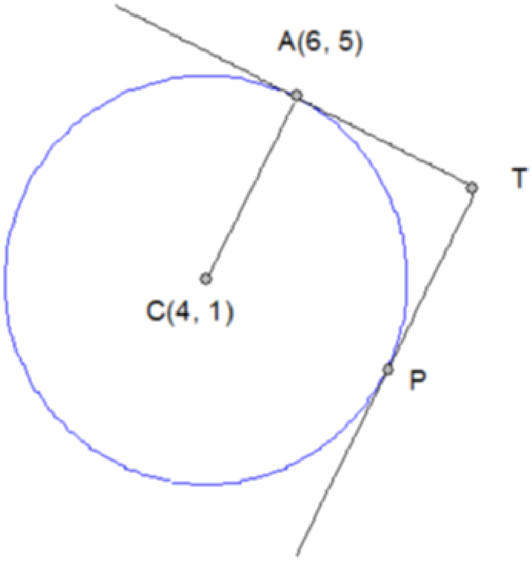
**Example (8):**
A circle is centred at (4, 1) and passes through the point $A$ (6,5).

i) Show that the equation of the tangent passing through $A$ and $T$ is $x + 2y = 16$.

ii) The line with equation $y = 2x - 17$ meets intersects the tangent in i) at point $T$.
Find the coordinates of $T$.

iii) The equation of the circle is $(x - 4)^2 + (y - 1)^2 = 20$.

Show algebraically that the line $y = 2x - 17$ is a tangent to the circle at $P$, and find the coordinates of point $P$.






i) The gradient of the radius $AC$ is

$\frac{5-1}{6-4} = \frac{4}{2} = 2$ , and so the gradient of the tangent through $A$ (6,5) is $-\frac{1}{2}$.

The equation of the tangent through $A$ is therefore $y - 5 = -\frac{1}{2}(x - 6) \Rightarrow 2y - 10 = 6 - x$

$\Rightarrow x + 2y - 10 - 6 = 0 \Rightarrow x + 2y = 16$.

ii) The point of intersection of the lines $x + 2y = 16$ and $y = 2x - 17$ at $T$ can be found by solving the equations simultaneously, substituting for $y$ in the first:

$x + 2y = 16 \Rightarrow x + 2(2x - 17) = 16 \Rightarrow x + 2(2x - 17) = 16 \Rightarrow 5x - 34 = 16 \rightarrow 5x = 50 \Rightarrow x = 10$.
The $x$-coordinate of $T$ is therefore 10; substituting in $y = 2x - 17$ gives $y = 20\text{-}17 = 3$.

$\therefore$ Coordinates of $T$ are (10, 3).

iii) The equation of the circle is $(x - 4)^2 + (y - 1)^2 = 20$
$\Rightarrow (x - 4)^2 + ((2x - 17) - 1)^2 = 20$ (substitute equation of line through $P$ for $y$)
$\Rightarrow (x - 4)^2 + (2x - 18)^2 = 20$
$\Rightarrow x^2 - 8x + 16 + 4x^2 - 72x + 324 = 20$ (expand)
$\Rightarrow 5x^2 - 80x + 340 = 20$ (collect)
$\Rightarrow 5x^2 - 80x + 320 = 0$
$\Rightarrow x^2 - 16x + 64 = 0$ (divide throughout by 5)
$\Rightarrow (x - 8)^2 = 0$ (factorise)

The resulting quadratic has a coincident root of $x = 8$, so the line $y = 2x - 17$ is a tangent to the circle at point $P$. Substituting $x = 8$ into the line's equation gives $y = -1$.

$\therefore$ Coordinates of point $P$ are (8, -1).

**Example (9).**

Points *A* (5, 15), *B* (13,9), *C* (-8, 2) and *D* (-2, 14) lie on a circle.

i) a) Find the coordinates of the midpoint of the chord *AB* and hence the equation of the perpendicular bisector of this chord.

i) b) Hence also show that this bisector passes through the origin.

ii) Repeat part i) a), using the chord *CD.*

iii) Using the results from i) and ii), find the centre, and hence the equation, of the circle. Show also that the radius of the circle is $k\sqrt{5}$ units where *k* is an integer.

i) a) The midpoint of *AB* is $\left(\frac{5+13}{2}, \frac{15+9}{2}\right)$ or (9, 12).

The gradient of the chord *AB* is $\frac{9-15}{13-5} = -\frac{3}{4}$, so the gradient of the perpendicular bisector is $\frac{4}{3}$.

The equation of the perpendicular bisector is hence $y - 12 = \frac{4}{3}(x - 9) \Rightarrow 3y - 36 = 4x - 36$

$\Rightarrow 4x - 36 - 3y + 36 = 0 \Rightarrow$ $\mathbf{4x - 3y = 0.}$

i) b) Substituting $(x, y) = (0, 0)$ gives $4(0) - 3(0) = 0$, satisfying the equation.

ii) a) The midpoint of *CD* is $\left(\frac{-8-2}{2}, \frac{2+14}{2}\right)$ or (-5, 8).

Gradient of chord *CD* = $\frac{14-2}{(-2)-(-8)} = 2$, $\therefore$ gradient of perpendicular bisector is = $-\frac{1}{2}$.

Perpendicular bisector equation is thus $y - 8 = -\frac{1}{2}(x + 5) \Rightarrow 2y - 16 = -x - 5$

$\Rightarrow 2y - 16 + x + 5 = 0 \Rightarrow$ $\mathbf{x + 2y - 11 = 0.}$

Because the perpendicular bisector of a chord is a diameter, we can find the centre of the circle by simultaneously solving the equations of the perpendicular bisectors.

$4x - 3y = 0$      *A*
$x + 2y - 11 = 0$      *B*

$4x + 8y - 44 = 0$      4*B*
$4x - 3y = 0$      *A*

$11y = 44$      4*B*-*A*

Hence $y = 4$, and substituting into equation *A* gives $4x - 12 = 0$, or $x = 3$.

$\therefore$ The centre of the circle is (3,4).

The equation of the circle is $(x - 3)^2 + (y - 4)^2 = r^2$ . To find $r$, we substitute the coordinates of any of the given points on the circle.

Thus, using point *A* (5,15), we have $r^2 = (5 - 3)^2 + (15 - 4)^2$ , or $2^2 + 11^2$, or 125.

$\therefore$ The equation of the circle is $(x - 3)^2 + (y - 4)^2 = 125$.

Since $125 = 25 \times 5$, the radius of the circle = $\sqrt{125} = \sqrt{25} \times \sqrt{5} = 5\sqrt{5}$ .

$\therefore$ The radius of the circle is $k\sqrt{5}$ units where $k = 5$.