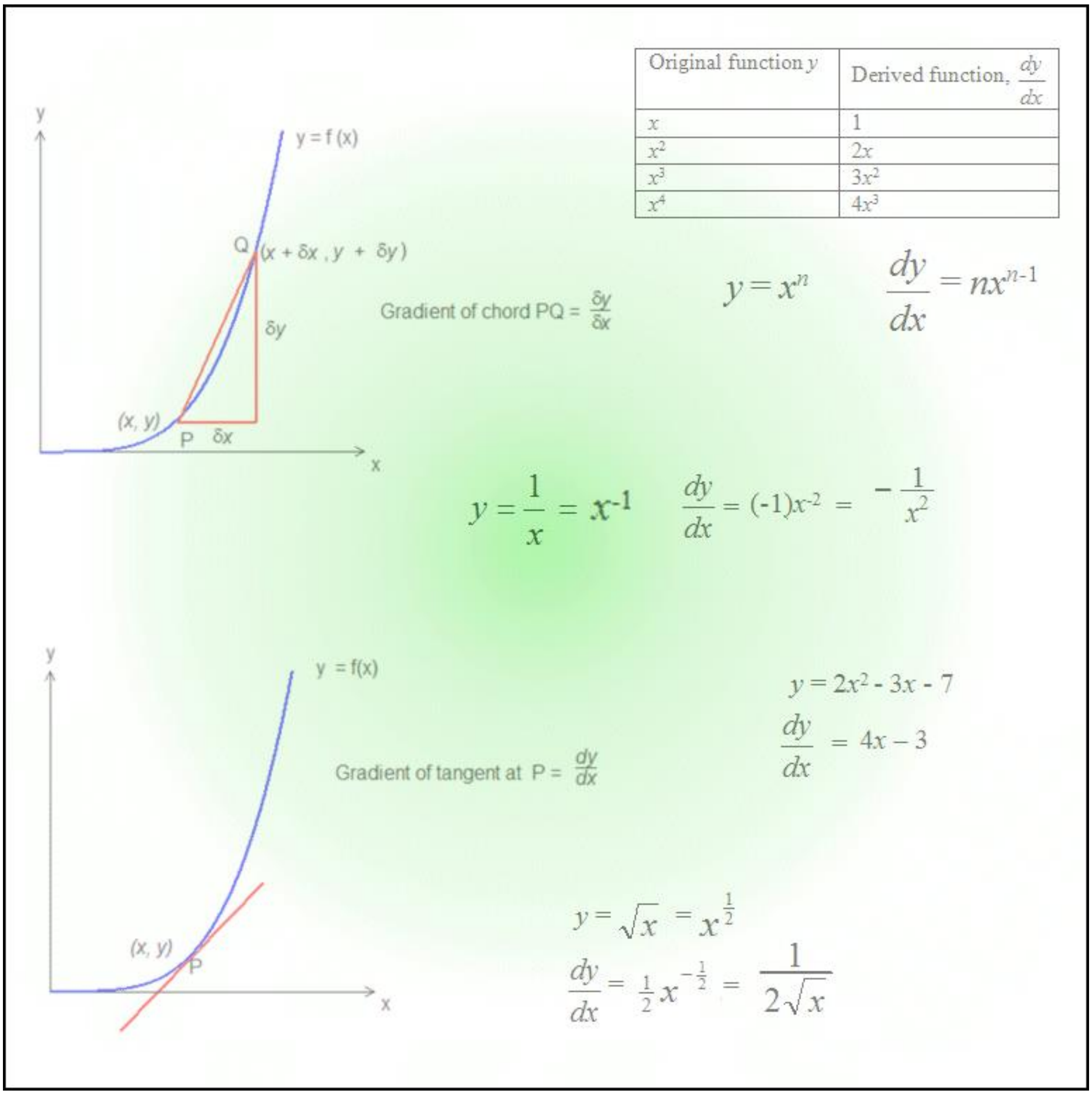
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 1 / AS

INTRODUCTION TO DIFFERENTIATION



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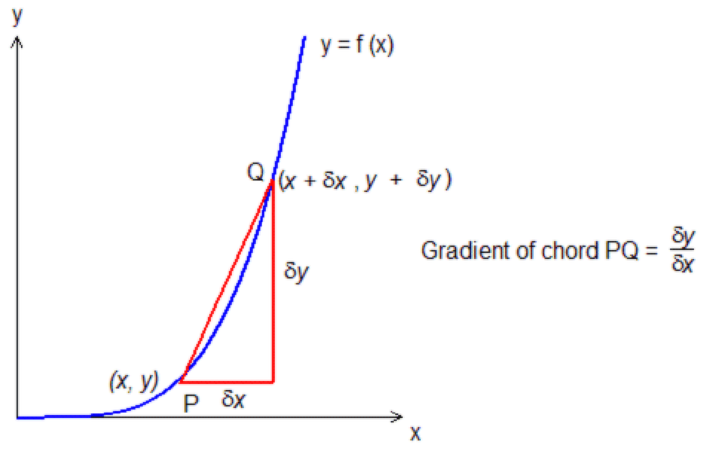
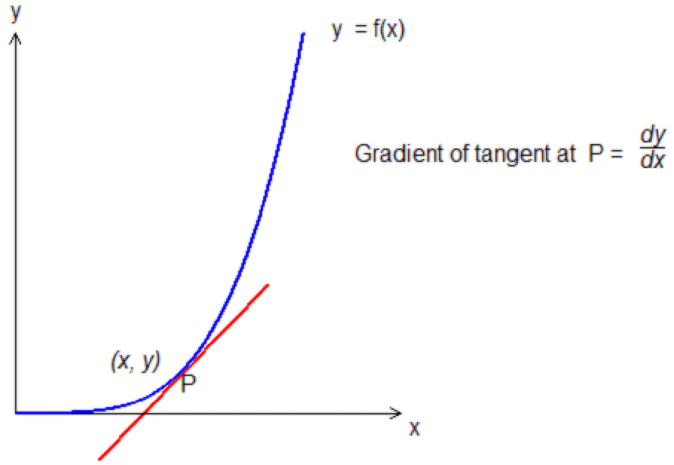
DIFFERENTIATION.

The gradient of a curve.

The idea of a gradient was brought about when studying linear functions. Now, linear functions have a constant gradient. The function $y = 2x - 5$, for example, has a gradient of 2 regardless of the value of x.

The gradient of a curve, by contrast, changes continuously along its length.





Take the chord PQ in the upper diagram. Its gradient is given by $\frac{\delta y}{\delta x}$ (delta y over delta x) where δx is a small change in x and δy is the corresponding change in y.

Note that δy and δx are not products – the symbol δ means " a small difference in".

As the chord PQ becomes smaller and smaller, the ratio $\frac{\delta y}{\delta x}$ approximates to the gradient of the curve at P more and more closely, until the point P and Q coincide and the chord becomes a tangent at P.

As Q moves towards P, the value of $\delta x \to 0$ and $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, i.e. $\frac{\delta y}{\delta x}$ tends to $\frac{dy}{dx}$.

$\frac{dy}{dx}$ is the **gradient function** and represents the **derivative** of y with respect to x.

Some gradient functions can be worked out using the method on the previous page – known as **differentiation from first principles**.

Example (1): Differentiate $y = x^2$ from first principles.

Going back to the diagram on page 2, if we set $y = x^2$, then a small change in x (here δx) will cause a corresponding change in y, namely δy.

Since $y = x^2$, it follows that $y + \delta y = (x + \delta x)^2$.

$\therefore\ y + \delta y = x^2 + 2x(\delta x) + (\delta x)^2$.

Subtracting the original function gives $\delta y = 2x(\delta x) + (\delta x)^2$, and dividing throughout by δx, we have

$$\frac{\delta y}{\delta x} = 2x + \delta x .$$

As δx tends to zero, $\frac{\delta y}{\delta x} \to \frac{dy}{dx} = 2x$. Hence the **derived function** of $y = x^2$ is $y = 2x$.

The derived function of a polynomial.

The method of differentiation from first principles was just a demonstration – we have standard rules to work out gradient functions far more rapidly than that !

Any polynomial function $y = x^n$, where n is a constant, has a gradient function of

$$\frac{dy}{dx} = nx^{n-1}$$

In other words, you multiply by the power, and then reduce the power by 1.

Original function y	Derived function, $\frac{dy}{dx}$
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$

Also, note the following:

The derivative of a constant function is zero.

If a function is multiplied by a constant, then its derivative is multiplied by the same constant.
For example, the derivative of x^2 is $2x$, so the derivative of $5x^2$ is $10x$.

If a function consists of separate terms added together, then the derivative of the sum is the sum of the derivatives of the separate terms.
For example, the derivative of $3x^3 - x^2$ is $9x^2 - 2x$.

An **increasing** function is one where the derivative is positive.
A **decreasing** function is one where the derivative is negative.

Many functions can be both increasing and decreasing – for example, $y = x^2$ is a decreasing function for negative values of x, but an increasing function for positive values of x.

Examples (2): Find the derived function of i) $2x$; ii) 8; iii) $4x^2$; iv) x^7; v) $2x^3 + 7x^2 - 5x + 4$.

i) The derivative of $2x$ is 2 (remember $x = x^1$).

ii) The derivative of 8 is 0 (8 is a constant – the derivative of any constant function is 0).

iii) The derivative of $4x^2$ is $8x$. (the result is 4 times $2x$, the derivative of x^2).

iv) The derivative of x^7 is $7x^6$ (multiply by the power, here 7, and reduce 7 by 1.)

v) The derivative of $2x^3 + 7x^2 - 5x + 4$ is $6x^2 + 14x - 5$ (each term's derivative summed together).

Example (3): Find the gradient of the curve $y = 4x^3$ at the point (2, 32).

Here, $\frac{dy}{dx} = 12x^2$, so the gradient at the point (2, 32) is 12×2^2 or 48.

Example (4): Find the coordinates of the point on the curve $y = 2x^2 - 3x - 7$ where the gradient is 5.

The gradient, $\frac{dy}{dx}$, is $4x - 3$, and so we solve $4x - 3 = 5$ to find the x-coordinate of the required point, namely 2. Substituting $x = 2$ into the original function gives the full coordinates of (2, -5).

Fractional and negative powers.

The rule of finding derivatives of polynomials can also be applied to fractional and negative powers.

Examples (5): Differentiate i) $\sqrt{x}$; ii) $\frac{1}{x}$; iii) $\frac{1}{x^2}$; iv) $\sqrt[3]{x}$.

i) Firstly, rewrite $\sqrt{x}$ as $x^{\frac{1}{2}}$, and then apply the rule of multiplying by the power and reducing the power by 1.

The derivative is thus $\frac{1}{2}x^{-\frac{1}{2}}$ or $\frac{1}{2\sqrt{x}}$
(recall the laws of indices on fractional and negative powers).

ii) The expression can be rewritten as $y = x^{-1}$ and by applying the usual rule, the gradient function is (-1)x^{-2} or $-\frac{1}{x^2}$.

iii) Since $\frac{1}{x^2} = x^{-2}$, differentiation gives $-2x^{-3} = -\frac{2}{x^3}$.

iv) Rewriting $\sqrt[3]{x}$ as $x^{\frac{1}{3}}$, the standard rule gives a derivative of $\frac{1}{3}x^{-\frac{2}{3}}$ or $\frac{1}{3x^{\frac{2}{3}}}$.

Example (6): Find the gradient of the curve $y = \sqrt{x}$ at the point (16, 4)

From Example (5) i) , $y = x^{\frac{1}{2}}$ has a derived function of $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ or $\frac{1}{2\sqrt{x}}$.

$\therefore$ At (16, 4), the gradient is therefore $\frac{1}{2\sqrt{16}}$ or $\frac{1}{8}$.

Example (7): Find the gradient of the curve $y = \frac{1}{x}$ at the point $(5, \frac{1}{5})$

From Example (5) ii) $y = \frac{1}{x}$ has a derivative of $\frac{dy}{dx}$ = (-1)x^{-2} or $\frac{-1}{x^2}$.

$\therefore$ At $(5, \frac{1}{5})$ the gradient is $-\frac{1}{25}$.

Derivatives in function notation.

If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ (f dash x)

If $y = kf(x)$ where k is a constant, then $\frac{dy}{dx} = kf'(x)$.

Another way of saying this is $\frac{d}{dx}\left(k\left(f(x)\right)\right) = k\frac{d}{dx}(f(x))$.

If $y = f(x) \pm g(x)$ where $f(x)$ and $g(x)$ are separate functions of x, then $\frac{dy}{dx} = f'(x) \pm g'(x)$

An alternative notation for differentiation from first principles is

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

Here, h corresponds to the small change in x, namely δx.

Example (8): If $f(x) = x^3 - 7x + 4$, find $f'(x)$ and $f'(2)$

$f'(x) = 3x^2 - 7$, and therefore $f'(2) = (3 \times 2^2) - 7 = 5$.
Sometimes a function needs to be expressed in the right form before it can be differentiated.

Example (9): Differentiate $f(x) = (2x - 7)(x + 4)$

A product cannot be differentiated term by term, and so it must first be expanded into a form that can.
$(2x - 7)(x + 4) = 2x^2 + x - 28$, which can be differentiated to give $f'(x) = 4x + 1$.

(There is a product rule for differentiation, but we will not study it until Year 13)

Example (10): Differentiate $f(x) = \frac{x^4 + 1}{x^2}$

A quotient cannot be differentiated term by term, so it must be rewritten as

$$\frac{x^4+1}{x^2} = x^2 + \frac{1}{x^2}$$

Both terms can now be differentiated to give $f'(x) = 2x - \frac{2}{x^3}$.

Note that $\frac{1}{x^2} = x^{-2}$ and differentiation gives $\frac{-2}{x^3} = -2x^{-3}$.

(Again, there is a quotient rule for differentiation, but we will not study it until Year 13)

The following example returns to the ideas behind differentiation from first principles.

Example(11): A curve passes through points P (2, 17) and Q (2.1, 18.23).

i) Find the gradient of the chord PQ.

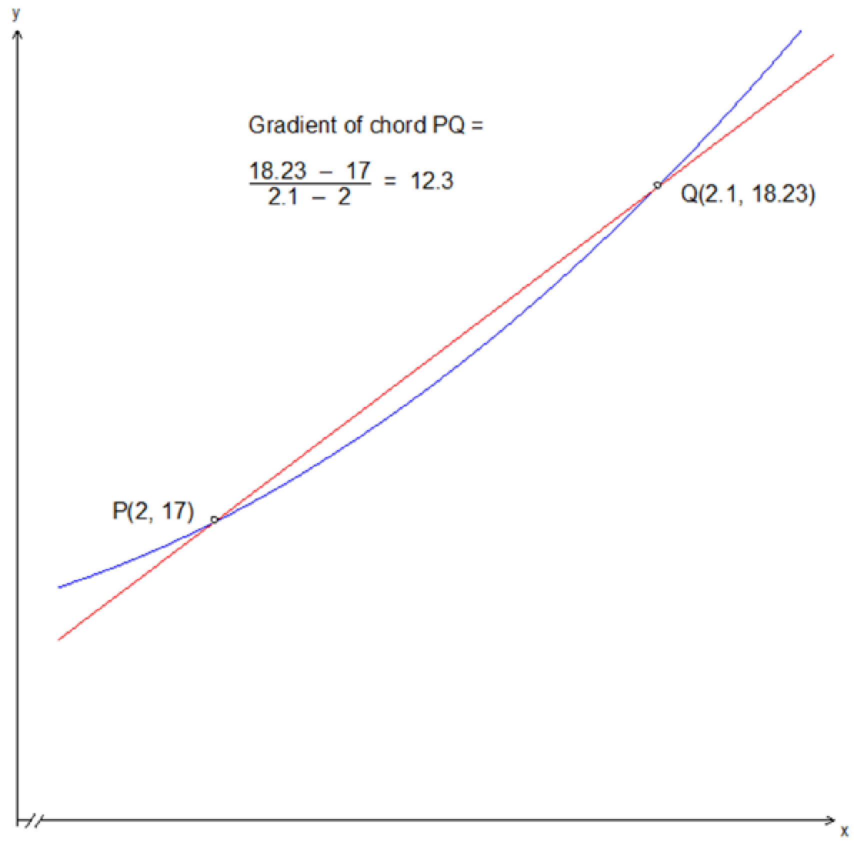
ii) Point Q is then moved ten times closer to P, to the point (2.01, 17.1203) . What is the gradient of the chord PQ now ?

iii) Suggest what happens to the chord and its gradient as Q is moved ever closer to P.

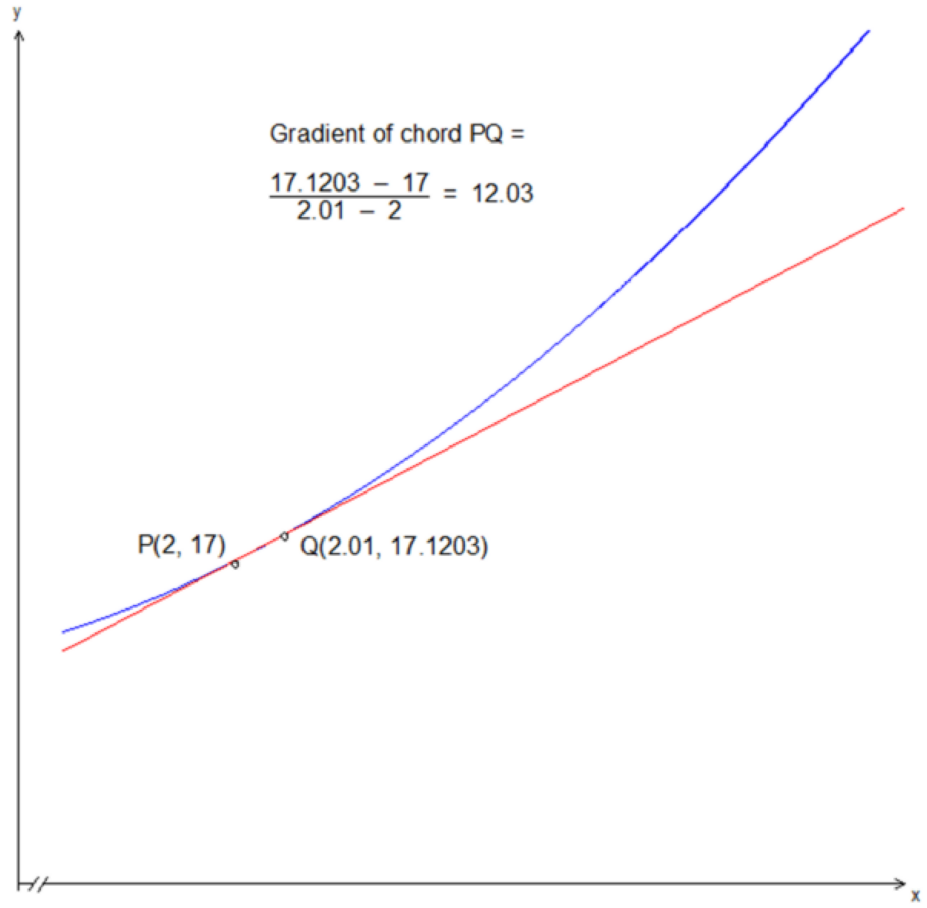
iv) The curve has an equation of $y = ax^2 + b$ where a and b are constants. Find the equation of the curve.

i) The chord PQ has a gradient of 12.3. (See diagram below – deliberately exaggerated).

Notice how the gradient of the chord at P is not particularly close to that of the tangent at that point.



ii) After point Q is moved to (2,01, 17.1203), the gradient of the chord PQ is 12.03.
Also, the chord is a much closer approximation to the tangent at point P now that PQ is ten times smaller.



iii) The gradient at P has changed from 12.3 to 12.03 as Q has become closer to P. Also, the chord approaches the tangent to P more closely the smaller the length of PQ.

This suggests that, when P and Q coincide, the chord PQ becomes a tangent at P and the gradient tends to a value of 12.

iv) The curve has an equation of $y = ax^2 + b$, and so its derivative $\frac{dy}{dx} = 2ax$.

From iii), $\frac{dy}{dx} = 12$ when $x = 2$, so $2a = 6$ and thus $a = 3$.

From this we can work out that the curve has an equation of $y = 3x^2 + b$.

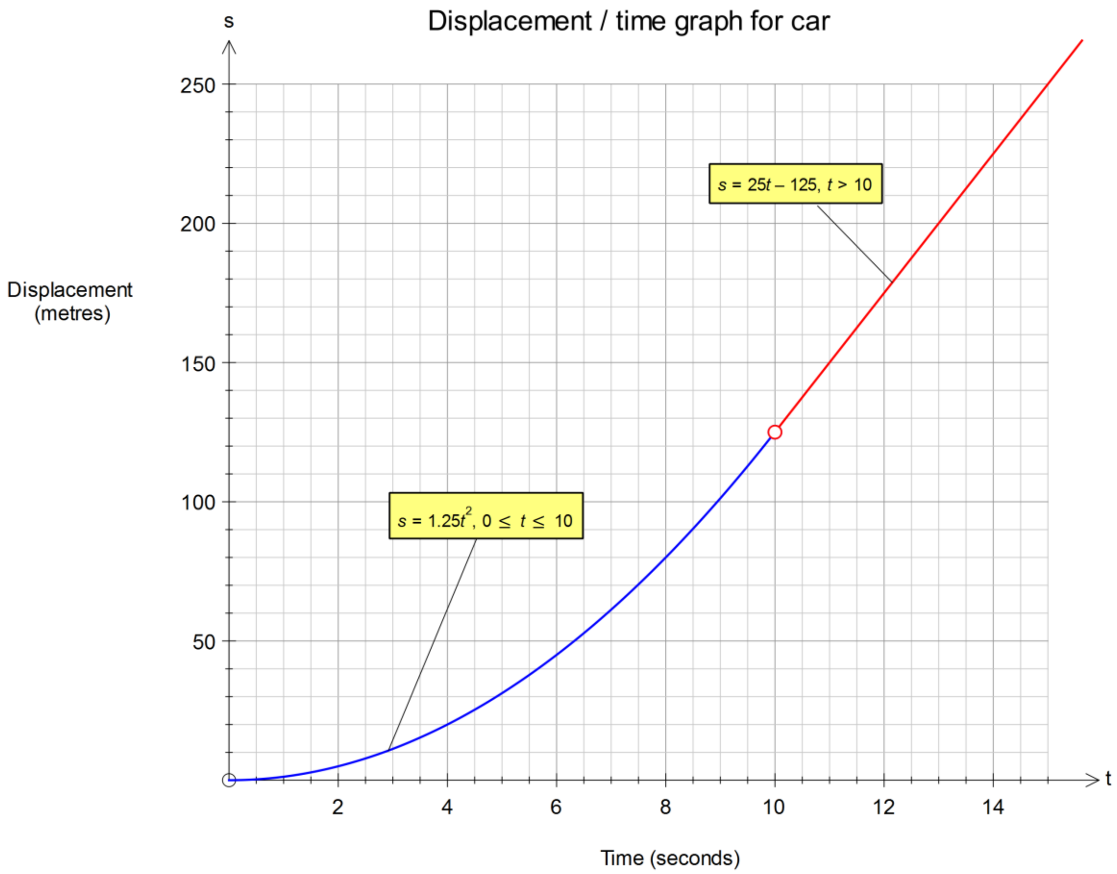
To find b, substitute $x = 2$, $y = 17$ $\Rightarrow$ $3x^2 + b = 17$ $\Rightarrow$ $12 + b = 17$ $\Rightarrow$ $b = 5$.

The equation of the curve is therefore $\boldsymbol{y = 3x^2 + 5}$.

Applications in Mechanics.

We have seen the relationship between quantities such as displacement, velocity and acceleration at GCSE when we had studied travel graphs. .

Example (12) : A car is being driven along a track, and the graph below illustrates its displacement (distance from the starting point) in metres for the first 15 seconds.



For the first ten seconds, the graph's equation is a quadratic, $s = 1.25t^2$, but after that, it becomes linear, with the equation $s = 25t - 125$.

We can use derivative notation to illustrate the relationships between displacement, s, and velocity, v.

Because velocity is the rate of change of displacement with respect to time, t, we say $v = \frac{ds}{dt}$.

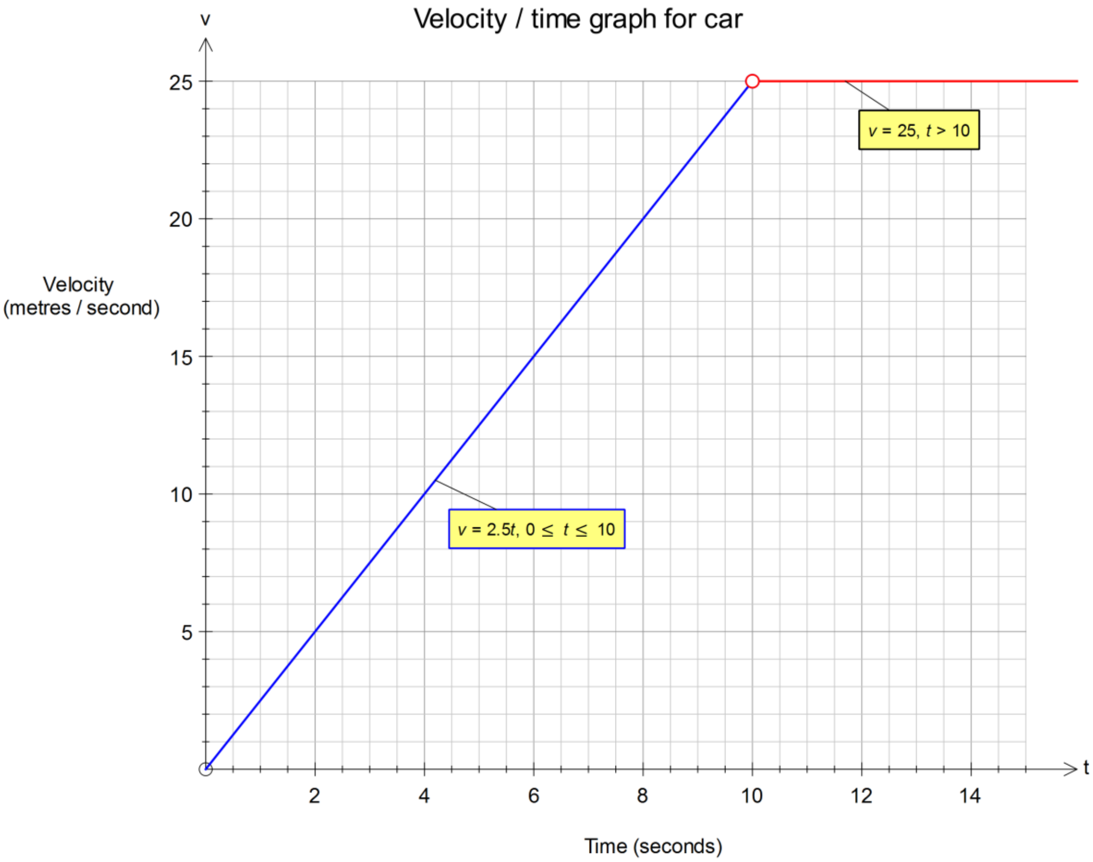
We can therefore find expressions for the velocity of the car by differentiation.

When $s = 1.25t^2$, $v = \frac{ds}{dt} = 2.5t$ (for the first 10 seconds).

When $s = 25t - 125$, $v = \frac{ds}{dt} = 25$ (thereafter) .

For the first ten seconds, the velocity (in m/s) varies depending on the time; after that, it is constant.

The corresponding velocity-time graph is shown below.



It can be seen that the velocity increases at a linear rate of 2.5 m/s per second for the first ten seconds, and then remains at 25 m/s afterwards.

In other words, the car is accelerating at a rate of 2.5 m/s per second for the first ten seconds, after which the acceleration falls to zero.

The rate of change of velocity (with respect to time) is also the rate of acceleration, so we also say $a = \frac{dv}{dt}$.

Differentiating the velocity expressions we have:

When $v = 2.5t$, $a = \frac{dv}{dt} = 2.5$ (for the first 10 seconds).

When $v = 25$, $a = \frac{dv}{dt} = 0$ (thereafter) .

Since $v = \frac{ds}{dt}$, we can also say $a = \frac{d^2s}{dt^2}$, i.e. we differentiate the displacement function once to find the velocity, and twice to find the acceleration.

Example (13): A particle moves in a straight line which passes through the fixed point *O*.

The particle's displacement, s, from *O* is given by $s = 12t^2 - 2t^3$
where t is the time in seconds and $0 \le t \le 6$.

i) Find an expression for the velocity of the particle in metres per second at time t seconds.

ii) Find the particle's displacement when $t = 4$, and show that its velocity is zero at that point.

iii) At what time does the particle have zero acceleration ?

i) We differentiate s to find the velocity ; $v = \frac{ds}{dt} = 24t - 6t^2$.

ii) When $t = 4$, $s = (12 \times 16) - (2 \times 64) = 64$, i.e. the particle is 64 metres from *O*.

Also, $v = (24 \times 4) - (6 \times 16) = 96 - 96 = 0$, so the particle's velocity is zero at $t = 4$.

iii) Differentiating again, $a = \frac{d^2s}{dt^2} = 24 - 12t$.

We therefore solve $24 - 12t = 0$, giving $t = 2$.
$\therefore$ The particle has zero acceleration after 2 seconds.

Example (14): A ball is released into the air at a velocity of 25 m/s, from an initial height of 2 m.

Its height is given by the formula $h = 2 + 25t - 5t^2$, where t is the time elapsed in seconds.

(Ignore the actual dimensions of the ball, i.e. treat it as a particle.)

i) Find expressions for the velocity and acceleration of the ball.

ii) Find the height and velocity of the ball after 4 seconds. Explain the latter result.

iii) Find the maximum height attained by the ball, to the nearest metre.

iv) Use the quadratic formula to show that the ball falls back to the ground after just over 5 seconds .

i) The velocity of the ball is $v = \frac{dh}{dt} = 25 - 10t$ m/s.

The acceleration is $a = \frac{d^2h}{dt^2} = \frac{dv}{dt} = -10$ m/s^2.

ii) When $t = 4$, $h = 2+100-80 = 22$, so the ball is 22m above ground level after 4 seconds.

The velocity, v , = 25 - 40 or -15 m/s. The context of the question makes it clear that the *upwards* direction is *positive* , therefore the negative velocity signifies a *downwards* direction.

iii) The ball reaches its maximum height when $v = \frac{dh}{dt} = 0$, i.e. when $25 - 10t = 0$.

This is when $t = 2.5$ seconds. Substituting $t = 2.5$ into the height formula gives $h = 2 + 62.5 - 31.25$, or 33. The maximum height reached by the ball is thus 33 metres.

iv) We need to solve $h = 0$, namely $2 + 25t - 5t^2 = 0$. We use the general quadratic formula

$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, or $t = \frac{-25 \pm \sqrt{625 + 40}}{-10}$. The solutions are -0.39 and 5.08.

Only the positive result is applicable here, so the ball falls back to the ground after 5.1 seconds.