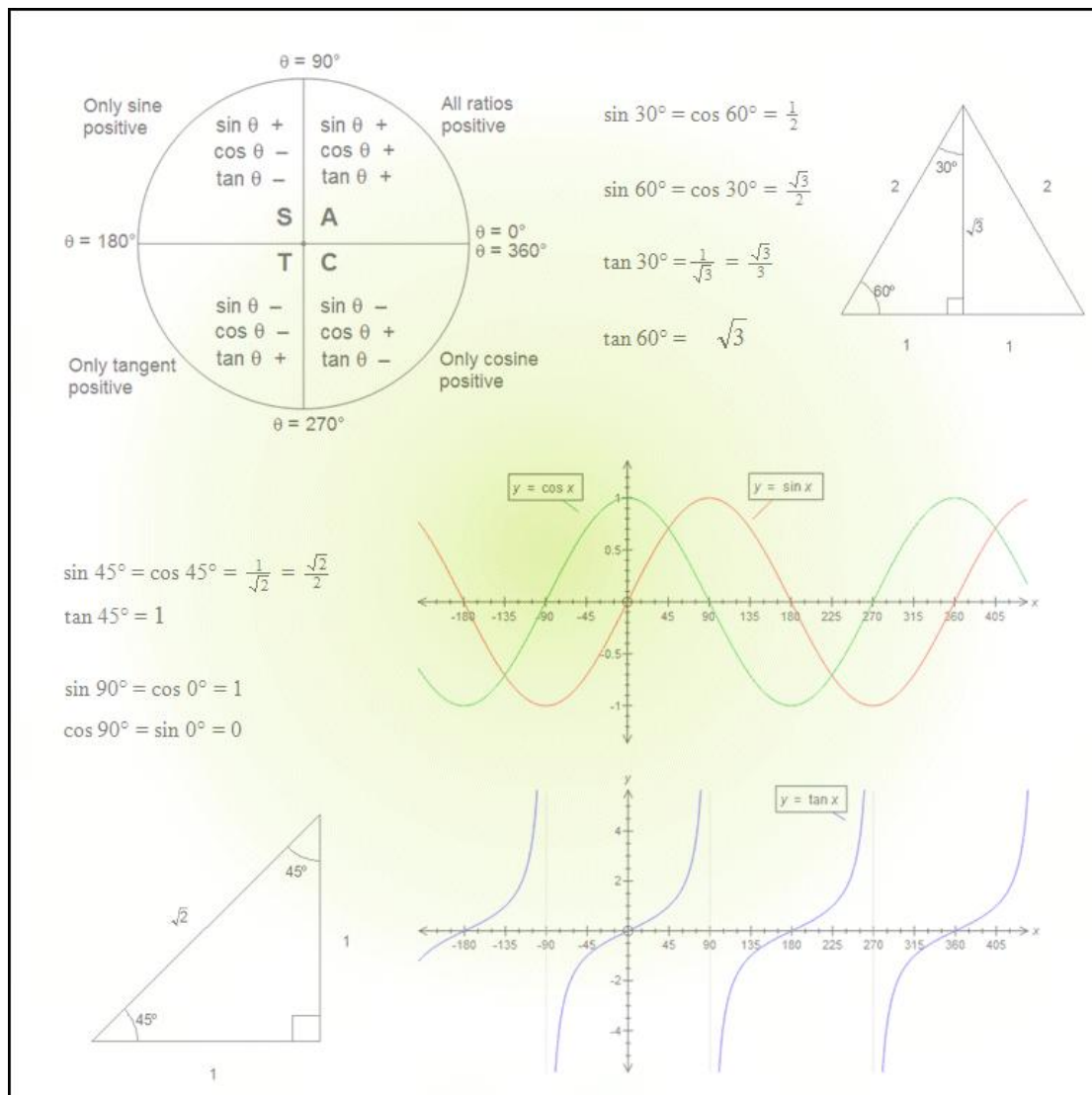


M.K. HOME TUITION

Mathematics Revision Guides

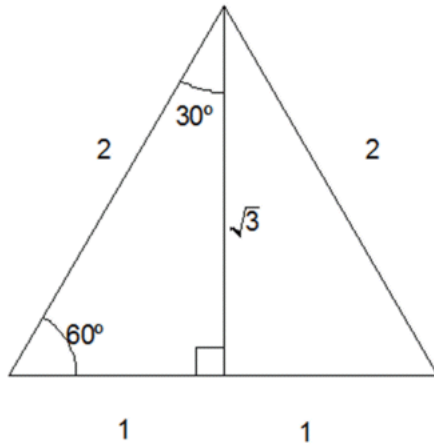
Level: A-Level Year 1 / AS

TRIGONOMETRIC RATIOS AND GRAPHS



Trigonometric ratios of special angles.

The trigonometric ratios of certain angles can be deduced by using Pythagoras' theorem.

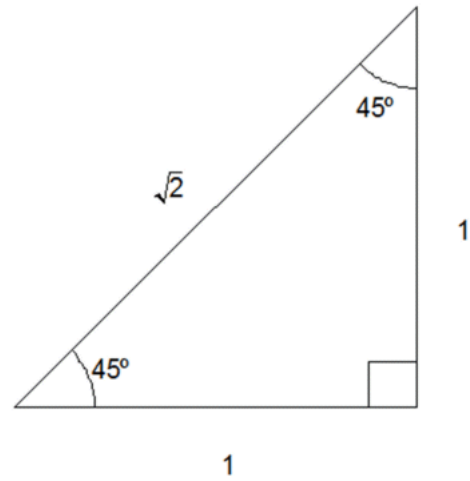


$$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$$

$$\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\tan 60^\circ = \sqrt{3}$$



$$\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = 1$$

$$\sin 90^\circ = \cos 0^\circ = 1$$

$$\cos 90^\circ = \sin 0^\circ = 0$$

$$\tan 0^\circ = 0$$

Trigonometric ratios for any angle.

The sine, cosine and tangent are all positive when the angle in question, A , is between 0° and 90° . When we come to more general cases, the signs of the ratios change depending on the quadrant the angle is in.

Any angle A outside the range $0^\circ \leq A < 360^\circ$ can be reduced to one within the range by subtracting the appropriate multiple of 360° from it, thus $\sin 760^\circ = \sin 40^\circ$ (subtract $2 \times 360^\circ$ from it) and $\tan (-290^\circ) = \tan 70^\circ$ (add 360° to it).

Sometimes it is more convenient to use the range $-180^\circ \leq A < 180^\circ$, especially when dealing with negative values of the trig ratios.

The unit circle.

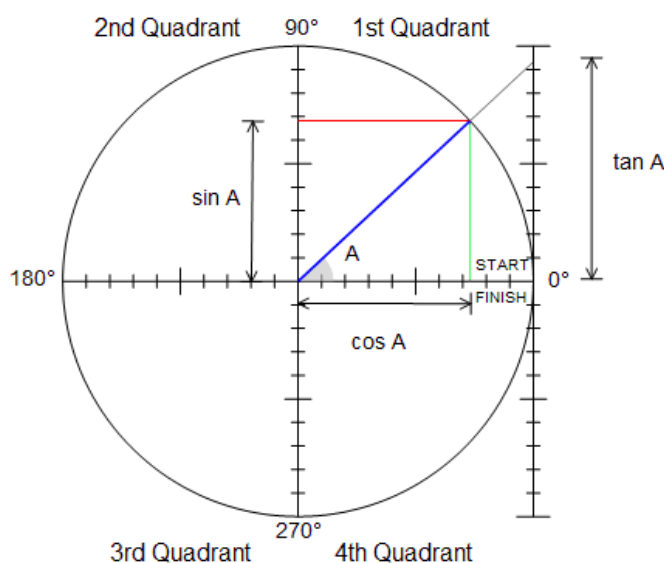
The circle shown has a radius of 1 unit, and is centred on the origin. The x - and y -axes divide the circle into four quadrants.

Angles are measured anticlockwise in the range $0^\circ \leq A < 360^\circ$ with the zero point coinciding with point $(1, 0)$ on the x -axis.

The value of $\sin A$ is the projection of the vertical displacement onto the y -axis.

The value of $\cos A$ is the projection of the horizontal displacement onto the x -axis.

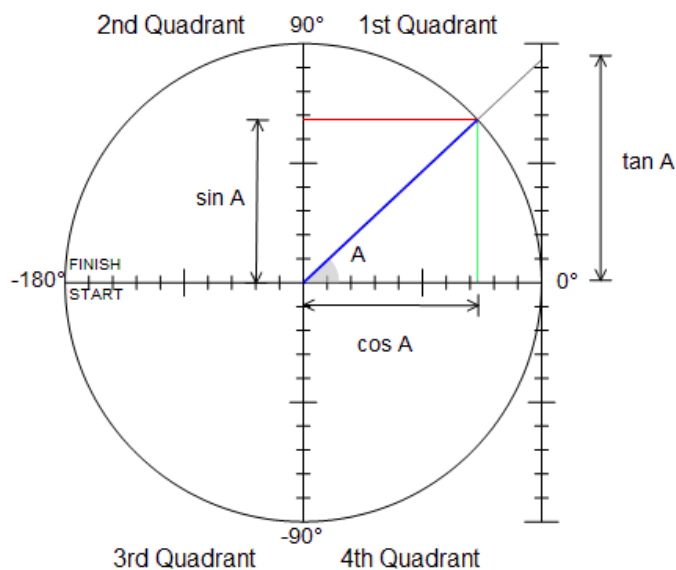
The value of $\tan A$ is the projection of the radius onto the tangent to the circle at $(1, 0)$.



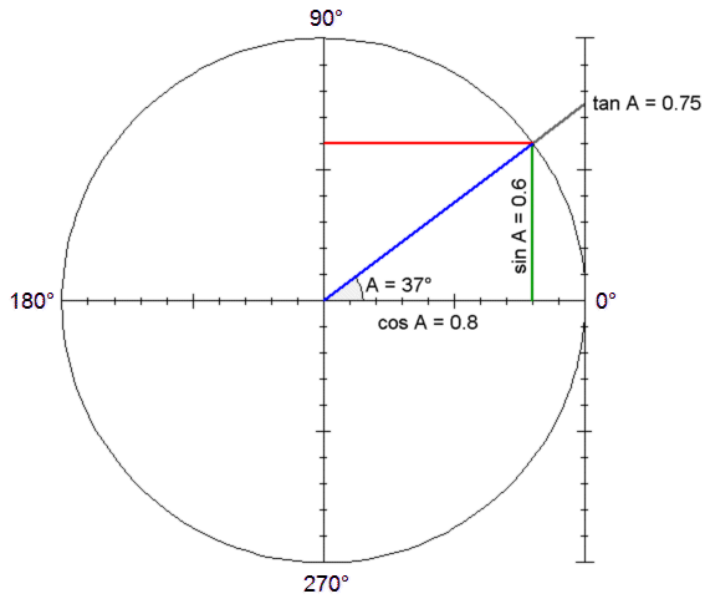
In this example, $\sin A$ is slightly below 0.7, $\cos A$ slightly above 0.7, and $\tan A$ slightly above 0.9.

This is the alternative unit circle for the angle range $-180^\circ \leq A < 180^\circ$. Angles are still measured anticlockwise, but this time from the point $(-1, 0)$.

Angles in the first and second quadrant are unchanged from the previous definition, but those in the third and fourth quadrants are decreased by 360° .



Angle between 0° and 90° (first quadrant).



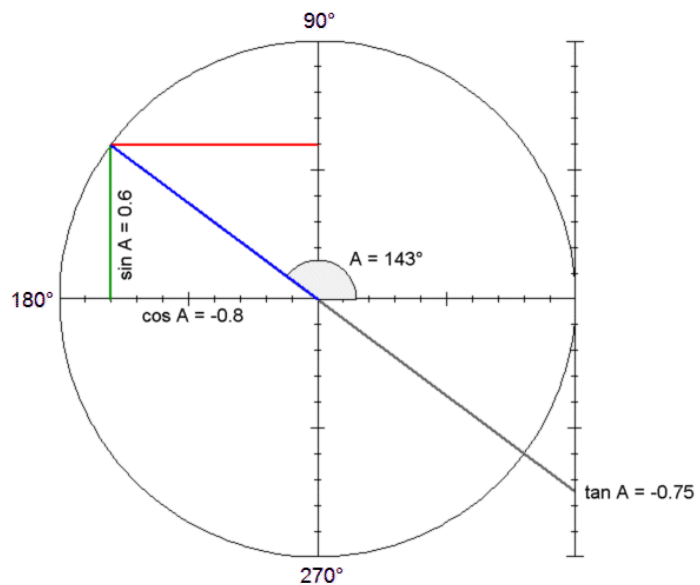
All three trigonometric ratios are positive here - the sine, cosine and tangent.

Angle between 90° and 180° (second quadrant).

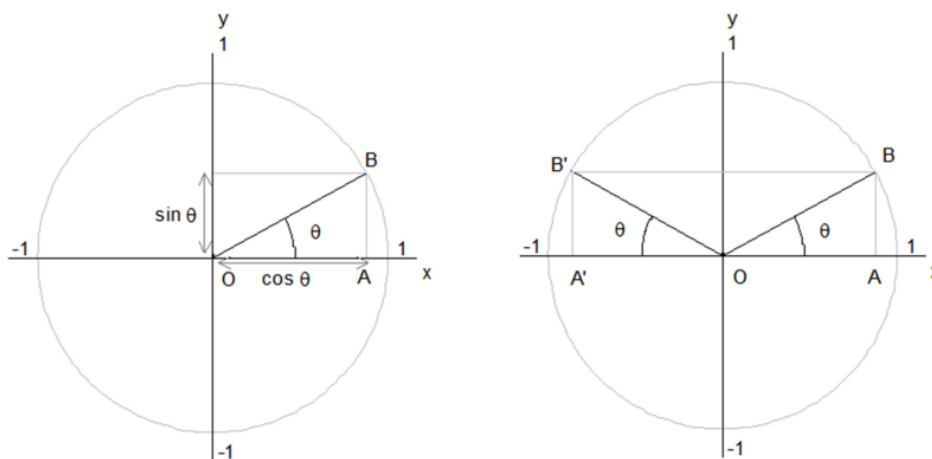
In the second quadrant, the sine remains positive but the cosine is now negative.

Because $\tan A = \frac{\sin A}{\cos A}$, it follows that

the tangent is also negative.



The relationship between angles in the first and second quadrants is shown below.



By symmetry, angle A'OB' in the right-hand diagram is identical to angle AOB, therefore angles AOB and AOB' add up to 180°.

It can also be seen that $\sin AOB' = \sin AOB$ and that $\cos AOB' = -\cos AOB$. By division, it can also be seen that $\tan AOB' = -\tan AOB$.

For any angle θ in the first quadrant, the trig ratios of the angle $(180 - \theta)$ in the second quadrant are related as follows:

$$\sin (180 - \theta)^\circ = \sin \theta^\circ ; \cos (180 - \theta)^\circ = -\cos \theta ; \tan (180 - \theta)^\circ = -\tan \theta.$$

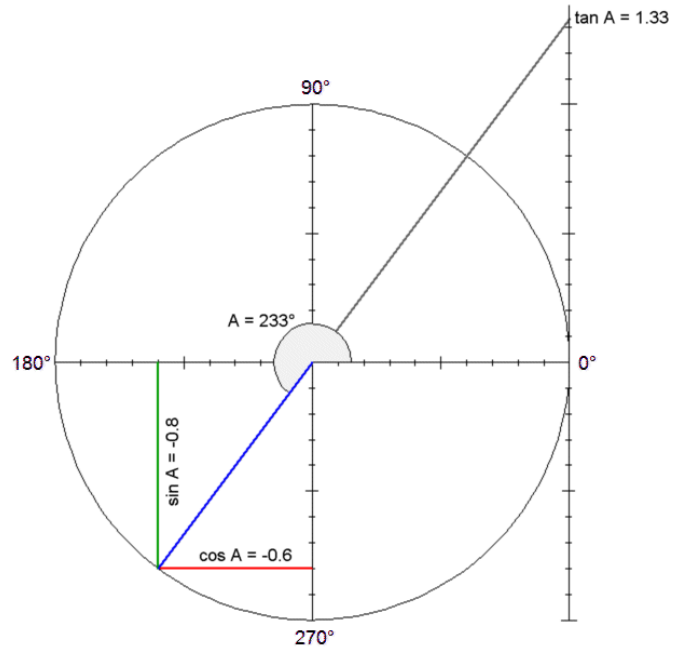
Example (1) : Given $\sin 35^\circ = 0.574$, $\cos 35^\circ = 0.819$ and $\tan 35^\circ = 0.700$ (to 3 d.p.), state the corresponding ratios for the angle of 145° .

Since $145^\circ = (180 - 35)^\circ$,
 we have $\sin 145^\circ = 0.574$, $\cos 145^\circ = -0.819$ and $\tan 145^\circ = -0.700$ (to 3 d.p.)

Angle between 180° and 270° (third quadrant).

In the third quadrant, both the sine and cosine are negative, and therefore the tangent is now positive.

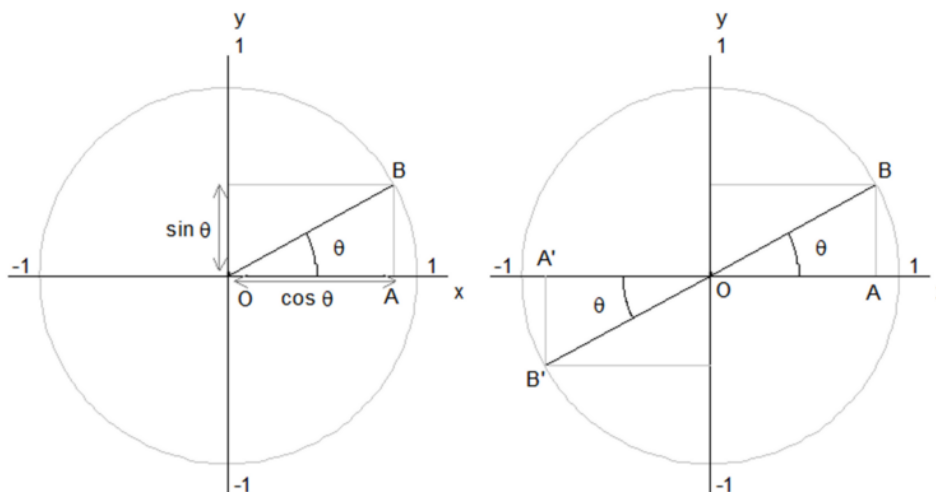
The relationship between angles in the first and third quadrants is shown below.



By symmetry, angle $A'OB'$ in the right-hand diagram is identical to angle AOB , and is on the opposite side of the origin, therefore $\text{angle } AOB' = \text{angle } AOB + 180^\circ$.

It can also be seen that $\sin AOB' = -\sin AOB$ and that $\cos AOB' = -\cos AOB$. By division, it can also be seen that $\tan AOB' = \tan AOB$.

For any angle θ in the first quadrant, the trig ratios of the angle $(180 + \theta)$ in the third quadrant are related as follows:



$$\sin (180 + \theta)^\circ = -\sin \theta^\circ ; \cos (180 + \theta)^\circ = -\cos \theta ; \tan (180 + \theta)^\circ = \tan \theta.$$

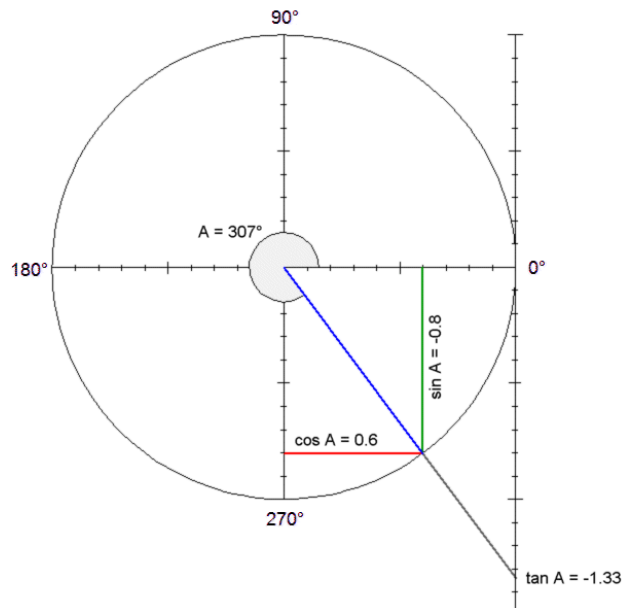
Example (2) : Given $\sin 65^\circ = 0.906$, $\cos 65^\circ = 0.423$ and $\tan 65^\circ = 2.145$ (to 3 d.p.), state the corresponding ratios for the angle of 245° .

Since $245^\circ = (180 + 65)^\circ$,
 we have $\sin 245^\circ = -0.906$, $\cos 245^\circ = -0.423$ and $\tan 245^\circ = 2.145$ (to 3 d.p.)

Angle between 270° and 360° (fourth quadrant).

In the fourth quadrant, the sine is negative but the cosine positive, so the tangent is now negative.

The relationship between angles in the first and fourth quadrants is shown below.

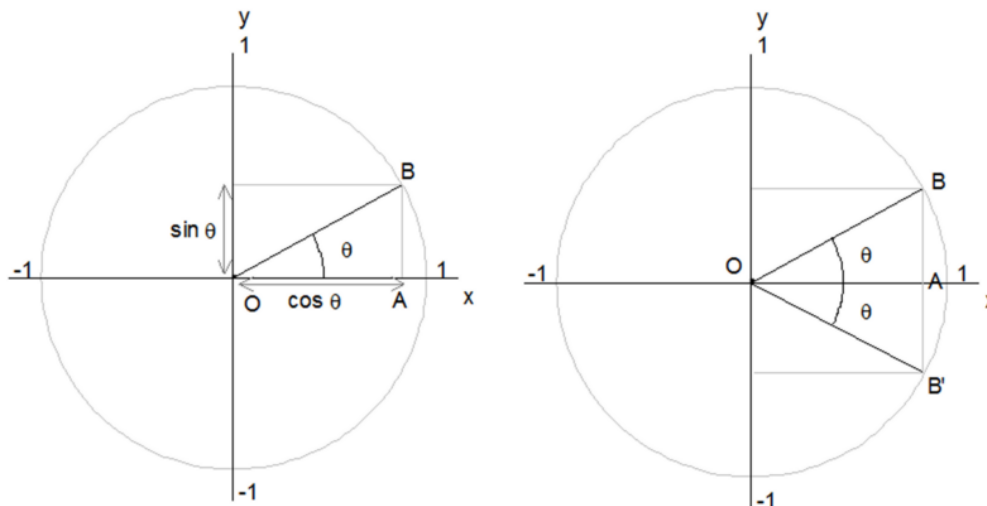


By symmetry, angle AOB' (clockwise) in the right-hand diagram is identical to angle AOB, therefore angles AOB and AOB' (measured anticlockwise) add up to 360°.

It can also be seen that $\sin AOB' = -\sin AOB$ and that $\cos AOB' = \cos AOB$. By division, it can also be seen that $\tan AOB' = -\tan AOB$.

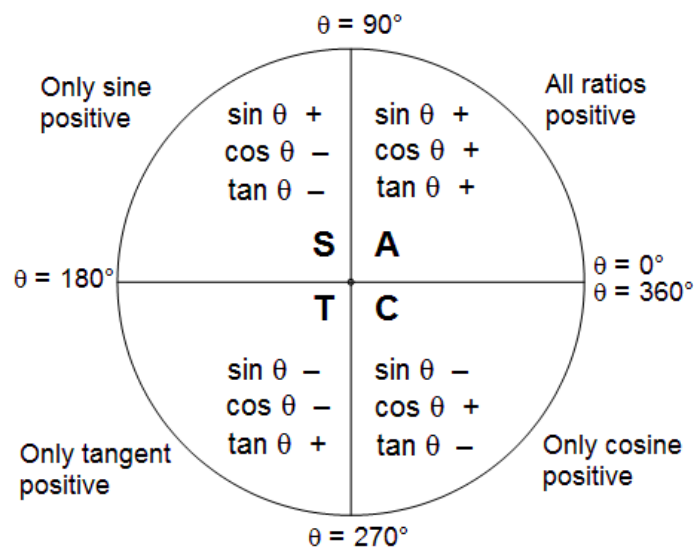
For any angle θ in the first quadrant, the trig ratios of the angle $(360 - \theta)$ in the fourth quadrant are related as follows:

$$\sin (360 - \theta)^\circ = -\sin \theta^\circ ; \cos (360 - \theta)^\circ = \cos \theta ; \tan (360 - \theta)^\circ = -\tan \theta.$$



Example (3) : Given $\sin 50^\circ = 0.766$, $\cos 50^\circ = 0.643$ and $\tan 50^\circ = 1.192$ (to 3 d.p.), state the corresponding ratios for the angle of 310° .

Since $310^\circ = (360 - 50)^\circ$,
 we have $\sin 310^\circ = -0.766$, $\cos 310^\circ = 0.643$ and $\tan 310^\circ = -1.192$ (to 3 d.p.)



These properties can be summarised by the 'CAST rule' above. Start at the lower right (4th quadrant) and write the word CAST in the circle anti-clockwise to remember which ratios are positive.

C- Cosine; A – All; S – Sine; T – Tangent

First quadrant: angle between 0 and 90°.

For any angle θ in the 1st quadrant, its trig ratios are all positive.

Second quadrant: angle between 90° and 180°.

For any angle θ in the 2nd quadrant, we have the trig ratios of the angle $(180^\circ - \theta)$ related as follows:
 $\sin (180^\circ - \theta) = \sin \theta$; $\cos (180^\circ - \theta) = -\cos \theta$; $\tan (180^\circ - \theta) = -\tan \theta$.

Third quadrant: angle between 180° and 270°.

For any angle θ in the 3rd quadrant, we have the trig ratios of the angle $(180^\circ + \theta)$ related as follows:
 $\sin (180^\circ + \theta) = -\sin \theta$; $\cos (180^\circ + \theta) = -\cos \theta$; $\tan (180^\circ + \theta) = \tan \theta$.

Fourth quadrant: angle between 270° and 360°.

For any angle θ in the 4th quadrant, we have the trig ratios of the angle $(360^\circ - \theta)$ related as follows:
 $\sin (360^\circ - \theta) = -\sin \theta$; $\cos (360^\circ - \theta) = \cos \theta$; $\tan (360^\circ - \theta) = -\tan \theta$.

The following ratios hold at the boundaries of the quadrants:

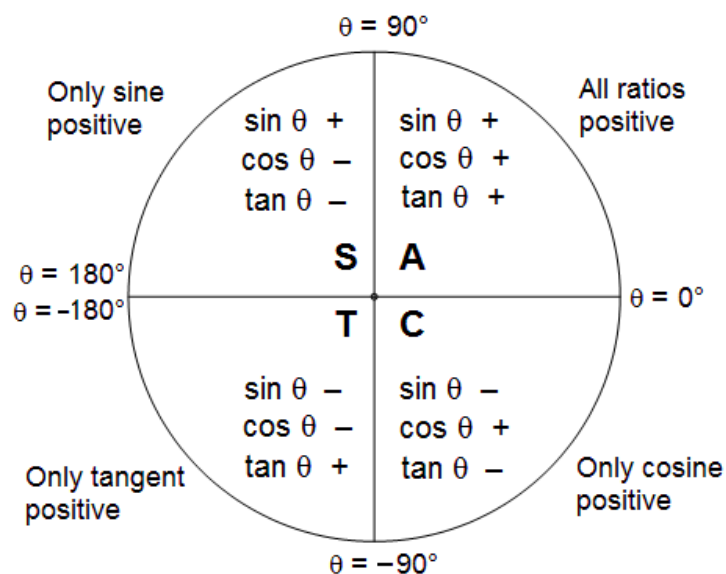
$$\sin 0^\circ = 0; \cos 0^\circ = 1; \tan 0^\circ = 0$$

$$\sin 90^\circ = 1; \cos 90^\circ = 0; \tan 90^\circ \text{ is undefined}$$

$$\sin 180^\circ = 0; \cos 180^\circ = -1; \tan 180^\circ = 0$$

$$\sin 270^\circ = -1; \cos 270^\circ = 0; \tan 270^\circ \text{ is undefined}$$

Alternative CAST diagram for the angle range -180° to 180° :



First quadrant: angle between 0 and 90° .

For any angle θ in the 1st quadrant, its trig ratios are all positive.

Second quadrant: angle between 90° and 180° .

For any angle θ in the 2nd quadrant, we have the trig ratios of the angle $(180^\circ - \theta)$ related as follows:

$$\sin (180^\circ - \theta) = \sin \theta ; \cos (180^\circ - \theta) = -\cos \theta ; \tan (180^\circ - \theta) = -\tan \theta.$$

Third quadrant: angle between -180° and -90° .

For any angle θ in the 3rd quadrant, we have the trig ratios of the angle $(-180^\circ - \theta)$ related as follows:

$$\sin (-180^\circ - \theta) = -\sin \theta ; \cos (-180^\circ - \theta) = -\cos \theta ; \tan (-180^\circ - \theta) = \tan \theta.$$

Fourth quadrant: angle between -90° and 0° .

For any angle θ in the 4th quadrant, we have the trig ratios of the angle $(-\theta)$ related as follows:

$$\sin (-\theta) = -\sin \theta ; \cos (-\theta) = \cos \theta ; \tan (-\theta) = -\tan \theta.$$

The following ratios hold at the boundaries of the quadrants:

$$\sin 0^\circ = 0 ; \cos 0^\circ = 1 ; \tan 0^\circ = 0$$

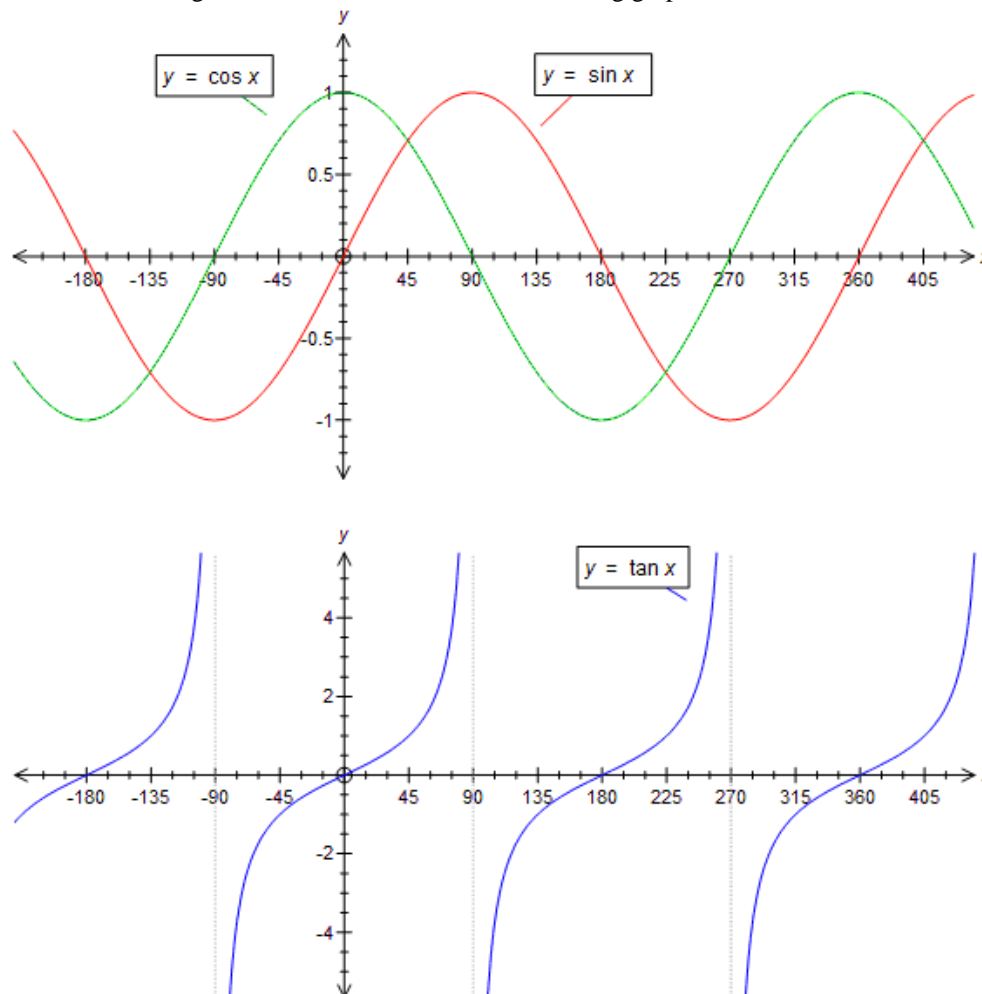
$$\sin 90^\circ = 1 ; \cos 90^\circ = 0 ; \tan 90^\circ \text{ is undefined}$$

$$\sin -180^\circ = 0 ; \cos -180^\circ = -1 ; \tan -180^\circ = 0$$

$$\sin -90^\circ = -1 ; \cos -90^\circ = 0 ; \tan -90^\circ \text{ is undefined}$$

Trigonometric Graphs.

The three main trigonometric functions have the following graphs:



The graphs of $\sin x^\circ$ and $\cos x^\circ$ are similar to each other; in fact they are shown together for comparison. Both functions can only take values in the range -1 to +1, and both repeat themselves every 360° . Indeed, the graph of $\cos x^\circ$ is the same as that of $\sin x^\circ$ translated by -90° in the x -direction.

The graph of $\tan x^\circ$ is quite different. It repeats every 180° , and moreover the function is undefined for certain values of x , such as $90^\circ, 270^\circ$, and all angles consisting of an odd number of right angles. When x approaches 90° from below, $\tan x^\circ$ becomes very large and positive; when x approaches 90° from above, $\tan x^\circ$ becomes very large and negative. The tangent graph therefore has asymptotes at $90^\circ, 270^\circ$, and all angles $90^\circ + 180n^\circ$ where n is an integer.

The graph of $\sin x^\circ$ has rotational symmetry of order 2 about the origin, and is therefore **odd**. There are other points of rotational symmetry at every 180° , and reflectional symmetry in the lines $x = 90^\circ, 270^\circ$ and other odd multiples of 90° , both positive and negative.

The graph of $\cos x^\circ$ has reflectional symmetry in the y -axis, and is thus **even**. There are other reflection lines at $x = 180^\circ, 360^\circ$ and all other multiples of 180° and rotational symmetry of order 2 about the points $(90^\circ, 0), (270^\circ, 0)$ and other points where the graph crosses the x -axis, including negative x .

The graph of $\tan x^\circ$ has rotational symmetry of order 2 about the origin, and is therefore **odd**. Other centres of rotational symmetry of the same order lie along the x -axis where $x = 90^\circ, 180^\circ$ or any multiple of 90° , both positive and negative.

Transformations of Trigonometric Graphs.

Example (4): If $f(x) = \sin x^\circ$, describe the transformations mapping $f(x)$ to
 i) $g(x) = 2 + \sin x^\circ$; ii) $h(x) = \sin 3x^\circ$.

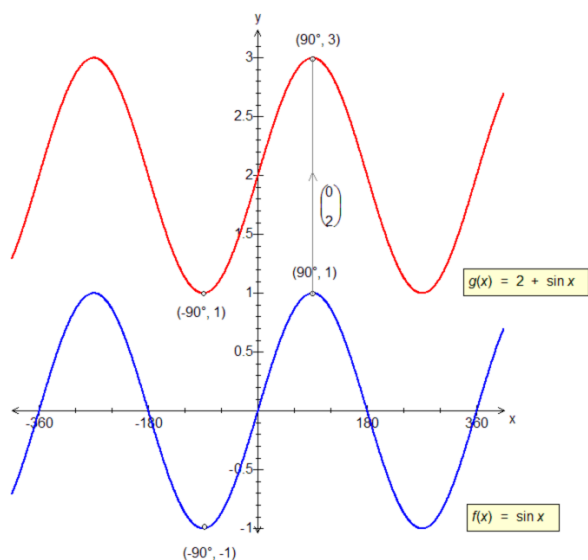
What can be said about the range and periodicity of $g(x)$ and $h(x)$?

The graph of $g(x) = 2 + \sin x^\circ$ is a y -shift of $f(x)$ by + 2 units, i.e. the graph of $\sin x^\circ$ is translated by vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

The range of $f(x)$ is $-1 \leq f(x) \leq 1$, but since $g(x)$ is a translation with vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, its range is also transformed to $1 \leq g(x) \leq 3$.

$\therefore 2 + \sin x^\circ$ can only take values between 1 and 3 inclusive.

The periodicity has not been affected by the translation, though; the period of $g(x)$ is 360° , exactly the same as that of $f(x)$.

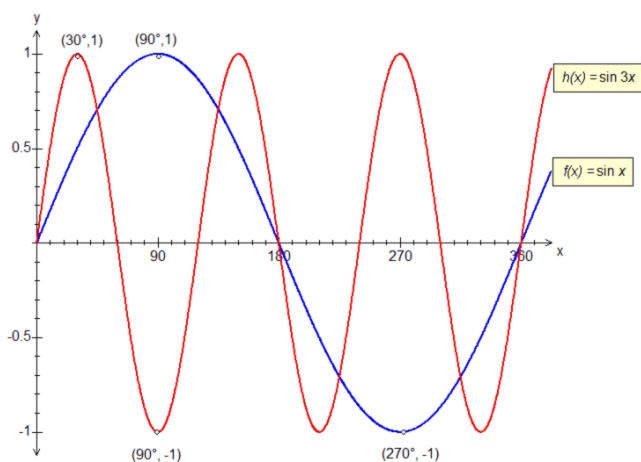


The graph of $h(x) = \sin 3x^\circ$ is an x -stretch of $f(x)$ with scale factor $\frac{1}{3}$, i.e. the graph of $\sin x^\circ$ is **compressed** by a factor of 3 in the x -direction.

The period of $f(x)$ is 360° , but since $h(x) = \sin 3x^\circ$ is an x -stretch of scale factor $\frac{1}{3}$, its period is one-third that of $f(x)$.

\therefore the period of $\sin 3x^\circ$ is 120° .

Also, $h(x)$ has three maxima and three minima between 0° and 360° , whereas $f(x)$ has only one of each.

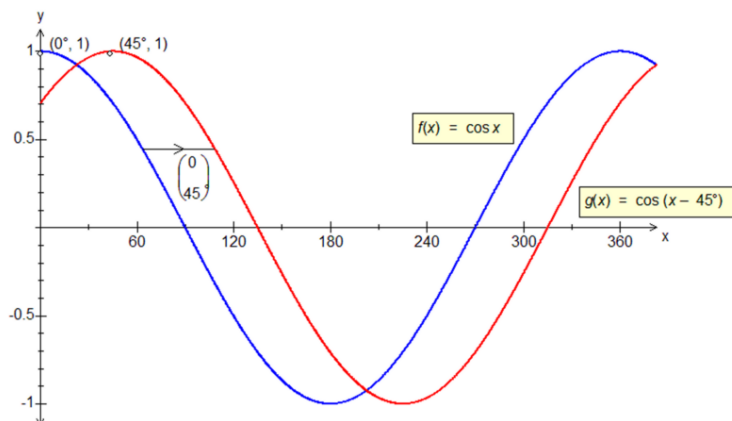


The ranges of $f(x)$ and $h(x)$ are no different however; -1 to 1 inclusive in each case.

Example (5): If $f(x) = \cos x$, describe the transformations mapping $f(x)$ to i) $g(x) = \cos(x - 45^\circ)$; ii) $h(x) = 2 \cos x$. What can be said about the ranges and periodicities of $g(x)$ and $h(x)$?

The graph of $g(x) = \cos(x - 45^\circ)$ is an x -shift of $f(x)$ by $+45^\circ$, i.e. the graph of $\cos x^\circ$ is translated by vector $\begin{pmatrix} 45^\circ \\ 0 \end{pmatrix}$. The shapes of the two graphs appear different, but this is only because we are looking at a different 360° 'snapshot' of the cosine function.

The range of $g(x)$ is $-1 \leq g(x) \leq 1$, the same as that of $f(x)$.

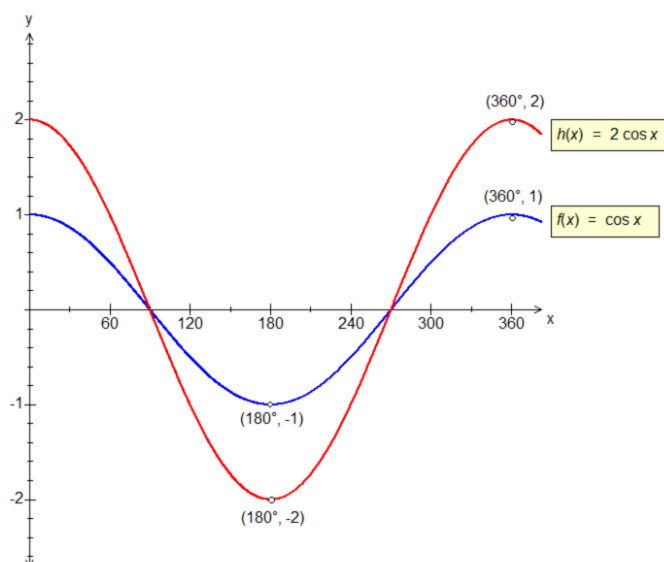


The graph of $h(x) = 2 \cos x^\circ$ is a y -stretch of scale factor 2, i.e. the graph of $\cos x^\circ$ is **stretched** by a factor of 2 in the y -direction.

The range of $f(x)$ is $-1 \leq f(x) \leq 1$, but since $h(x)$ is a y -stretch of factor 2, its range is also transformed to $-2 \leq h(x) \leq 2$.

$\therefore 2 \cos x^\circ$ can only take values between -2 and 2 inclusive.

The periodicity has not been changed by the y -stretch, though; the period of $h(x)$ is still 360° .



Example (6): If $f(x) = \tan 4x^\circ$, what is the periodicity of $f(x)$?

The period of $\tan x^\circ$ is 180° , but since $f(x)$ is an x -stretch of scale factor $\frac{1}{4}$, the period of $f(x)$ will likewise be one quarter that of $\tan x^\circ$, i.e. 45° .
 \therefore the period of $\tan 4x^\circ$ is 45° .

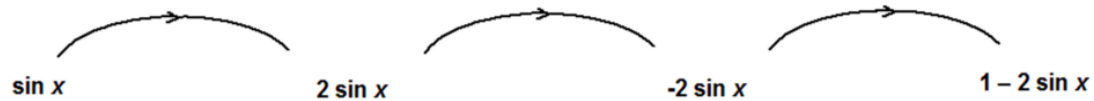
The above results show that the period of a trig function is not changed by any transformation except by an x -stretch. The range, on the other hand, is altered when performing a y -translation or a y -stretch.

Again, composite transformations of trig functions are handled similarly to those of polynomials.

Example (7): The graph of $y = \sin x$ is transformed to that of $y = 1 - 2 \sin x$.

Describe the transformations in detail, and plot each of the three steps on a separate graph, with x taking values between 0° and 360° .

What range of values can $y = 1 - 2 \sin x$ take, and how is it related to the transformations ?



Step 1: a y -stretch of scale factor 2 to transform $y = \sin x$ to $y = 2 \sin x$.

Step 2: a reflection in the x -axis to transform $y = 2 \sin x$ to $y = -2 \sin x$.

(Steps 1 and 2 could be said to consist of a single y -stretch of scale factor -2.)

Step 3: a translation with vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to transform $y = -2 \sin x$ to $y = 1 - 2 \sin x$.

The original function $y = \sin x$ can only take values between -1 and 1, i.e. the range is $-1 \leq y \leq 1$.

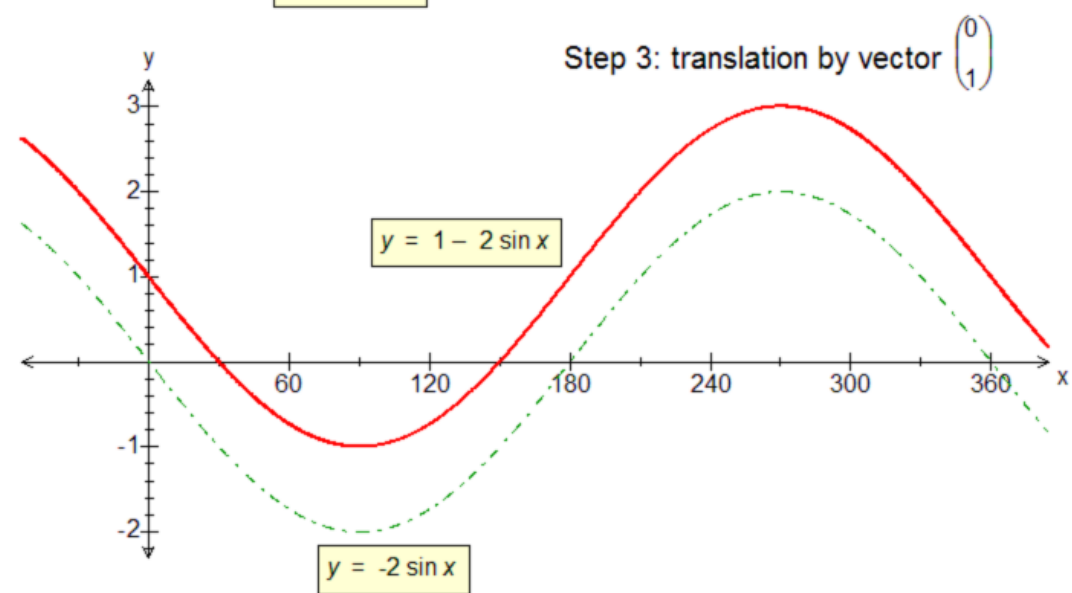
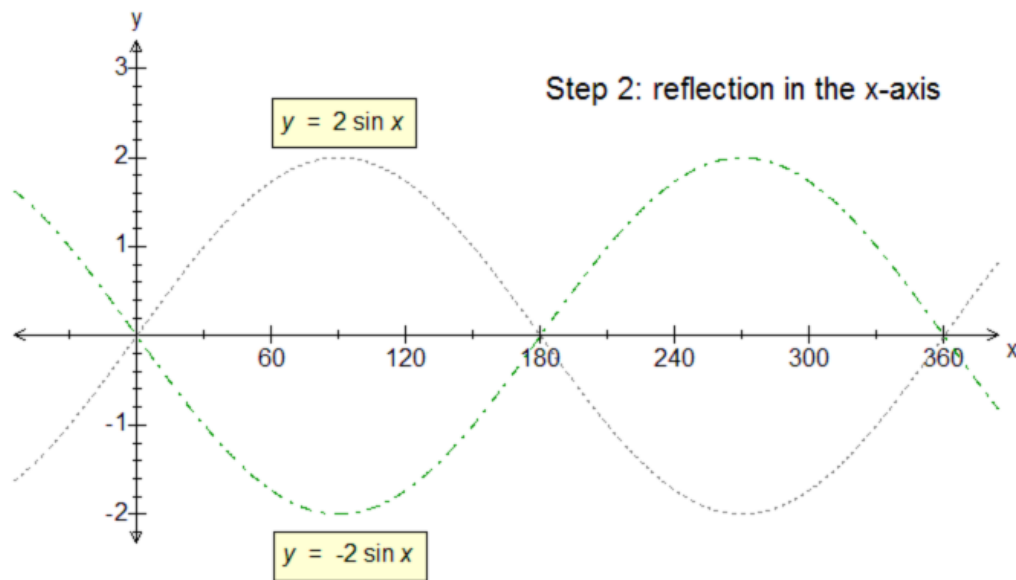
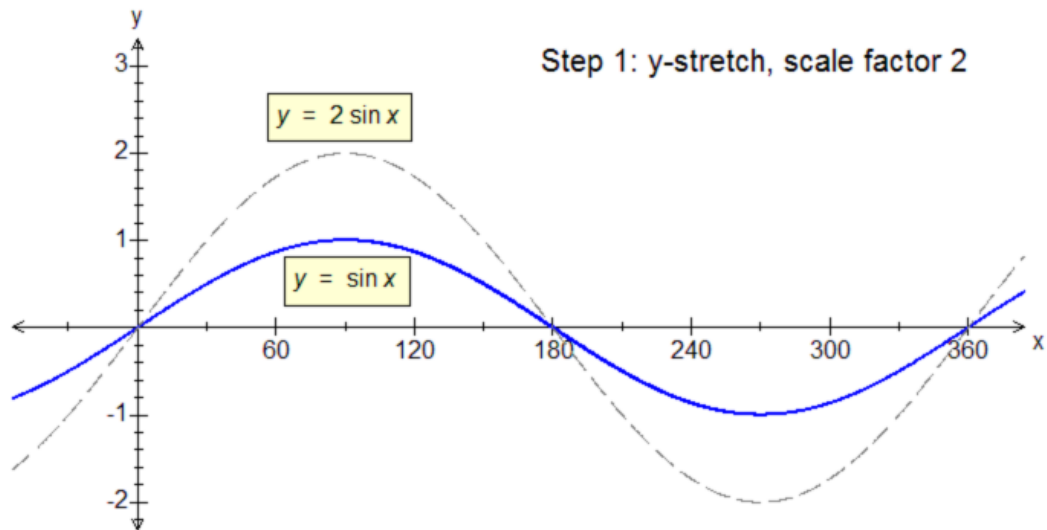
After Step 1, all the y -coordinates are doubled, so the range becomes transformed to $-2 \leq y \leq 2$.

Step 2 has no effect on the range of $y = -2 \sin x$, as it still remains between -2 and 2.

In Step 3, all the y -coordinates are increased by 1, and so the range is transformed again to $-1 \leq y \leq 3$.

\therefore the function $y = 1 - 2 \sin x$ takes a range of values from -1 to 3 inclusive.

See the next page for the graphs.



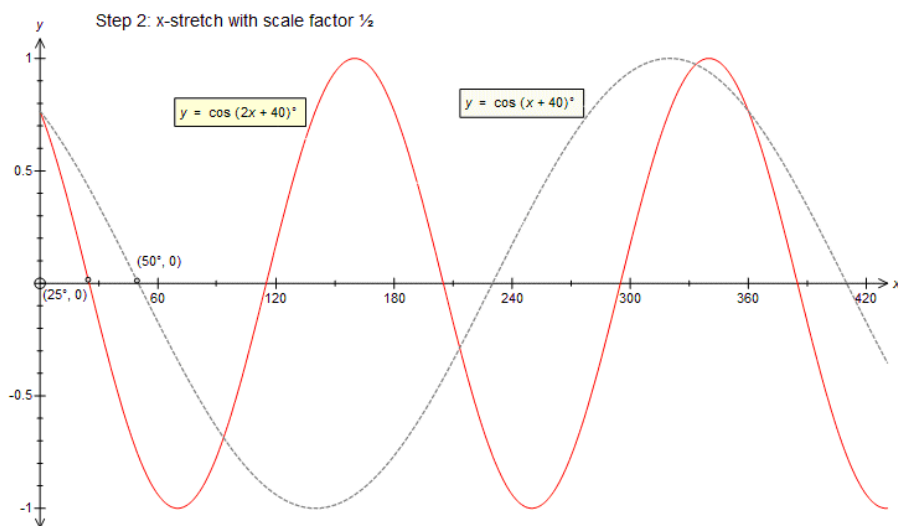
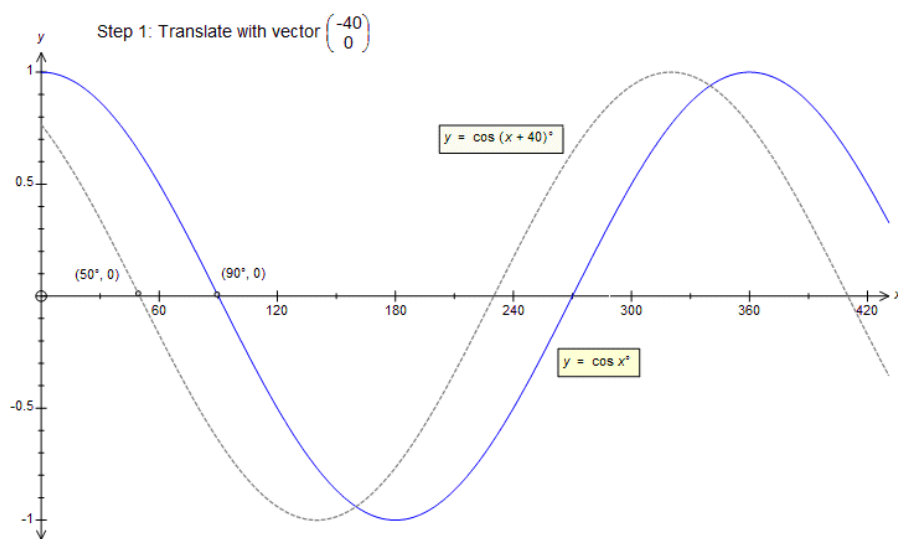
Example (8):

- i) What sequence of transformations maps the graph of $y = \cos x^\circ$ to the graph of $y = \cos (2x + 40^\circ)$?
 ii) The point $(90^\circ, 0)$ on the graph of $y = \cos x^\circ$ is mapped to (p, q) after the combined transformation.
 Find the values of p and q .

i) Firstly, we translate in the x -direction by the vector $\begin{pmatrix} -40 \\ 0 \end{pmatrix}$ - this maps $\cos x^\circ$ to $\cos (x + 40^\circ)$.

Secondly, an x -stretch with scale factor $\frac{1}{2}$ maps $\cos (x + 40^\circ)$ to $\cos (2x + 40^\circ)$.

Note: a common error is to perform the x -stretch *before* the x -translation.



- ii) Because the combined transformation only affects x -coordinates of any points, the value of q remains unchanged at 0.

To find the value of p , we solve $(2x + 40^\circ) = 90^\circ$ for x , giving $x = 25^\circ$.

$$\therefore (p, q) = (25^\circ, 0)$$