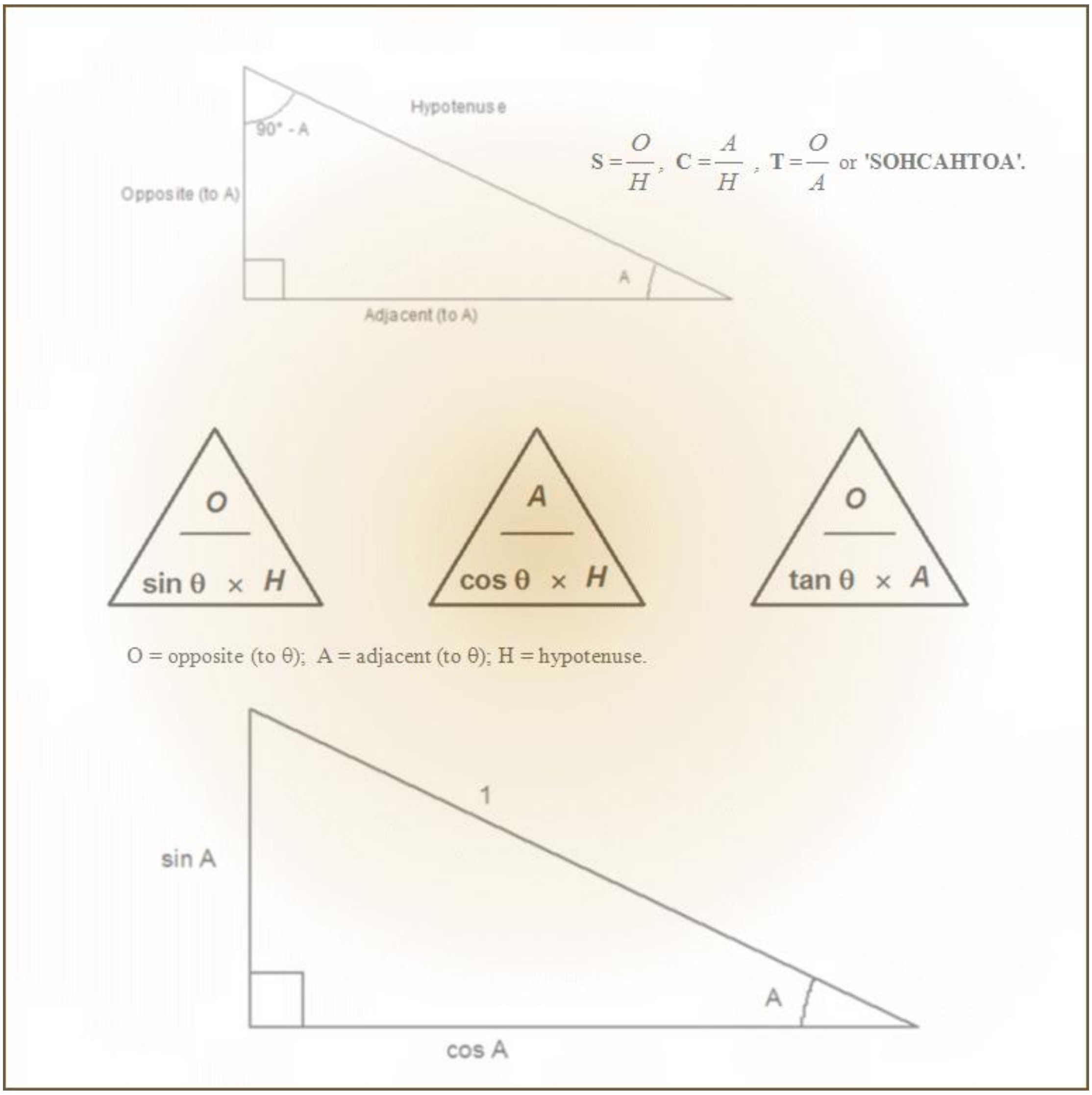
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 1 / AS

TRIGONOMETRIC RATIOS AND SOLVING SPECIAL TRIANGLES - REVISION



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Solving special triangles.

(Revision from GCSE Higher Tier)

Right – angled triangles - Pythagoras' theorem.
Pythagoras' theorem states that for any right-angled triangle, **the square on the hypotenuse is equal to the sum of the squares of the other two sides.**
It thus follows that $c^2 = a^2 + b^2$, or $c = \sqrt{a^2 + b^2}$, where c is the hypotenuse.

The above form is used when the hypotenuse is unknown, but it can be adapted to find an unknown side when the hypotenuse is known.

Then, $a^2 = c^2 - b^2$ or $a = \sqrt{c^2 - b^2}$ if a is unknown, and similarly

$b^2 = c^2 - a^2$ or $b = \sqrt{c^2 - a^2}$ if b is unknown.

Example (1): A right-angled triangle has a hypotenuse of 15cm and another side of 12cm. What it the length of the third side ?

The missing side has a length of $\sqrt{15^2 - 12^2} = \sqrt{225 - 144} = \sqrt{81} = 9$ cm.

Example (2): A right-angled triangle has the lengths of its two shorter sides equal to 8cm and 15cm. What is the length of the hypotenuse ?
The length of the hypotenuse is $\sqrt{15^2 + 8^2} = \sqrt{225 + 64} = \sqrt{289} = 17$ cm.

Area of a right-angled triangle.

The formula for the area of a triangle is given by

area = ½ (base × height).

This is particularly simple when the triangle is right-angled; all that is needed are the lengths of the sides containing the right angle.

Example (3). Find the areas of the triangles from examples (1) and (2) above.

We are only given one side containing the right angle to begin with.in the case of the triangle from Example (1). After calculating the missing side as being 9 cm long, we can work out the area as ½ (9 × 12) or 54 sq. cm.

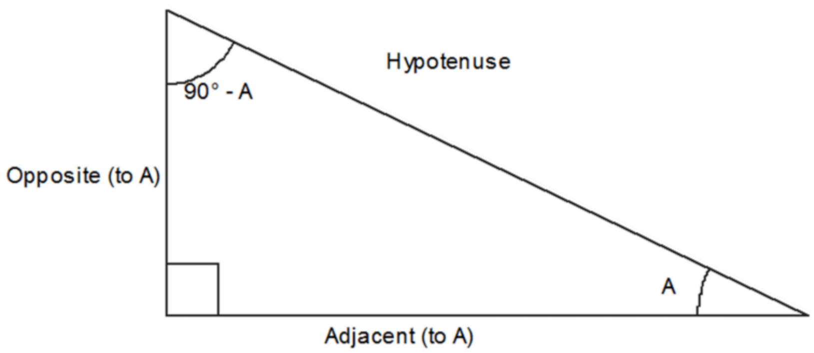
The two sides containing the right angle are already given in the triangle from Example (1). The area is therefore ½ (8 × 15) or 60 sq. cm.

The earlier examples showed how to work out the length of one side, given the two other sides. Here, we will show how to solve any right-angled triangle given sufficient information.

Solving right-angled triangles using the trigonometric ratios.

The hypotenuse is the longest side of the triangle, and is the side opposite the right angle.

Using angle A as the reference, the other two sides of the triangle are the 'opposite' which is opposite A, and the 'adjacent' which runs between A and the right angle.



The three sides are all related by the following ratios; the **sine**, **cosine** and **tangent** of A. They are abbreviated to **sin**, **cos** and **tan**.

The sine of angle A (sin A) is the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$

The cosine of angle A (cos A) is the ratio $\frac{\text{adjacent}}{\text{hypotenuse}}$

The tangent of angle A (tan A) is the ratio $\frac{\text{opposite}}{\text{adjacent}}$

It also follows that tan A = $\frac{\sin A}{\cos A}$.

In brief, **S** = **O/H**, **C=A/H and T=O/A**, or **'SOHCAHTOA'.**

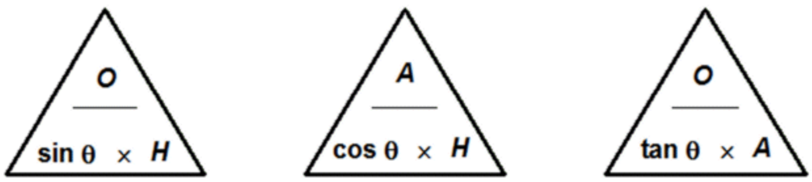
The formulae above can also be rearranged to find missing sides by changing the subject.

Hypotenuse = $\frac{\text{opposite}}{\sin A}$ or $\frac{\text{adjacent}}{\cos A}$

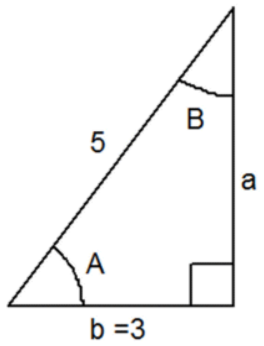
Opposite = hypotenuse × sin A or adjacent × tan A

Adjacent = hypotenuse × cos A or $\frac{\text{opposite}}{\tan A}$

Or we can use formula triangles as below: (the symbol θ = Greek ‘theta’ denotes the angle)



O = opposite; A = adjacent; H = hypotenuse
θ = the given angle



In the triangle shown left, we can work out side *a*, the angles A and B, and the area using both Pythagoras and the trig ratios. (Note also the convention: side *a* is opposite angle A, and side *b* opposite angle B)

Side *a* is $\sqrt{5^2 - 3^2} = \sqrt{25-9} = \sqrt{16} = 4$ units long.

Now that we have both the base and the height, the area can be easilt worked out as ½ (3 × 4) or 6 square units.

Since side *b* is opposite angle B, sin B = $\frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5}$

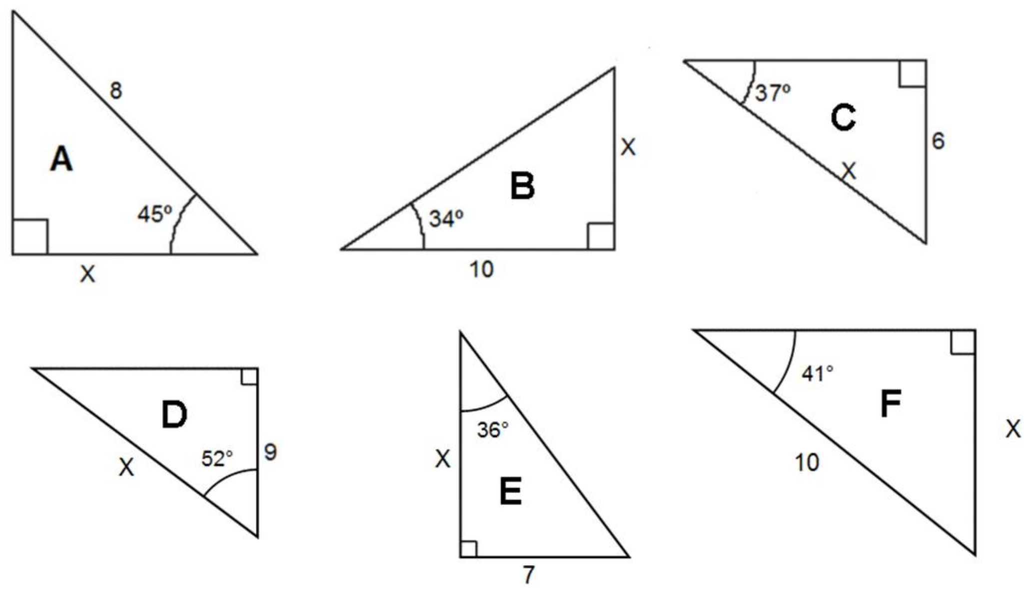
and therefore angle B = 36.9° to 1 d.p.

Angle A can be worked out by realising that side *b* is adjacent to it,

and using cos A = $\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5}$, giving B = 53.1° to 1 d.p.

Alternatively, we could have realised that angle B = 90° – A (angles of a triangle add up to 180°)

Example (3a): Find the sides labelled X in the triangles below. All lengths are in centimetres. Give results to 2 decimal places.



Triangles not accurately drawn !

In triangle **A**, we are given the angle of 45° and the hypotenuse of 8 cm. Side X is not opposite the 45° angle, so it is the adjacent.

The length of X (the adjacent) is therefore (hypotenuse × cos 45°), or 8 cos 45°, or 5.66 cm.

We have the angle and the adjacent in triangle **B**, but we need to find the opposite. Use the formula opposite = (adjacent × tan 34°), giving the length of X as 10 tan 34° or 6.75 cm.

In triangle **C**, we are given the angle (37°) and the opposite (6 cm), but we are required to work out the hypotenuse X.

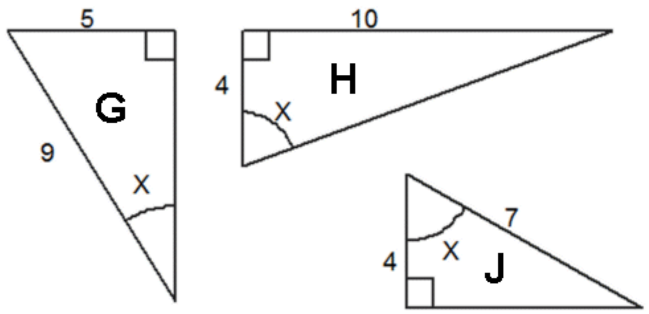
Hence the hypotenuse = $\dfrac{6}{\sin 37°}$, so X = 9.97 cm.

In triangle **D**, we are given the angle of 52° and the adjacent (9 cm),
so X = hypotenuse = $\dfrac{9}{\cos 52°}$ = 14.62 cm.

Triangle **E** has the opposite (7 cm) and the angle (36°) given,
so X = adjacent = $\dfrac{7}{\tan 36°}$ = 9.63 cm.

Finally, triangle **F** has the angle (41°) and the hypotenuse (10 cm) given,
so X = opposite = 10 sin 41°= 6.56 cm.

Example (3b): Find the angles labelled X in the triangles below. All lengths are in centimetres. Give results to 1 decimal place.



Triangles not accurately drawn !

This time we are looking for angles, not sides, so we make use of **inverse trig functions**.

These functions are used to find an unknown angle given one of its ratios.
They are $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$. For example, $\sin^{-1}(0.5) = 30°$ and $\tan^{-1}(1) = 45°$.

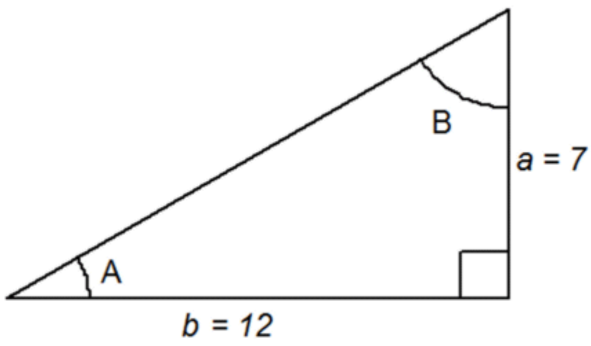
The input to $\sin x$, $\cos x$ and $\tan x$ is in each case an angle, and the output a number.
With the inverse functions, the input is a number and the output is an angle.

Triangle **G** has the opposite (5 cm) and hypotenuse (9cm) known, therefore $\sin X = \frac{5}{9}$,
in other words $X = \sin^{-1}\left(\frac{5}{9}\right) = 33.7°$.

The opposite (10cm) and adjacent (4cm) are known in triangle **H**, so $\tan X = \frac{10}{4}$,
or $X = \tan^{-1}\left(\frac{10}{4}\right) = 68.2°$.

The adjacent and hypotenuse are known in triangle **J**, so $\cos X = \frac{4}{7}$,
i.e. $X = \cos^{-1}\left(\frac{4}{7}\right) = 55.2°$.

Example (4): Find the unknown sides and angles in the triangle below, together with its area.



The area can be worked out at once as we have the base and height given – it is ½ (7 × 12) or 42 square units.

Side *a* is opposite angle A, and side *b* is adjacent to angle A,

$\therefore$ tan A = $\frac{\text{opposite}}{\text{adjacent}} = \frac{7}{12}$

giving A = 30.3° to 1 d.p. and thus B = 59.7° to 1 d.p.

The hypotenuse, *c*, can be worked out either by

i) using the value of A obtained earlier and the formula : hypotenuse = $\frac{\text{opposite}}{\sin A}$

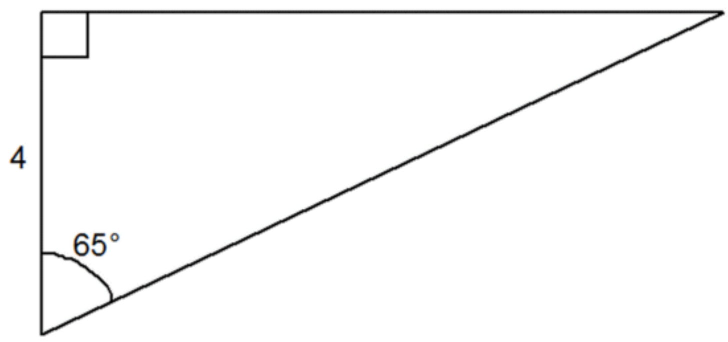
giving $c = \frac{7}{\sin 30.3°}$ or 13.9 units to 1 d.p.

ii) using Pythagoras:

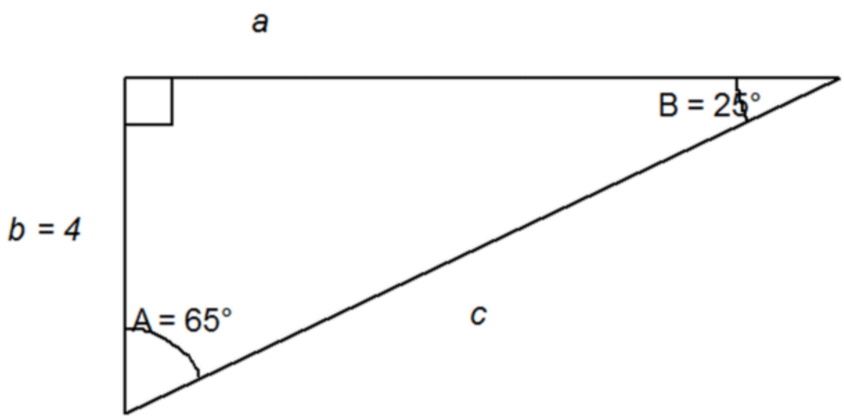
$$c = \sqrt{a^2 + b^2} = \sqrt{49 + 144} = \sqrt{193} = 13.9 \text{ units to 1 d.p.}$$

(Here, Pythagoras gives a more accurate result, as the value of 30.3° has been rounded.)

Example (5): Find the unknown sides and angles in the triangle below along with its area, giving the results to two decimal places.



This time, we are give just one angle and one side, and neither the sides nor the angles have been lettered. The first step is to label the side and angles using side *a* opposite angle A and so forth. Angle B can be worked out at once by subtracting 65 from 90.



We have the length of the adjacent to A, namely 4, be not that of the opposite, *a*.

We use the formula $\tan A = \frac{\text{opposite}}{\text{adjacent}}$ and rewrite it as

opposite = tan A × adjacent.

This gives $a = \tan 65° \times 4$ units, or 8.58 units to 2 d.p.. (Use more decimal places, i.e. 8.578 or full calculator accuracy when working out intermediate results).

We now have the required information to work out the area - it is half the product of the opposite and the adjacent, or ½ (4 × 8.578), or 17.16 square units.

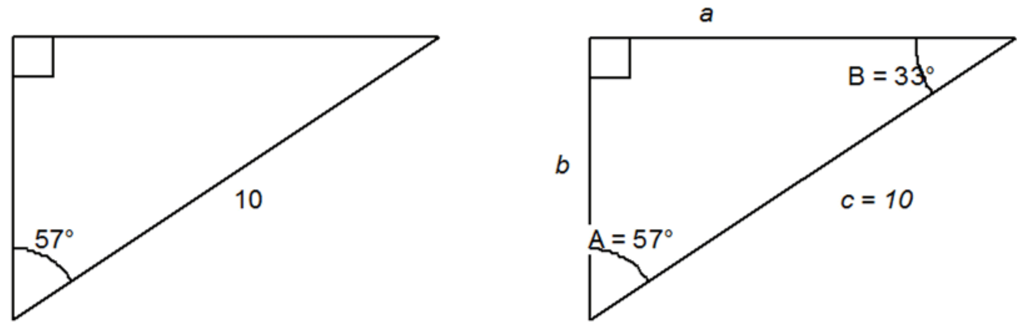
(We could have equally used angle B and treated *b* as the opposite, but this time the tangent formula would be rewritten as

adjacent = $\frac{\text{opposite}}{\tan B}$, giving $a = \frac{4}{\tan 25°}$, or 8.58 units to 2 d.p. as before.)

From this, we can calculate the length of the hypotenuse:

$$c = \sqrt{a^2 + b^2} = \sqrt{8.578^2 + 16} = \sqrt{89.58} = 9.46 \text{ units to 2 d.p.}$$

Example (6): Find the unknown sides and angles in the triangle below, along with its area, giving the results to two decimal places.



Label sides and angles, remembering that 57° + 33° = 90°

This time, we know the length of the hypotenuse c, so we use the formula $\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$ and rewrite it as

opposite = sin A × hypotenuse.

This gives a = sin 57° × 10 units, or 8.39 units to 2 d.p.

There are three ways to work out side b:

i) We can treat it as the opposite side to angle B and apply the same formula as before, i.e.
gives b = sin 33° × 10 units, or 5.45 units to 2 d.p.

ii) We can treat it as the adjacent side to angle A and use
$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$ and rewrite it as

adjacent = cos A × hypotenuse , giving b = cos 57° × 10 units, or 5.45 units to 2 d.p as before.

iii) Use Pythagoras after having worked out the length of side a.

$b = \sqrt{c^2 - a^2} = \sqrt{100 - 8.387^2} = \sqrt{29.66} = 5.45$ units to 2 d.p.

The area is half the product of a and b, or ½ (10 sin 57° × 10 cos 57°) or (50 × 0.8387 × 0.5446) = 22.84 square units.

Three important properties can be gleaned from the last example:

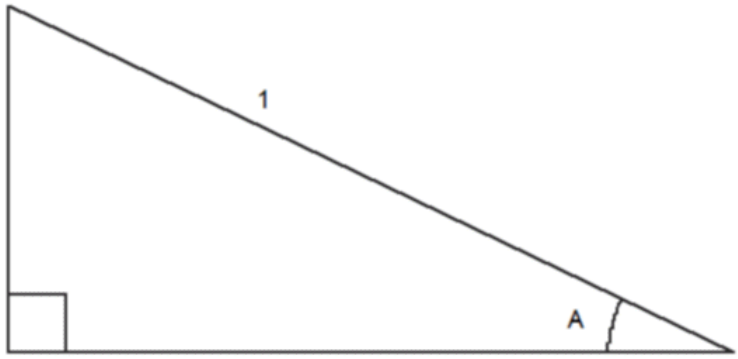
Given the hypotenuse c and an angle A, another formula for the area is $\frac{1}{2}(c^2 \sin A \cos A)$.

The sine of an angle is equal to the cosine of its difference from 90°, i.e. sin A = cos (90° – A)

Also, the tangent of an angle is equal to the reciprocal of the tangent of its difference from 90° (excluding multiples of right angles), i.e. tan A × tan (90° – A) = 1

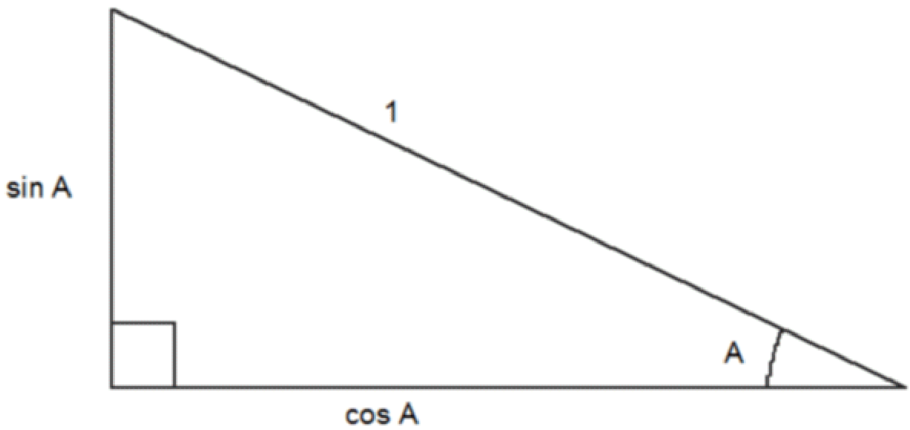
Example (7) - an important trig identity.

Find a formula relating sin A and cos A in the triangle below. Use Pythagoras' Theorem here.



The opposite side has a length of sin A × hypotenuse, but since the hypotenuse is 1 unit long, its length is simply sin A units.

Similarly, the adjacent has a length of cos A units.



Pythagoras' theorem states that the square of the hypotenuse is equal to the sum of the squares on the other two sides, therefore

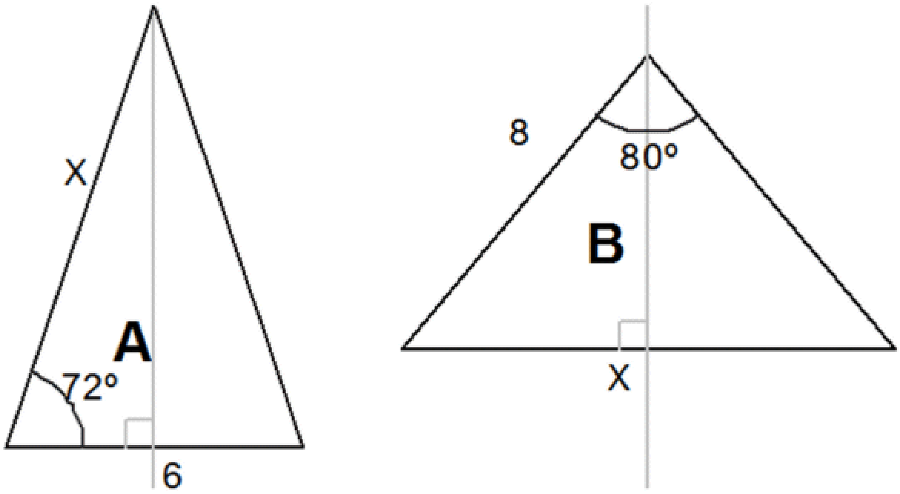
$\mathbf{\cos^2 A + \sin^2 A = 1.}$

(This identity holds for all angles A.)

Solving isosceles triangles.

It is possible to use the methods used for right-angled triangles to find the area, unknown sides and angles in an isosceles triangle, because it has a line of symmetry bisecting it into two congruent right-angled triangles.

Example (8): Solve triangles **A** and **B** and calculate their areas.



Triangle **A** is bisected to give two right-angled triangles. We need to find the hypotenuse of one of those halves, being given the adjacent (here 3 - half the base).

hypotenuse = $\frac{\text{adjacent}}{\cos 72°}$, or $\frac{3}{\cos 72°}$ which gives the length of **X** as 9.71 units to 2 d.p.

(Do not forget to divide the base of the triangle by 2 !)

We are required to find the base of triangle **B,** namely **X**. Here we are given the hypotenuse and the angle, but we need to find the opposite. The line of symmetry bisects the angle of 80°, so we use 40° in the calculation.

opposite = hypotenuse × sin 40°, therefore the length of **X** is double that (by symmetry), namely 16 sin 40° or 10.28 units to 2 d.p.

(Don't forget to double the length of the opposite !)

To obtain the areas of the triangles **A** and **B**, we will need to find their perpendicular heights.

The height of **A** can be worked out as opposite = adjacent × tan 72°, or 3 tan 72° or 9.23 units, and thus the area of the whole triangle is 3 × 9.23 or 27.70 square units. The ½ has been taken out because we are measuring the area of **two** combined right-angled triangles.

Another way of working out the area is to use the general formula:

Area = ½*ab* sin C where *a* and *b* are two sides and C the included angle. This formula will be used under ‘Solving General Triangles’, but we can use it here.

Using the base and the one other side as sides *a* and *b*, and the angle of 72° between them, we have

Area = ½ (6 × $\frac{3}{\cos 72°}$ × sin 72°) = $\frac{9\sin 72°}{\cos 72°}$ or 27.70 sq.units.

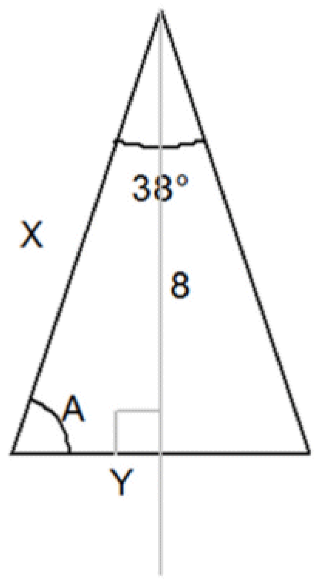
The height of **B** can be worked out as adjacent = hypotenuse × cos 40°, or 8 cos 40° or 6.13 units.
This gives the area of the whole triangle as ½ × 10.28 × 6.13 or 31.51 square units.

Again, the formula gives a quicker result, if we use the two sides of 8 units and the angle of 80° between them.

Area = ½ (8^2 × sin 80°) = 32 sin 80° = 31.51 sq.units.

Sometimes an isosceles triangle might have an angle and its perpendicular height known, but none of its sides.

Example (9): Find angle **A** , side **X** and side **Y** in the isosceles triangle shown, as well as its area.



To find angle **A**, subtract half of 38° from 90° (by symmetry). **A** is therefore 71°.

To work out the sides, we can either use angle **A** or half of 38° (19°) for reference. Here angle **A** will be used.

The height of the triangle is therefore the opposite, but we need to find the adjacent and the hypotenuse, not forgetting to double the adjacent to give side **Y** by symmetry.

hypotenuse = $\frac{\text{opposite}}{\sin 71°}$, or $\frac{8}{\sin 71°}$ which gives the length of **X** as 8.46 units to 2 d.p.

adjacent = $\frac{\text{opposite}}{\tan 71°}$, or $\frac{8}{\tan 71°}$, giving the length of **Y** as $\frac{16}{\tan 71°}$ units, or 5.51 units to 2 d.p.

Now that **Y** has been worked out, the area is ½ (5.51 × 8) or 22.04 square units.

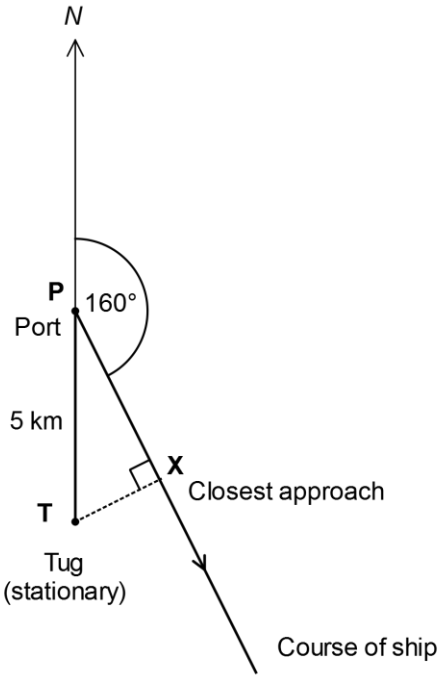
Example (10): A ship leaves port on a bearing of 160°, and there is also a stationary tug anchored 5 km south of the port. Calculate the shortest distance between the moving ship and the tug, and also the ship's distance from port when this situation occurs.

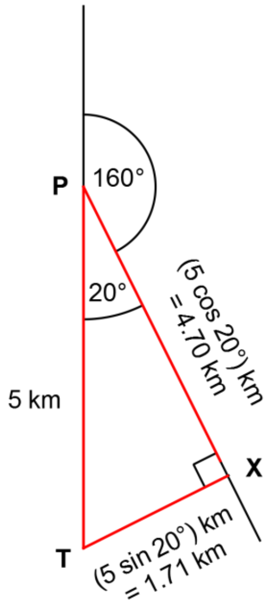
Let the port be at **P** and the tug at point **T,** 5 km due south of the port. The ship's course is in the direction of the arrow at a bearing of 160°, and the point **X** is where the ship and the tug are at their closest to each other.

The shortest path from a point to a line is also perpendicular to the line, so the angle between **PX** and **XT** is a right angle. Additionally, the angle **TPX** = 20° (180° in a straight line).

PT is the hypotenuse of the triangle **PXT**, **TX** is opposite the 20° angle and **PX** is adjacent to it.

Therefore the shortest distance between the ship and the tug, **TX** = (5 sin 20°) km = **1.71 km**.
When the ship and the tug are at their closest, the distance between the port and the ship is
PX = (5 cos 20°) km = **4.70 km.**

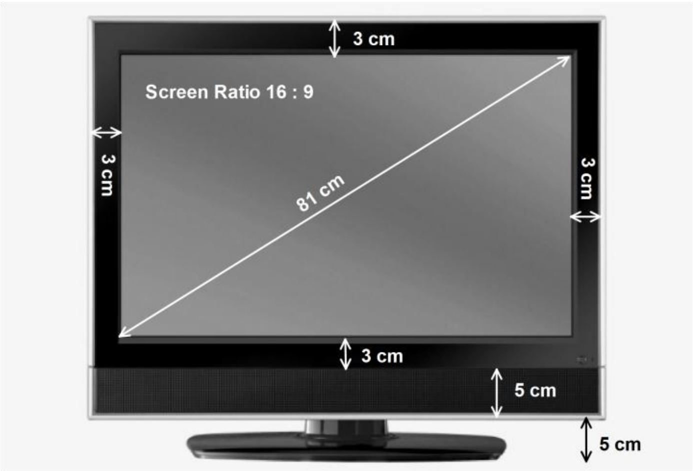




Example (11): Jim is looking at a television set but is not sure whether it would fit inside his cabinet. The interior dimensions of the cabinet are 80 cm by 60 cm.

The diagonal of the visible screen of the set is 81 cm, and the side lengths are in the ratio 16 : 9. There is a bezel (border) of 3 cm around all the edges of the screen and a 5 cm loudspeaker area below the lower edge bezel. There is also a stand which adds a further 5 cm to the overall height.

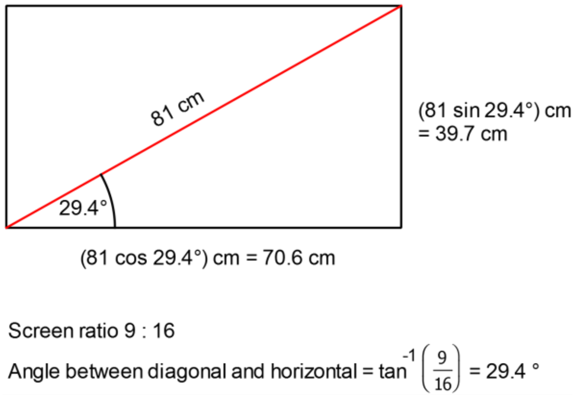
Will the TV set fit inside Jim's cabinet ? Justify your answer, showing all calculations.



We do not know the length or the width of the visible screen, but we are given their ratio, 9 : 16.
The tangent of the angle between the horizontal and the diagonal is therefore $\frac{9}{16}$,
i.e. the angle is $\tan^{-1}\left(\frac{9}{16}\right) = 29.4°$.

The screen diagonal of 81 cm is therefore the hypotenuse of the right-angled triangle (right), the length of the screen is the adjacent side, and the height of the screen is the opposite side.

Length of screen = (81 cos 29.4°) cm = **70.6 cm.**
Height of screen = (81 sin 29.4°) cm = **39.7 cm.**



We need to add the two bezel widths of 3 cm each to obtain the total length of the set, namely **76.6 cm.**

Similarly we need to add the two bezel heights of 3 cm each, the loudspeaker area of 5 cm, and the stand of 5 cm – a total of 16 cm – to the height of the screen. The total height of the set is **55.7 cm.**

The dimensions of the TV set as a whole are 76.6 cm by 55.7 cm, so the set will fit inside Jim's TV cabinet with a few centimetres to spare either way. .