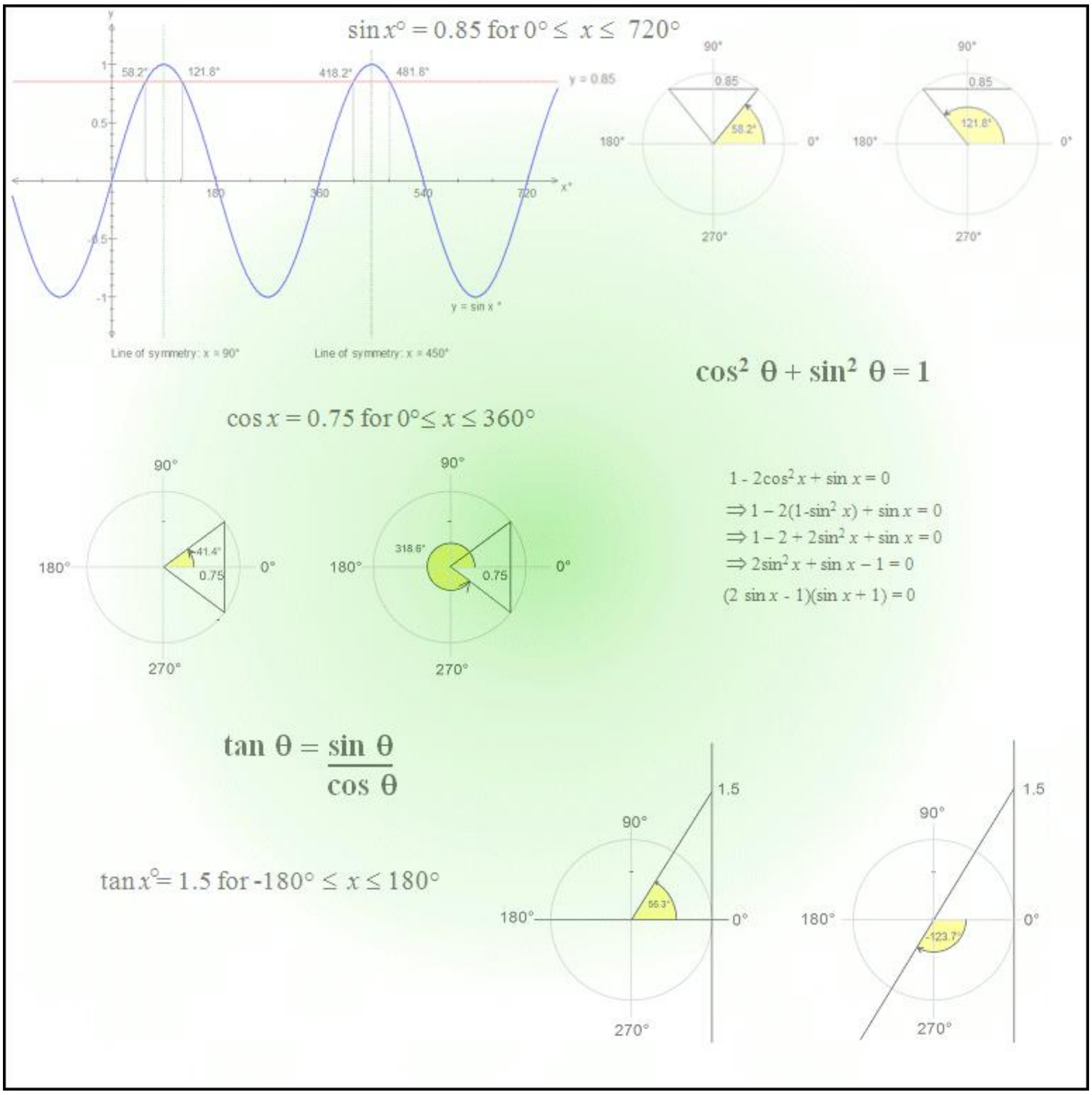
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 1 / AS

SOLVING TRIGONOMETRIC EQUATIONS



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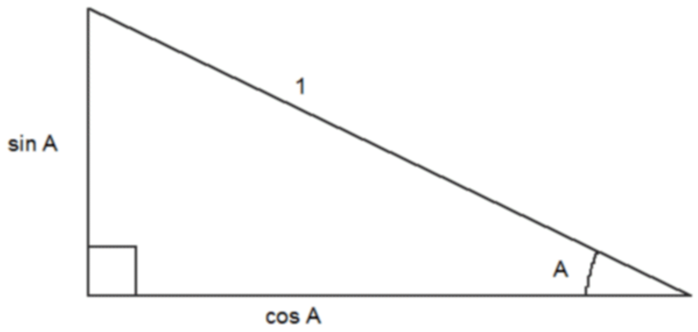
Trigonometric identities.

There are two very important trigonometric identities.

For all angles θ :

$\mathbf{\cos^2 \theta + \sin^2 \theta = 1}$ (This is the Pythagorean identity)



The square on the hypotenuse in the triangle above is 1.
The sum of the squares on the other two sides = $\cos^2 \theta + \sin^2 \theta$.

The length of the opposite side to $A = \sin \theta$ and the length of the adjacent to $A = \cos \theta$.

Since the tangent is the opposite divided by the adjacent, it also follows that

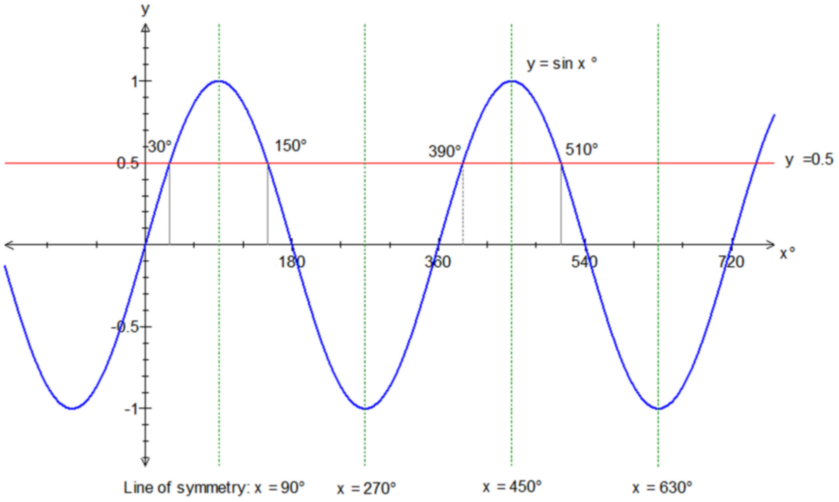
$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

These two identities are important when simplifying expressions or solving various types of equations.

SOLVING TRIGONOMETRIC EQUATIONS

Any trigonometric equation of the form sin $x°$ = a (where $-1 \le a \le 1$), cos $x°$ = a (where $-1 \le a \le 1$) or tan $x°$ = a can have an infinite variety of solutions.

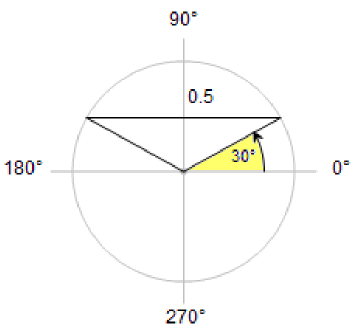
To solve such equations using a calculator, it must be remembered that only *one* of an infinite number of possible values will be given – we must find *all* solutions within the range of the question.

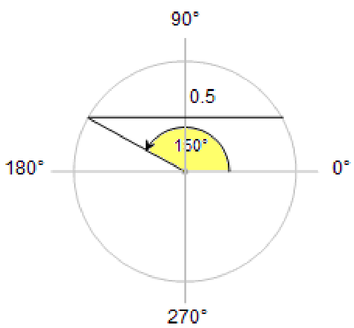


The graph above shows several of the possible solutions of sin $x°$ = 0.5. Apart from the one of 30° given on the calculator (the principal value), the possible solutions also include (180-30)° or 150° because of the reflective symmetry of the graph about the line x = 90°. Moreover, as the graph has a repeating period of 360°, other solutions are 390°, 510° and so on in the positive x-direction, and –210°, -330° and so forth in the negative x-direction.

We can also use CAST diagrams to show the two solutions in the range 0° - 360°, and we can add or subtract multiples of 360° as desired for any further values, should the question ask for them.

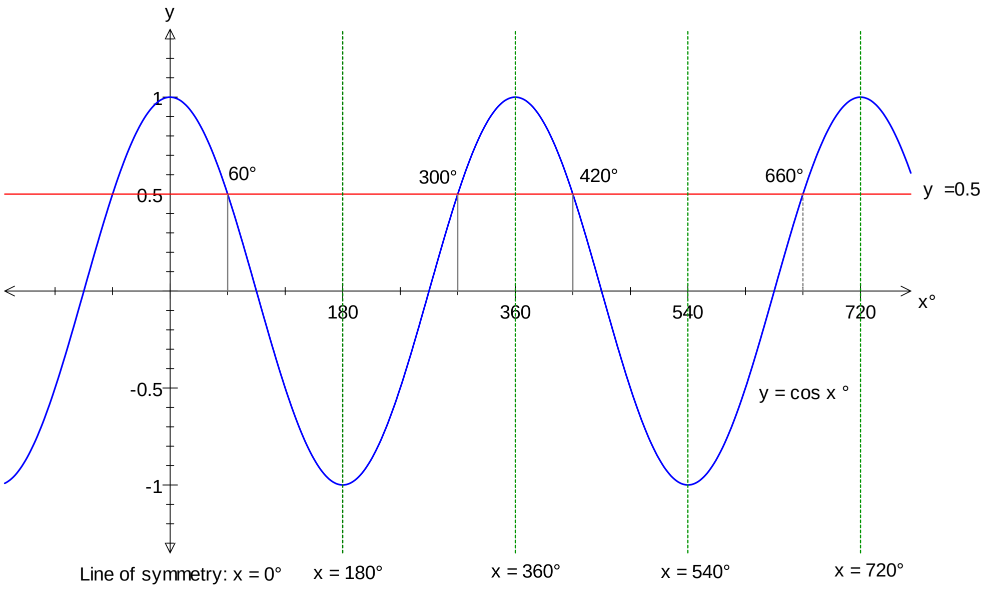
Positive angles are measured anticlockwise from the origin; negative ones clockwise.





In general, if sin $x°$ = a, then :
sin (180-*x*)° = *a*
sin $(180n + x)°$ = a for even integer n ; sin $(180n - x)°$ = a for odd integer n.

Therefore, for instance, sin $(360+x)°$, sin $(540-x)°$ and sin $(-180-x)°$ are also equal to sin x.

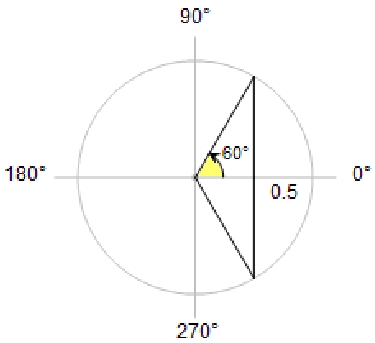


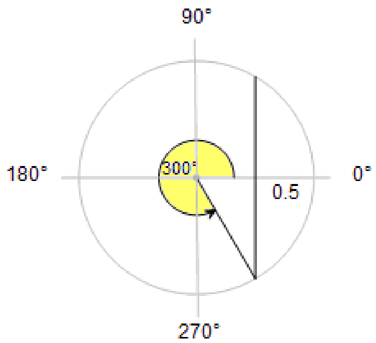
The cosine function also shows similar periodicity.

Several of the possible solutions of cos x° = 0.5 are shown in the graph above. Apart from the one of 60° given on the calculator(the principal value), the possible solutions also include 300° because the graph has reflective symmetry about the line x = 180°. Again, there is a repeating period of 360°, and so other solutions are 420°, 660° and so on in the positive x-direction, and -60’, –300°, -420° and so forth in the negative x-direction.

The CAST diagrams to show the two solutions in the range 0° - 360°. Again, we can add or subtract multiples of 360° if required.

The relationship between cos 60° and cos -60° is evident from the left-hand diagram.





In general, if cos x° = a, then :
cos (-x)° = a or **cos (360-x)° = a**
cos (360n + x)° = a for any integer n
cos (360n - x)° = a for any integer n.

Therefore, for instance, cos (360+x)°, cos (720-x)° and cos (-360-x)° are also equal to cos x.

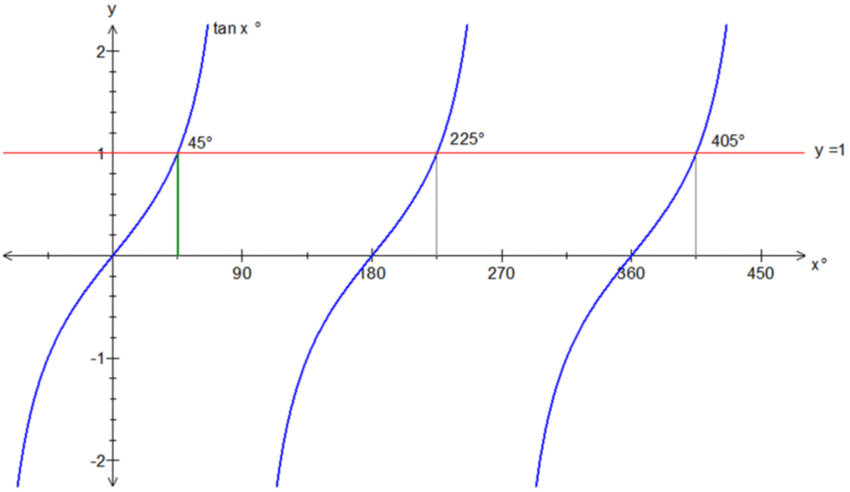
The symmetrical pairing of solutions in the range $0° \le x < 360°$ is common to both the sine and cosine graphs, for all angles on either side of the graphs' lines of symmetry.

(An exception occurs when sin x and cos x = ±1 ; then there is only one solution in that range.)

Thus sin 115° = sin 65° (both equidistant from line of symmetry at 90°), and cos 155° = cos 205° (both equidistant from line of symmetry at 180°).

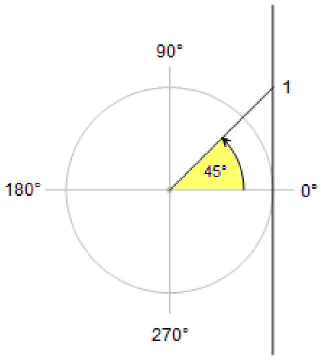
The tangent function, on the other hand, has a period of 180° rather than 360° as in the sine and cosine functions. Also, there is no linear symmetry, so there is no 'doubling-up' of solutions (as in sine and cosine examples).

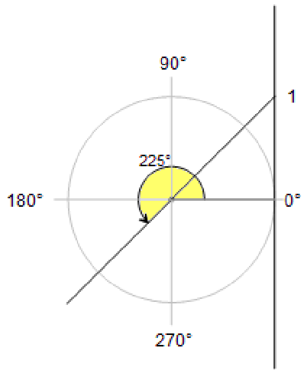
The graph shows several solutions of the equation tan x° = 1.



Apart from the one of 45° given on the calculator, there are other possible solutions because the graph has a repeating period of 180°. Those solutions are 225°, 405° and so on in the positive x-direction, and –135°, -315° and so forth in the negative x-direction.

The CAST diagrams for the tangent function are simpler in form than the others – you merely add multiples of 180° to obtain all the solutions needed.



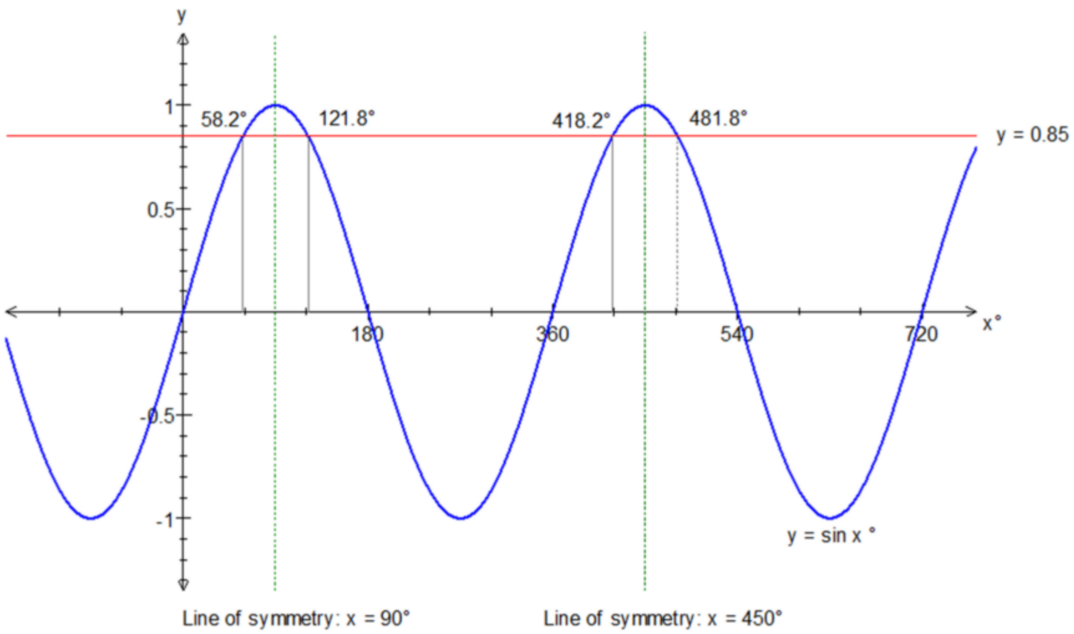


In general, if tan x° = a, then tan $(180n + x)$° = a for any integer n.

Example (1): Solve sin x° = 0.85 for 0° ≤ x ≤ 720°, giving answers in degrees to one decimal place.

The principal value is **58.2°**, but care must be taken to ensure that all solutions within the range are given.

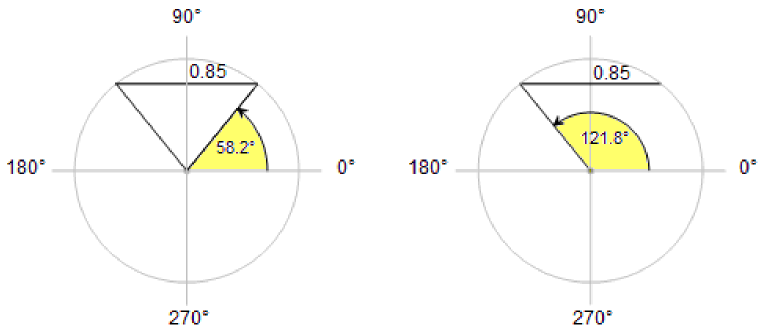
The substitution formulae can be used, but a sketch graph or CAST diagram will probably be easier in finding extra solutions.



The two important lines of symmetry here are x = 90° and x = 450°.
Since 58.2° is 31.8° *less* than 90°, the solution on the other side of that line of symmetry must be 31.8° *greater* than 90°, or **121.8°.**

Alternatively we can use sin (180-x)° = sin x°, and 180° - 58.2° = 121.8°.

CAST diagram illustration:



Having obtained the two solutions above, it is a simple matter of adding and subtracting multiples of 360° as required. Subtracting 360° is no help as there will be no new values found within the range, but adding 360° will give two other solutions:

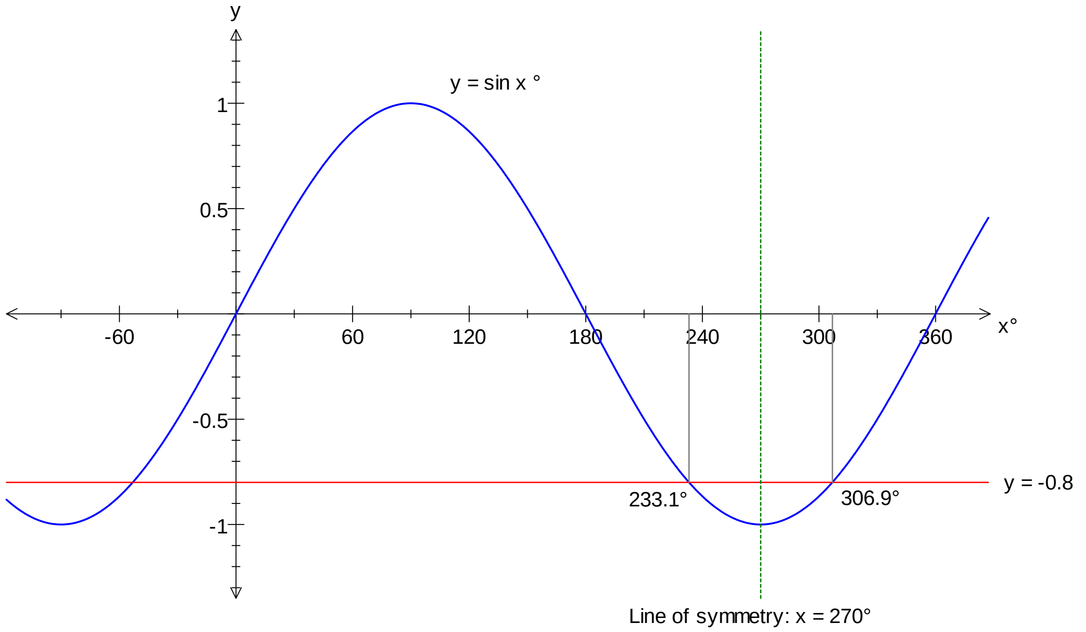
(58.2 + 360)° or **418.2°.**
(121.8 + 360)° or **481.8°.**

∴ the solutions of sin x° = 0.85 for 0° ≤ x ≤ 720° are **58.2°, 121.8°, 418.2°** and **481.8°**.

Example (2): Solve sin x° = -0.8 for 0° ≤ x ≤ 360°, giving answer in degrees to one decimal place.

The principal value, and the one given on a calculator, is **-53.1°**, from which we can derive the other solutions. Note that this solution is not in the quoted range, and so we must add an appropriate multiple of 360° (the period of sin x) to it.

Here, adding 360° gives one solution, i.e. **306.9°**.



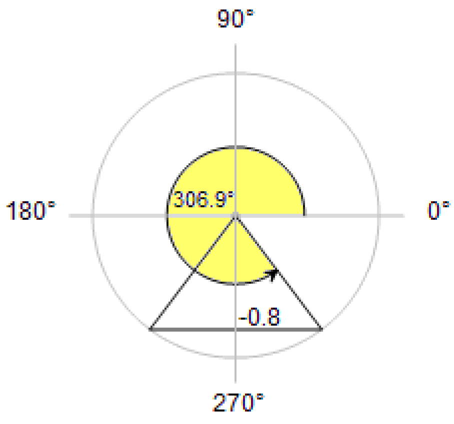
The important line of symmetry here is x = 270°.

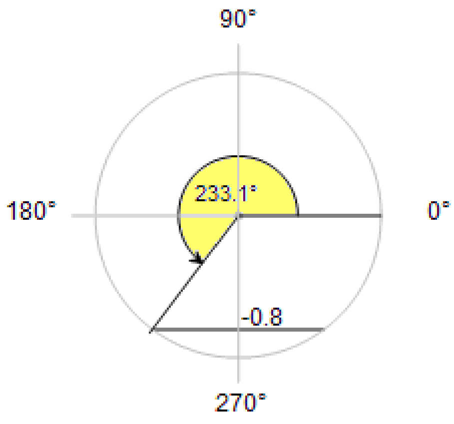
Since 306.9° is 36.9° *greater* than 270°, the solution on the other side of that line of symmetry must be 36.9° *less* than 270°, or **233.1°.**

∴ the solutions of sin x° = -0.8 for 0° ≤ x ≤ 360° are **233.1°** and **306.9°.**

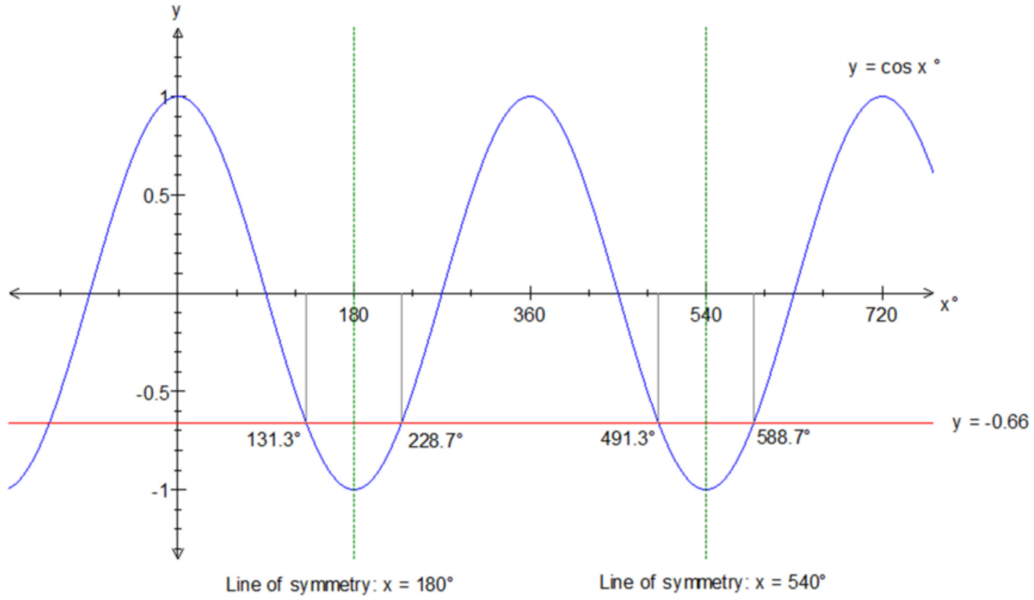
Again we could have used sin (180-x)° = sin x°, and 180° - (-53.1°) = 233.1°, although it is less obvious from the diagram.

CAST diagram illustration:





Example (3): Solve cos x° = -0.66 for 0° ≤ x ≤ 720°, giving answer in degrees to one decimal place.



The two important lines of symmetry here are x = 180° and x = 540°.

Here, the principal value, and the one given on a calculator, is 131.3°.

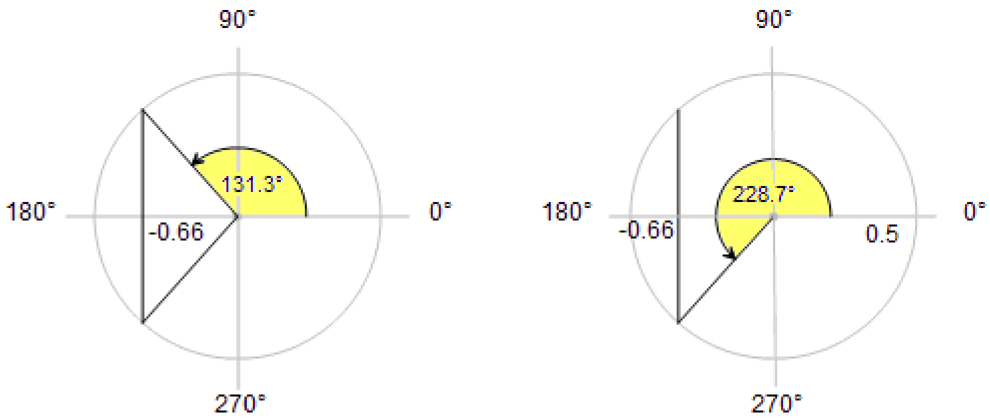
Since 131.3° is 48.7° *less* than 180°, the solution on the other side of that line of symmetry must be 48.7° *greater* than 180°, or **228.7°.**

We could have used cos (360-x)° = cos x°, and 360° - 131.3° = 228.7°.

Having obtained the two solutions above, it is a simple matter of adding and subtracting multiples of 360° as required. Subtracting 360° is no help as there will be no new values found within the range, but adding 360° will give two other solutions, i.e. **491.3°** and **588.7°.**

∴the solutions of cos x° = -0.66 for 0° ≤ x ≤ 720° are 131.3°, 228.7°, 491.3° and 588.7°.

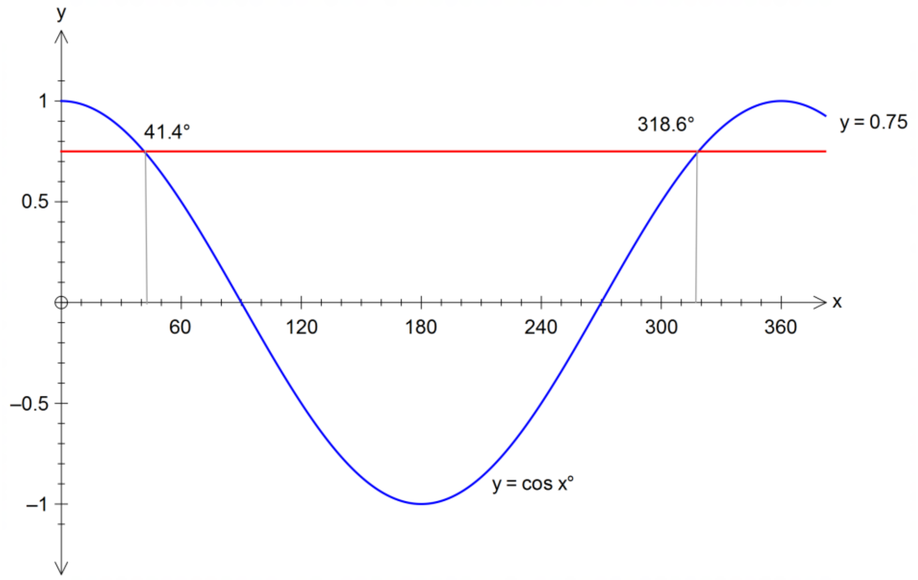
CAST diagram illustration:



Example (4): Solve cos x = 0.75 for $0 \le x \le 360°$, giving answers in degrees to 1 decimal place.

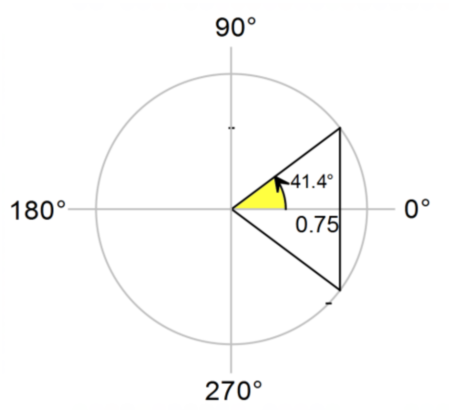
The principal value is **41.4°** , but to find the other solution, we can make use of the lines of symmetry at $x = 0$ and $x = 360°$. Since the principal value is 41.4° *greater* than zero, the other required value must be 41.4° *less* than 360°, or **318.6°**. (This is the same as using the identity cos (360°-x) = cos x.)

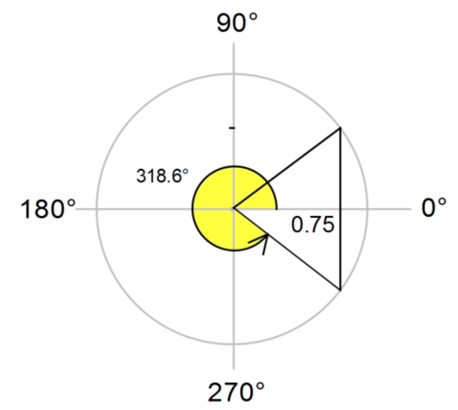
We could have also used the line of symmetry at x = 180°, and worked the value as 180° + (180° - 41.4°) , leading to the same conclusion.



∴ the solutions of cos x = 0.75 for $0 \le x \le 360°$ are 41.4° and 318.6°.

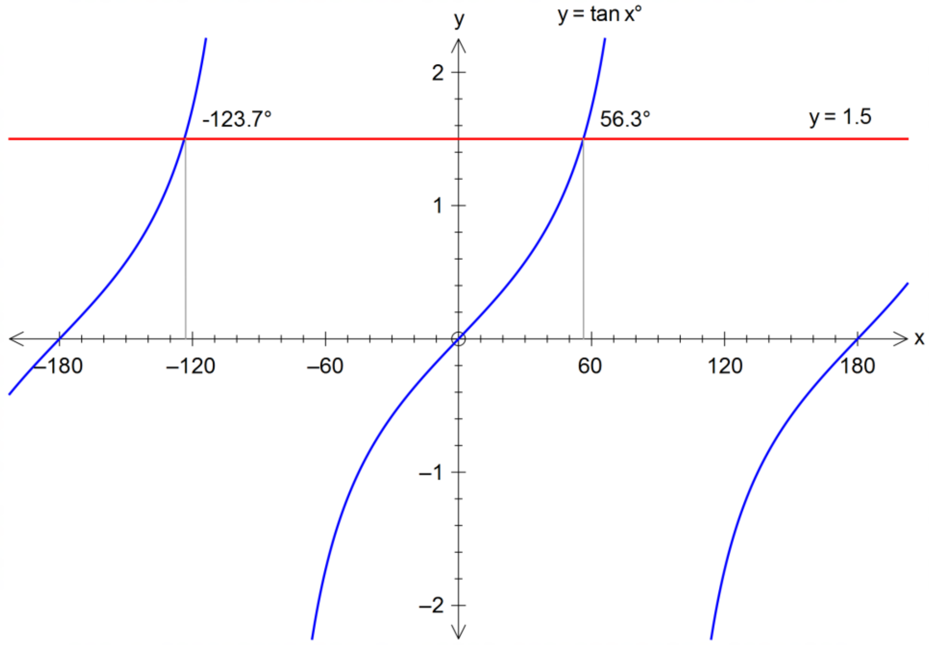
CAST diagram illustration:





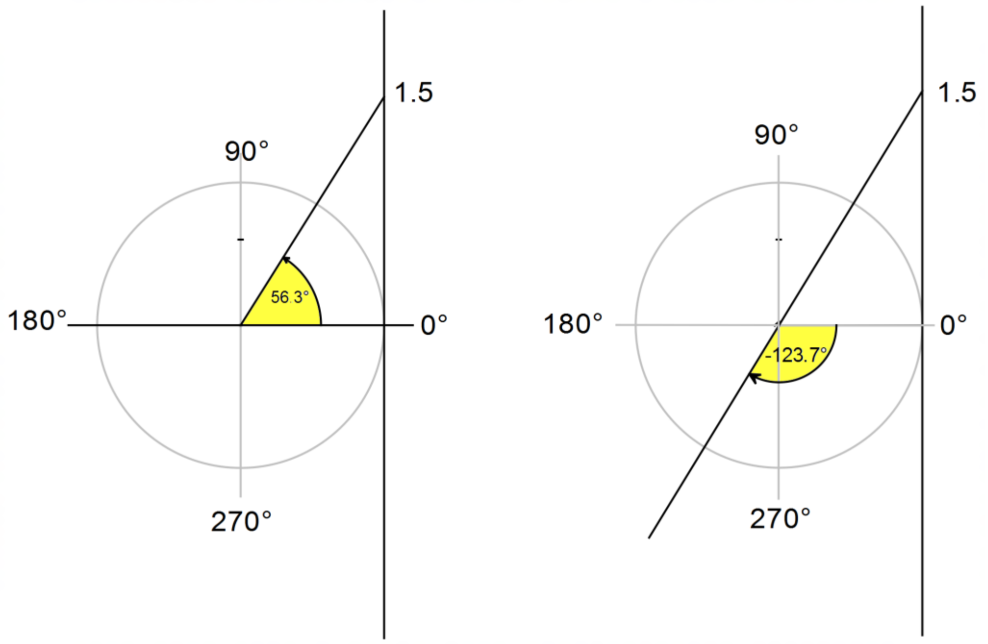
Solution of equations of the form tan $x = k$ is easier: we just add or subtract multiples of 180° to / from the principal value.

Example (5): Solve tan x = 1.5 for -180° ≤ x ≤ 180°, giving the answer in degrees to 1 decimal place.



The principal value is 56.3°, but because the tangent function repeats itself every 180°, there is another solution at (56.3 - 180)° or –123.7°.

CAST diagram illustration:



Note that the negative angle is denoted by a clockwise arrow.

Summary. (The principal solution is the one displayed on the calculator.)

The rules are designed to find solutions in the range -180° ≤ x < 180°.
This range has the virtue of having the principal solution coincide with the calculator display.

If a different range is stipulated, it is only a matter of adding multiples of 360° to the solutions so obtained (when solving equations of the form sin $x = k$ or cos $x = k$), or multiples of 180° (when solving equations of the form tan $x = k$). In each case, n is an integer.

Solving sin $x = k$.

Value of k	Principal soln.	Companion solution	Additional solutions
0	x = 0°	(none)	Add/subtract 180n °
1	x = 90°	(none)	Add/subtract 360n °
-1	x = -90°	(none)	Add/subtract 360n °
positive	0° < x < 90°	180° - x	Add/subtract 360n °
negative	-90° < x < 0°	-180° - x	Add/subtract 360n °

Solving cos $x = k$.

Value of k	Principal soln.	Companion solution	Additional solutions
0	x = 90°	(none)	Add/subtract 180n °
1	x = 0°	(none)	Add/subtract 360n °
-1	x = -180°	(none)	Add/subtract 360n °
positive	0° < x < 90°	- x	Add/subtract 360n °
negative	90° < x < 180°	- x	Add/subtract 360n °

Solving tan $x = k$.

Value of k	Principal soln.	Companion solution	Additional solutions
0	x = 0°	(none)	Add/subtract 180n °
positive	0° < x < 90°	(none)	Add/subtract 180n °
negative	-90° < x < 0°	(none)	Add/subtract 180n °

Sometimes we need to use the idea of transformations to solve slightly more complex trig equations.

Example (6): Use the results from Example (4) to solve cos $2x = 0.75$ for $0 \le x \le 180°$, giving your answers to the nearest degree.

Here the limits for x are $0 \le x \le 180°$, but we are asked to solve for $2x$. To ensure that no values are omitted, we must substitute A for $2x$ and multiply the limiting values by 2 to get the transformed limit of $0 \le A \le 360°$.

From Example (4), we see that the solutions of cos $A = 0.75$ for $0 \le A \le 2\pi$ are 56.3° and 123.7°. To convert the A-values back to x-values, we must divide by 2.

$\therefore$ the solutions of cos $2x = 0.75$ for $0 \le x \le 180°$ are 28° and 62° to the nearest degree.

Example (7): Solve $\tan(2x + 60°) = 1$ for $0 \le x \le 360°$.

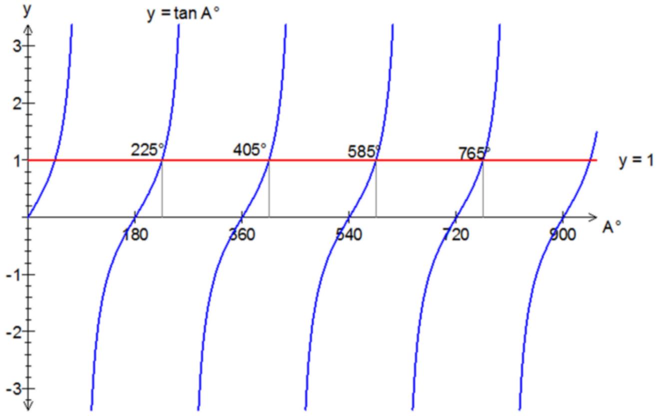
By letting A stand for $2x + 60°$, we first transform the x-limits to A-limits:

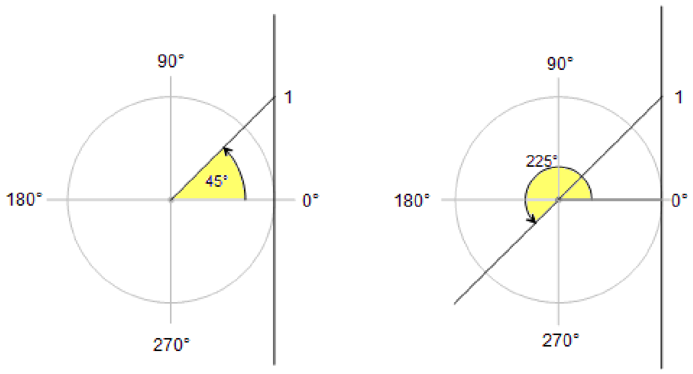
$0 \le x \le 360° \Rightarrow 60° \le A \le 780°$ (x-values doubled, 60° added).

When drawing sketch graphs or CAST diagrams, draw them with respect to the transformed variable A, and not the original variable x.

The substitution back to x-values must be done afterwards.

The principal value of A where tan A = 1 is 45°, but that value is not within the A-limits.
We therefore keep adding multiples of 180° to obtain A = 225°, 405°, 585° and 765°.





Since we doubled the x-values and then added 60° to get the A-values, we must perform the inverse operations to change the A-values back to x-values – i.e. subtract 60° and then halve the result.

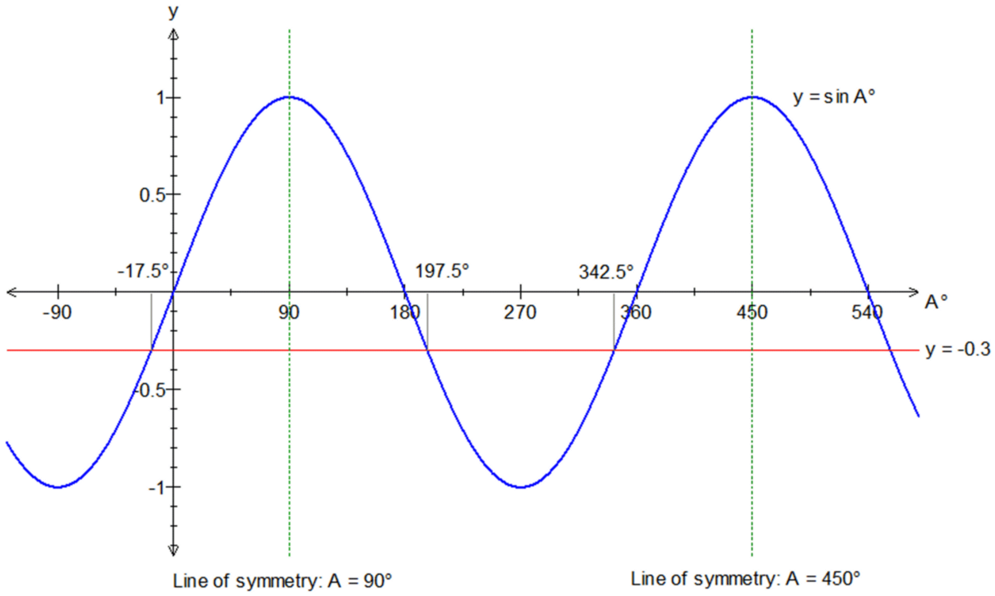
Therefore $A = 225° \Rightarrow x = ½(225\text{-}60)° = 82.5°$.
Similarly $A = 405° \Rightarrow x = 172.5°$, $A = 585° \Rightarrow x = 262.5°$ and finally $A = 765° \Rightarrow x = 352.5°$.

The solutions in the range are therefore x = 82.5°, 172.5°, 262.5° and 352.5°.

Example (8): Solve sin (3*x* - 40)° = -0.3 for 0 ≤ *x* ≤ 180° giving answers in degrees to one decimal place.

Again, we must substitute the *x*-limits of 0 ≤ *x* ≤ 180° with *A* – limits. By using *A* = 3*x*- 40, the *A*-limits transform to –40° ≤ *A* ≤ 500°.

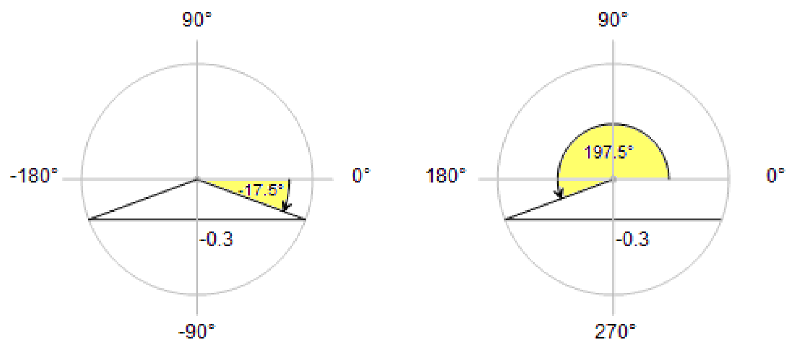
The graph below uses the *A*-limits.



The principal value of *A* is here –17.5°,. which is inside the *A*-limits.
This value is 107.5° to the left of the line of symmetry at *A*= 90°, so another solution will be 107.5° to the right, i.e. at *A* = 197.5°
(We could also have used sin (180-*x*)° = sin *x*°, and 180° - (-17.5°) = 197.5°.)

CAST diagram working:

Note that the negative angle is labelled with a clockwise arrow.



To complete the full set of solutions for *A* in the required range, we add multiples of 360° to the two above values. In fact the only additional one is *A* = (-17.5 + 360)° = 342.5°.

These three solutions will then need converting back from *A*-values to *x*-values by the inverse operation of

$x = \frac{A+40}{3}$, so when *A* = -17.5°, *x* = 7.5°.

Similarly, when *A* = 197.5°, *x* = 79.2° and when *A* = 342.5°, *x* = 127.5°.

Hence the solutions of sin (3*x* - 40)° = -0.3 for 0 ≤ *x* ≤ 180° are **7.5°, 79.2°** and **127.5°.**

Quadratic equations in the trigonometric functions are solved in the same way, but care must be taken when factorising them.

Example (9) : Solve the equation $2\cos^2 x - \cos x = 0$ for $-180° \le x \le 180°$.

This factorises at once into $(\cos x)(2\cos x - 1) = 0$.

$\cos x = 0$ when $x = 90°$ or $x = -90°$.

$2\cos x = 1$, or $\cos x = 0.5$, when $x = 60°$ or $x = -60°$.

The solutions to the equation are therefore $x = \pm 90°$ and $\pm 60°$.

Important: we cannot simply cancel out $\cos x$ from the equation as follows:

$2\cos^2 x - \cos x = 0 \Rightarrow 2\cos^2 x = \cos x \Rightarrow 2\cos x = 1 \Rightarrow x = \pm 60°$.

The final division of both sides by $\cos x$ has led to a loss of the solutions satisfying $\cos x = 0$, i.e. $x = \pm 90°$.

Example (10) : Solve the equation $\cos^2 x - \cos x - 1 = 0$ for $0 \le x \le 360°$. Give the answer in degrees to one decimal place.

This quadratic does not factorise and so the general formula must be used.

Substitute $x = \cos x$, a = 1, $b = -1$ and $c = -1$ into the formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\cos x = \frac{1 \pm \sqrt{5}}{2} \ .$$

These are the possible solutions, and are 1.618 and -0.618 to 3 decimal places.

The first value can be rejected, since the cosine function cannot take values outside the range $-1 \le \cos x \le 1$.

The only solutions are those where $\cos x = \dfrac{1-\sqrt{5}}{2}$. The principal value of x is 128.2°.

Since $\cos(360° - x) = \cos x$, another solution would be $(360 - 128.2)°$ or 231.8°.

$\therefore$ the solutions of $\cos^2 x - \cos x - 1 = 0$ where $0 \le x \le 360°$, are **128.2°** and **231.8°** (1 d.p.)

Example (11): Find the angle(s) between 0° and 360° satisfying the equation

$1 - 2\cos^2 x + \sin x = 0$

This equation can be manipulated into a quadratic in sin x by replacing $2\cos^2 x$ with $2(1 - \sin^2 x)$ using the Pythagorean identity.

$\Rightarrow 1 - 2(1 - \sin^2 x) + \sin x = 0$
$\Rightarrow 1 - 2 + 2\sin^2 x + \sin x = 0$
$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$

This is now a quadratic in sin x which factorises into $(2 \sin x - 1)(\sin x + 1) = 0$

Hence $\sin x = 0.5$ or -1.

This gives $x = 30°$ or $150°$ (where $\sin x = 0.5$); also $x = 270°$ (where $\sin x = -1$).

Example (12): Prove that $\dfrac{(1+\cos x)(1-\cos x)}{\sin x \cos x} = \tan x$.

The top line of the left-hand expression can be recognised as a 'difference of squares' result, namely $1 - \cos^2 x$. This in turn can be replaced by $\sin^2 x$ using the Pythagorean identity.

The expression on the left thus becomes $\dfrac{\sin^2 x}{\sin x \cos x}$, and dividing both sides by sin x we finally obtain $\dfrac{\sin x}{\cos x}$ or tan x.

Example (13): Show that the equation $\sin x - \cos^2 x - 5 = 0$ has no solutions.

Substituting $\cos^2 x$ by $1 - \sin^2 x$ gives $\sin x - (1 - \sin^2 x) - 5 = 0$

$\Rightarrow \sin x - 1 + \sin^2 x - 5 = 0$
$\Rightarrow \sin^2 x + \sin x - 6 = 0$
$\Rightarrow (\sin x + 3)(\sin x - 2) = 0$
$\Rightarrow \sin x = 2$ or -3.

The quadratic in sin x factorises all right, but the solutions are impossible because the sine function cannot take values outside the range $-1 \le \sin x \le 1$.

Example (14): Solve the equation $\sin x = \sqrt{3}\cos x$ for angles between 0° and 360°.

We can divide the equation by cos x to give $\dfrac{\sin x}{\cos x} = \sqrt{3}$, or $\tan x = \sqrt{3}$.

The principal value of x is 60°, but since the tangent graph repeats every 180°, another solution is x = 240°.