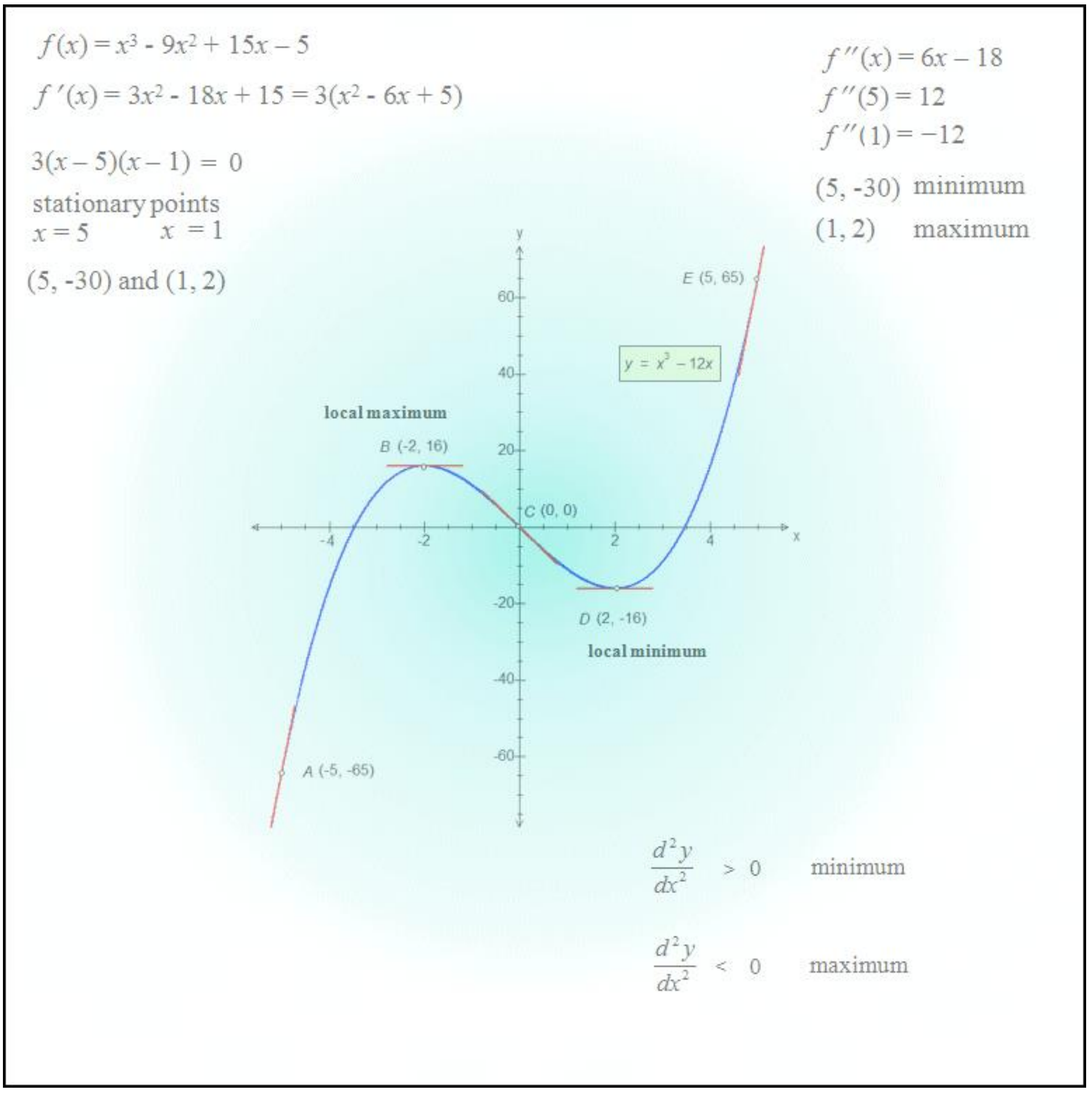
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 1 / AS

MAXIMA & MINIMA



Version : 3.1 Date: 10-01-2015

Turning Points - Maxima and Minima.

A graph of a function of x is said to have a turning point (or points) if its derivative can take a value of zero for some value(s) of x. Turning points are also known as stationary points.

Turning points of quadratic graphs.

All quadratic graphs have one turning point, i.e. one occurrence of a zero derivative.

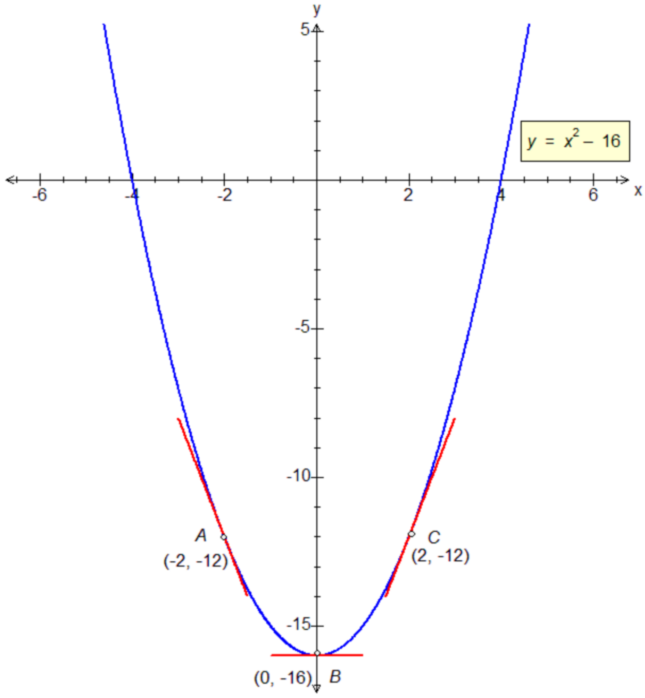
The graph of $y = x^2 - 16$ is shown on the right, and points A (-2, -12), B (0, -16) and C (2, -12) marked for reference.

At point A, the value of y is -12 and continues to decrease until it reaches a stationary minimum of -16 at point B.
Moving from B to C and beyond, the value of y then begins to increase.

Between A and B, the gradient is negative but decreases to zero by the time we reach B. Thereafter, the gradient becomes increasingly positive.

Note the following:
The gradient at B is zero, so point B is a turning point – in fact it is a minimum point as y cannot be any lower.

The gradient of the curve goes from negative through zero to positive as x increases.



The following graph is that of $y = 4 - x^2$, and this time we are interested in the points A(-2, 0), B (0, 4) and C (2,0)

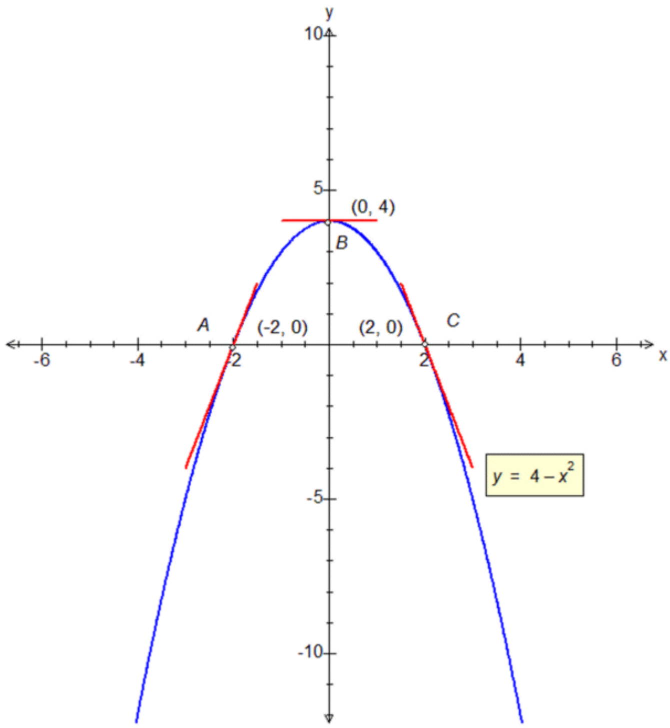
At point A, the value of y is 0 and continues to increase until it reaches a stationary maximum of 4 at point B.

Moving from B to C and beyond, the value of y then begins to decrease.

Between A and B, the gradient is positive but decreases to zero by the time we reach B. Thereafter, the gradient becomes increasingly negative.

The gradient at B is zero, so point B is a turning point – in fact it is a maximum point as y cannot be any higher.

The gradient of the curve goes from positive through zero to negative as x increases.



The last two examples illustrated the behaviour of a curve near a turning point:

- In the neighbourhood of a minimum point, the gradient changes from negative through zero to positive as x increases.

- In the neighbourhood of a maximum point, the gradient changes from positive through zero to negative as x increases.

The minimum and maximum points in the last example were also **global** since they were the only turning points.

Turning points of cubic graphs.

The derived function can give useful information about the behaviour of a curve. Two cubics are shown here.

Graph of $y = x^3$.

Points A (-2, -8), B (0, 0) and C (2, 8) are shown here.

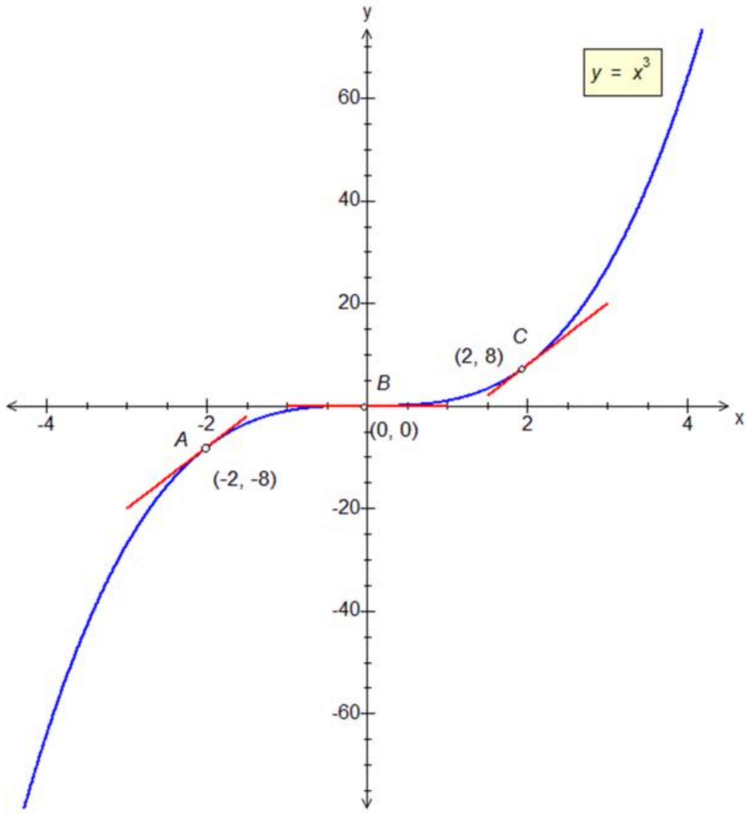
For this graph, the value of y is never decreasing, changing from -8 at A, through zero at B and to 8 at C.

Between A and B, the gradient is positive but decreases to zero by the time we reach B. Thereafter, the gradient becomes positive again.

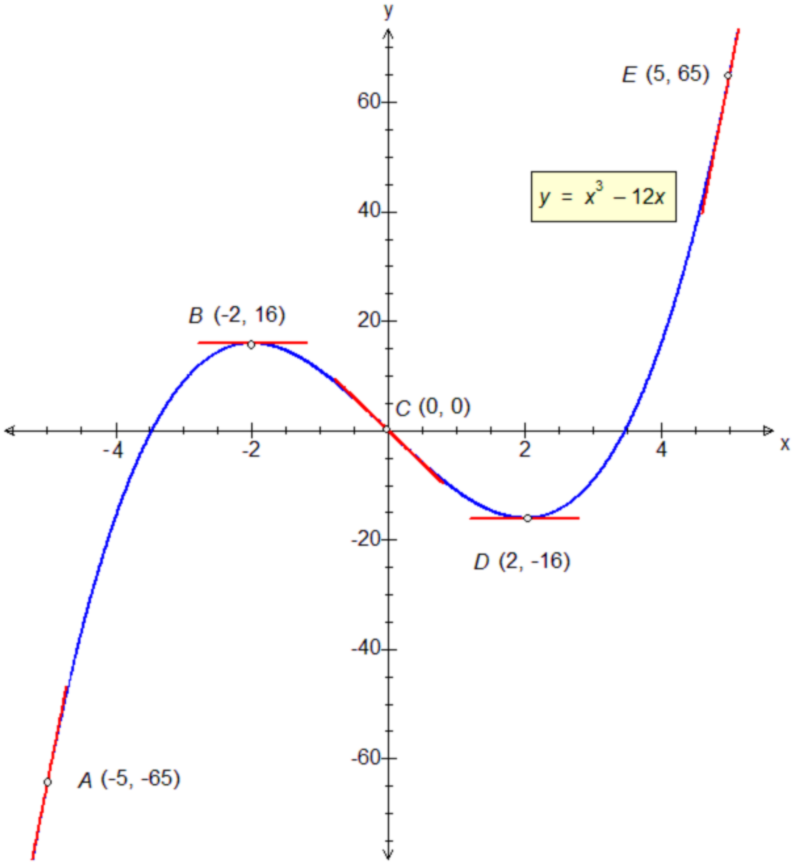
The gradient at B is zero, so point B is a turning point, but it is neither a minimum nor a maximum.

In fact it is a **stationary point of inflection.**

The sign of the gradient does not change on either side of the point of inflection.



Graph of $y = x^3 - 12x$.



Something more interesting happens here !

The value of y increases from A to B, with a positive but decreasing gradient, until at B the gradient is zero and we have a stationary point – a **local maximum** at that point.

The value of y then decreases from B through C, with a negative gradient, until at D the gradient is zero again and we have another stationary point – this time a **local minimum**.

From D to E, the function is increasing again, with an increasing positive gradient.

Notice how the gradient behaves on either side of the local maximum at point B. It changes from positive through zero at B and then becomes negative.

The pattern is reversed in the case of the local minimum point D. There the gradient changes from negative through zero at D and then becomes positive.

Hence, as x increases:

In the neighbourhood of a local **minimum**, the gradient changes sign **from negative to positive**.
In the neighbourhood of a local **maximum**, the gradient changes sign **from positive to negative**.
In the neighbourhood of a point of **inflection**, the gradient **does not change sign**.

The maximum point at B is referred to as a **local** maximum because the value of y, namely 16, at that point is not an absolute maximum. At point E, for instance, for instance, $y = 65$.

For the same reason, the minimum at D is local only; there $y = -16$, but at point A, $y = -65$.

Enumerating the turning points of a cubic curve.

Two cubic curves are shown below, and although one of them clearly has two turning points, the other does not. How is it possible to find out if a cubic curve has turning points without sketching its graph ?

The derivative of a cubic equation is a quadratic, and therein lies the clue.

The graph of $y = x^3 - 2x^2 - 4x - 4$ is shown on the right.

The derived function, $\frac{dy}{dx}$, is $3x^2 - 4x - 4$.

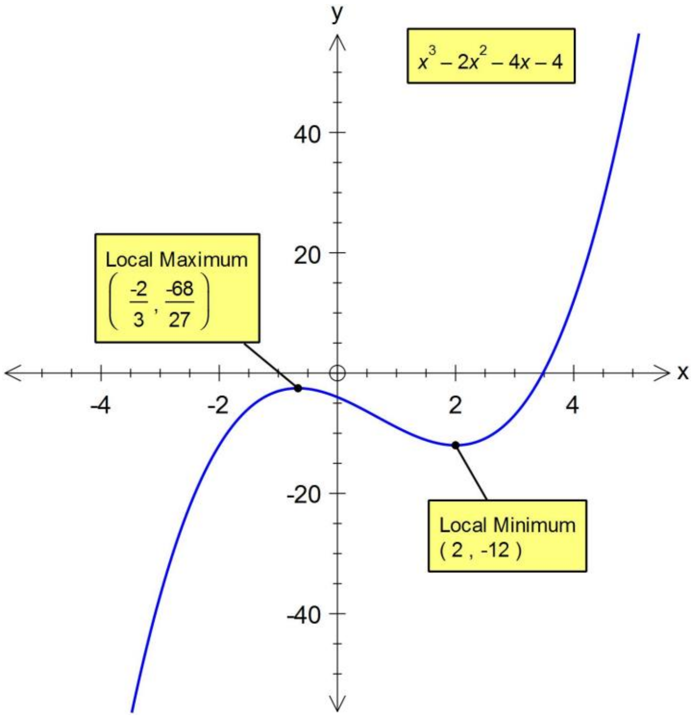
This can be factorised to $(3x + 2)(x - 2)$ to give the x-coordinates of the turning points, namely 2 and $-\frac{2}{3}$.

However, if we are only interested in the number of stationary points, we just need to find the discriminant of the quadratic derived function.

Here $a = 3$, $b = -4$ and $c = -4$.

The value of $b^2 - 4ac = 64$ (i.e. positive), so the equation $\frac{dy}{dx} = 0$ has two real roots.

These correspond to the stationary points of the cubic graph of $y = x^3 - 2x^2 - 4x - 4$.



By contrast, the graph of $y = x^3 + 4x^2 + 10x + 6$ appears to have no stationary points.

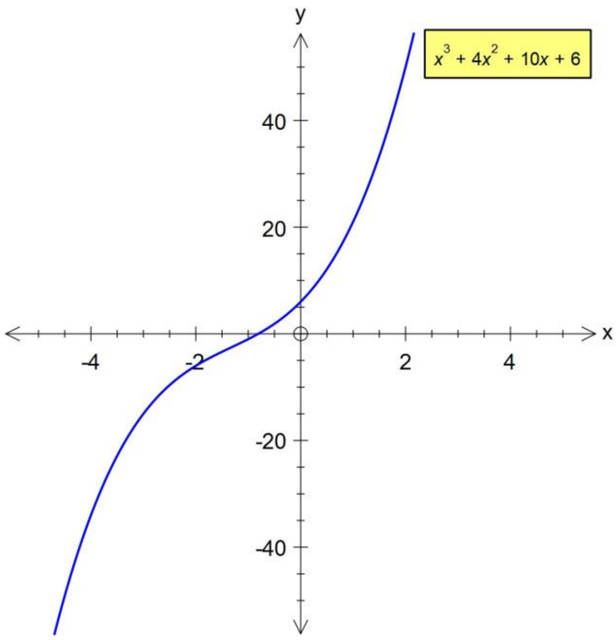
This time the derived function, $\frac{dy}{dx}$, is $3x^2 + 8x + 10$.

The discriminant of the derived function is $b^2 - 4ac$ where $a = 3$, $b = 8$ and $c = 10$.

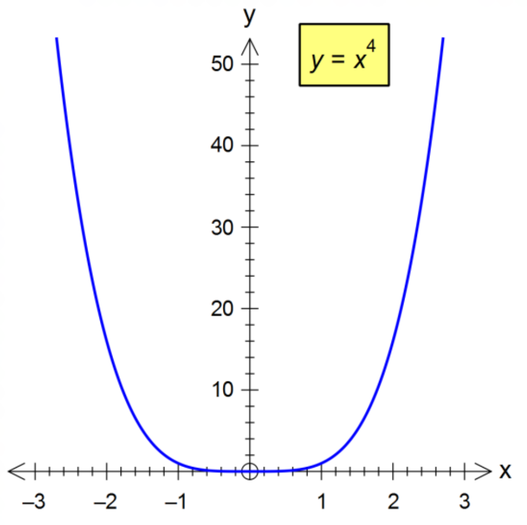
This time, $b^2 - 4ac = 64 - 120 = -56$.

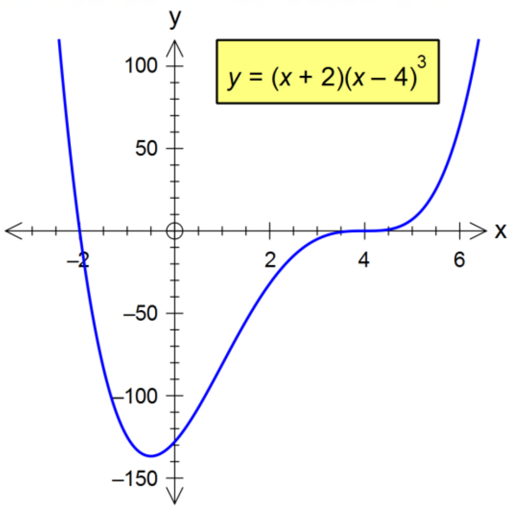
The negative discriminant implies no real roots of the derived function, and hence no turning points on the graph.

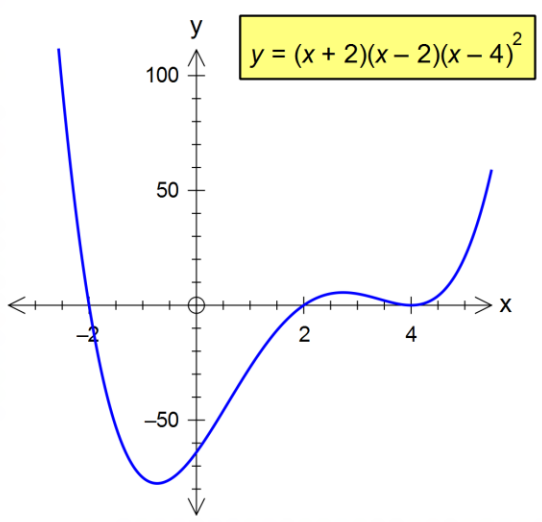
(Note: the case of equal roots of the derived function signifies a stationary point of inflection, such as with the graph of $y = x^3$ whose derivative $3x^2$ has coincident roots of $x = 0$.)

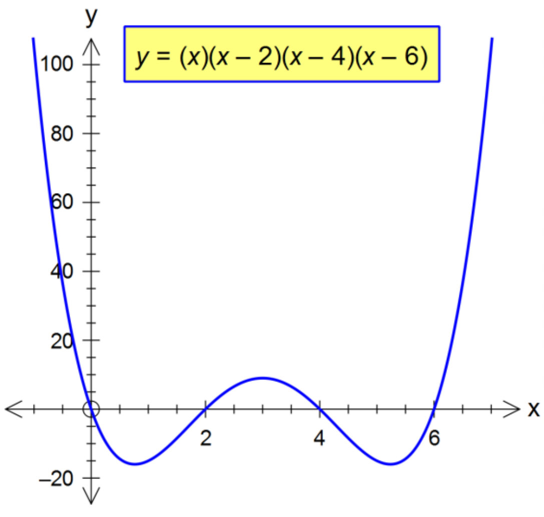


Turning points of quartic curves.









The four quartic curves above have different numbers of turning points, and the turning points also differ in nature. The basic curve of $y = x^4$ has only one turning point, namely a minimum.

The two lower curves both have three turning points – a minimum, a maximum and another minimum as x increases, whilst the curve at upper right has a local minimum and a stationary point of inflection.

The Second Derivative.

Recalling the function notation for derivatives, we have

If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ (f dash x)

Differentiating $y = f(x)$ a second time gives the second derivative, written as

$\frac{d^2y}{dx^2} = f''(x)$ (f double dash x)

Just as an increasing function is one where the gradient is positive, and a decreasing is one where the gradient is negative, the second gradient can give further information.

A **convex** function is one where the second derivative is positive.
A **concave** function is one where the second derivative is negative.

Again, many functions can be both convex and concave – for example, $y = x^3$ is a concave function for negative values of x, but a convex function for positive values of x.

The second derivative can also give information about the nature of any stationary points.
At a stationary point:

If $f''(x) > 0$, then the point is a local minimum.
If $f''(x) < 0$, then the point is a local maximum.

What if $f''(x) = 0$?

If $f''(x) = 0$ the result is inconclusive. It could mean a point of inflection, but could still be a maximum or a minimum. In that case, we must check the behaviour of the gradient in the neighbourhood of the stationary point, by choosing suitable values of x on each side of it.

If $f'(x)$ changes sign from -ve through 0 to +ve in the neighbourhood of the stationary point as x increases, then the point is a local minimum.

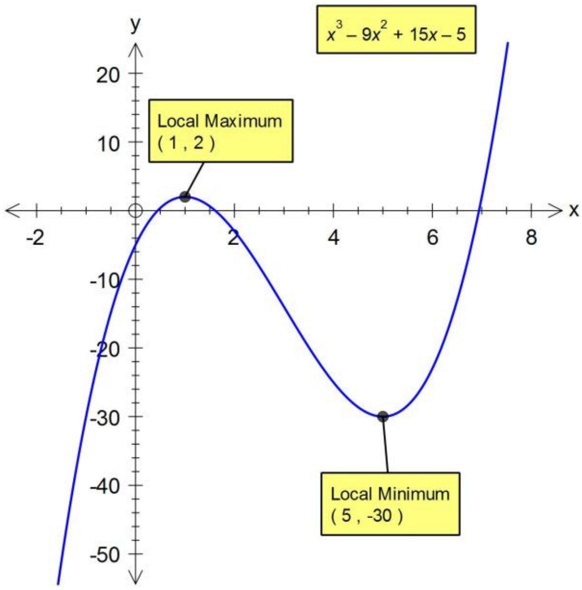
If $f'(x)$ changes sign from +ve through 0 to -ve in the neighbourhood of the stationary point as x increases, then the point is a local maximum.

If $f'(x)$ does not change sign on either side in the neighbourhood of the stationary point as x increases, then the point is a point of inflection.

Example (1): Find the stationary points of the curve
$f(x) = x^3 - 9x^2 + 15x - 5$ and use the second derivative to distinguish between them.

Firstly, we differentiate to find the stationary points; $f'(x) = 3x^2 - 18x + 15 = 3(x^2 - 6x + 5)$.

At a stationary point, $f'(x) = 0$, so it is a matter of solving the resulting quadratic .
The quadratic factorises to $3(x - 5)(x - 1)$, and therefore the stationary points of $f(x)$ occur where $x = 5$ or $x = 1$.



Substituting those values into $f(x)$ gives the stationary points as (5, -30) and (1, 2).

Taking second derivatives
we have $f''(x) = 6x - 18$.

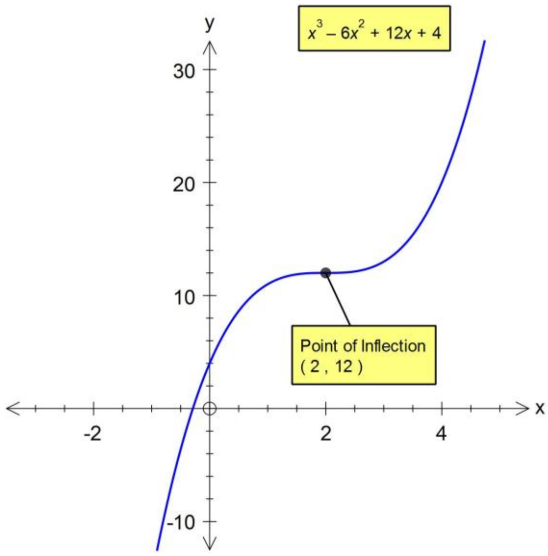
When $x = 5$, $6x - 18 = 12$. The second derivative is positive here, and so (5, -30) is a local **minimum**.

When $x = 1$, $6x - 18 = -12$. The second derivative is negative here, and so (1, 2) is a local **maximum**.

Example (2): Find the stationary points of the curve $f(x) = x^3 - 6x^2 + 12x + 4$. Which, if any, is a maximum or a minimum ?

Differentiating, we have

$f'(x) = 3x^2 - 12x + 12 = 3(x^2 - 4x + 4)$.



The quadratic factorises to $3(x - 2)^2$, giving only one stationary point where $x = 2$.
This stationary point is (2,12).

Taking the second derivative,
$f''(x) = 6x - 12$, but when $x = 2$, the second derivative is zero, and therefore (2, 12) is neither a maximum nor a minimum point and the test is inconclusive.

(In fact it is a stationary point of inflection).

Example (3): Show that the curve $g(x) = x^3 + 4x^2 + 7x - 4$ has no stationary points.

Differentiating, we have $g'(x) = 3x^2 + 8x + 7$.

Inspection of the quadratic shows that it is of the form $ax^2 + bx + c$ where $a = 3$, $b = 8$ and $c = 7$.
The discriminant $b^2 - 4ac$ is negative here, meaning that the quadratic has no real solutions.

The gradient of curve $g(x)$ cannot therefore take a value of 0 for any real x, hence the curve has no stationary points.

Example (4): The equation of a quartic (fourth-degree) curve is $f(x) = x(x - 2)(x - 4)(x - 6)$.

i) Show that $f(x) = x^4 - 12x^3 + 44x^2 - 48x$.
ii) Show that $f(x)$ has a turning point at $x = 3$ and find its y-coordinate.
iii) Find the x – coordinates of the other turning points of $f(x)$ using calculus, giving the results in the form $a \pm \sqrt{b}$ where a and b are integers.
iv) Show that each turning point found in part iii) has a y-coordinate of -16.
v) Determine the nature of each of the turning points.

i) We begin with $(x-2)(x-4) = x^2 - 6x + 8$.

Next, we multiply this product by $(x-6)$:

$(x-6)(x^2 - 6x + 8) = x(x^2 - 6x + 8) -6(x^2 - 6x + 8) = x^3 - 6x^2 + 8x - 6x^2 + 36x - 48$
$= x^3 - 12x^2 + 44x - 48$.

Finally we multiply by x: $x(x^3 - 12x^2 + 44x - 48) = x^4 - 12x^3 + 44x^2 - 48x$

Hence $\boldsymbol{f(x) = x^4 - 12x^3 + 44x^2 - 48x}$.

ii) Differentiation gives $f'(x) = 4x^3 - 36x^2 + 88x - 48$.
Substituting $x = 3$, $f'(3) = 108 - 324 + 264 - 48 = 0$, so $f(x)$ has a turning point at $x = 3$.

Therefore $4(x - 3)$, or $4x - 12$, is a factor of $f'(x) = 4x^3 - 36x^2 + 88x - 48$.

		x^2	$-6x$	$+4$
$4x - 12$	$4x^3$	$- 36x^2$	$+ 88x$	-48
	$4x^3$	$- 12x^2$		
		$-24x^2$	$+ 88x$	
		$-24x^2$	$+ 72x$	
			$16x$	-48
			$16x$	-48
				0

The quotient is $x^2 - 6x + 4$, which is a quadratic that cannot be factorised. (Note the surd references).

We will solve this equation by completing the square.

Now $x^2 - 6x + 4 = (x^2 - 6x + 9) - 5 = (x-3)^2 - 5$, so the solutions of $(x-3)^2 - 5 = 0$ are $(x-3)^2 = 5$,
or $x - 3 = \pm\sqrt{5}$, and $\boldsymbol{x = 3 \pm \sqrt{5}}$.

The x-coordinates of the stationary points are therefore 3, $3 + \sqrt{5}$ and $3 - \sqrt{5}$.

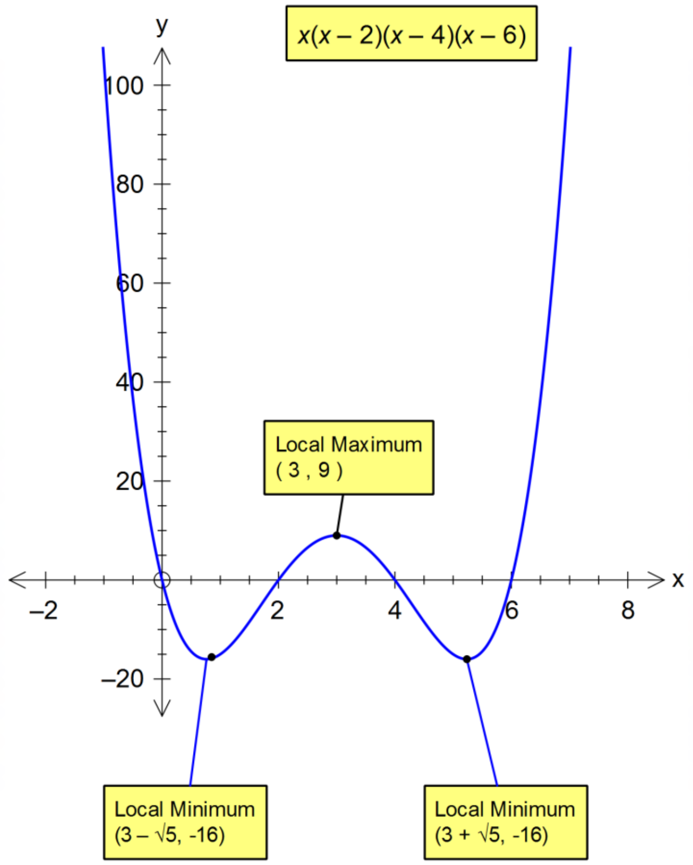
iv) If we substitute $x = 3 + \sqrt{5}$, or $\sqrt{5} + 3$, into the equation for $f(x) = x(x - 2)(x - 4)(x - 6)$,
$f(3 + \sqrt{5}) = (\sqrt{5} + 3)(\sqrt{5} + 1)(\sqrt{5} - 1)(\sqrt{5} - 3)$.

This rather formidable product can be rearranged into $(\sqrt{5} + 3)(\sqrt{5} - 3) \times (\sqrt{5} + 1)(\sqrt{5} - 1)$, with the "difference of two squares" form evident. This simplifies into $(5 - 9) \times (5 - 1) = -16$.

The same result is obtained for $x = 3 - \sqrt{5}$, or $(-\sqrt{5} + 3)$, because the square of $(-\sqrt{5})$ is also 5.

Hence the other two turning points have coordinates of $(3+\sqrt{5}, -16)$ and $(3+\sqrt{5}, -16)$.

v) Substituting into the equation for $f(x)$,
we have $f(3) = 3(3 - 2)(3 - 4)(3 - 6) = 9$,
so one turning point is (3, 9).



The second derivative is $f''(x) = 12x^2 - 72x + 88$.

$f''(3) = 108 - 216 + 88 = -20$;
it is negative, so (3, 9) is a local **maximum.**

$f''(3 + \sqrt{5}) = 12(14 + 6\sqrt{5}) - 72(3 + \sqrt{5}) + 88 = 40$;
it is positive , so $(3+\sqrt{5}, -16)$ is a local **minimum**.

$f''(3 - \sqrt{5}) = 12(14 - 6\sqrt{5}) - 72(3 - \sqrt{5}) + 88 = 40$;
it is positive , so $(3-\sqrt{5}, -16)$ is a local **minimum**.

Example (5): A quartic function is defined as $f(x) = (x + 2)(x - 2)(x - 4)^2$.

i) Find the x-coordinates of its turning points, giving your answers in an exact form.
ii) Determine the nature of each turning point.

We must first expand the expression as $(x^2 – 4)(x - 4)^2 = (x^2 – 4)(x^2 – 8x +16)$

$= x^4 - 8x^3 + 16x^2 – 4x^2 + 32^x – 64 = x^4 - 8x^3 + 12x^2 + 32x - 64$.

Hence $f(x) = x^4 - 8x^3 + 12x^2 + 32x – 64$ and the derivative $f'(x) = 4x^3 - 24x^2 + 24x + 32$.

Inspection of $f(x)$ shows that there is a constant factor of 4 plus a repeated root at $x = 4$ in the equation $f(x) = 0$, so there is also a stationary point at $x = 4$.

Therefore $f'(4) = 0$ and $4(x – 4)$, or $4x – 16$, is a factor of $f'(x)$. $4x^3 - 24x^2 + 24x + 32$

		x^2	$+2x$	-2
$4x - 16$	$4x^3$	$- 24x^2$	$+ 24x$	$+32$
	$4x^3$	$- 16x^2$		
		$-8x^2$	$+ 24x$	
		$-8x^2$	$+32x$	
			$-8x$	$+32$
			$-8x$	$+32$
				0

The quotient is $x^2 + 2x – 2$, which is a quadratic that cannot be factorised. (Note the ‘exact form’ hint !)

(We will solve this equation by completing the square).

Now $x^2 + 2x – 2 = (x^2 - 2x – 1) – 3 = (x-1)^2 -3$, so the solutions of $(x-1)^2 -3 = 0$ are $(x-1)^2 =3$,
or $x – 1 = \pm \sqrt{3}$, and $\boldsymbol{x = 1 \pm \sqrt{3}}$.

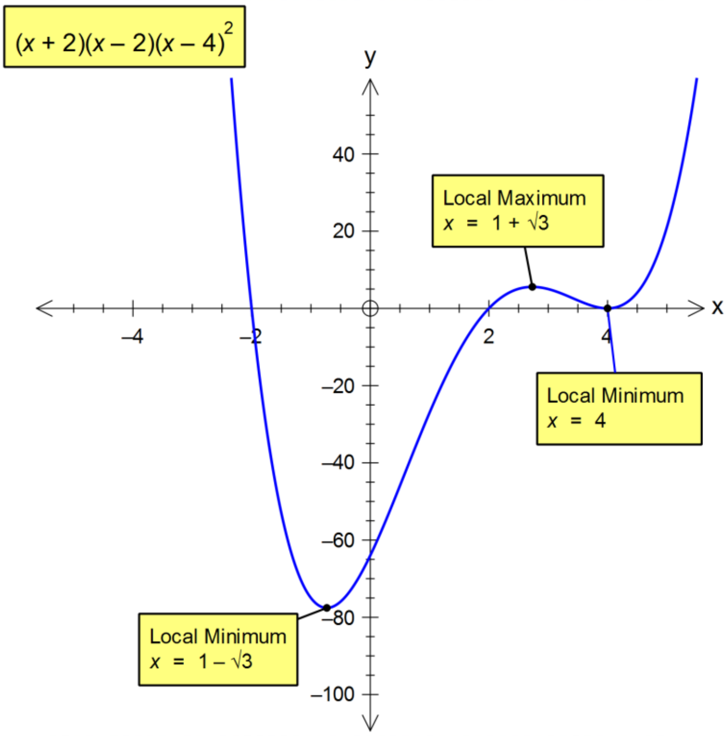
ii) We work out the second derivative;

$f''(x) = 12x^2 - 48x + 24 = 12(x^2 - 4x + 2)$.

$f''(4) = 24$ (i.e. > 0) , so there is a local minimum at $x = 4$.

$f''(1+\sqrt{3}) = 24(1-\sqrt{3})$ (i.e. < 0) , so there is a local maximum at $x = 1+\sqrt{3}$.

$f''(1-\sqrt{3}) = 24(1+\sqrt{3})$ (i.e. > 0) , so there is a local minimum at $x = 1-\sqrt{3}$.



Sometimes the working can be simplified using transformations of graphs.

Example (6): Let $f(x) = (x + 2)(x - 4)^3$.

i) Show that $g(x) = x^3(x + 6)$ is a translation of $f(x)$ via the vector $\begin{pmatrix}-4\\0\end{pmatrix}$.

ii) Find the turning points of $g(x)$ and determine their nature.
iii) Hence state the turning points of $f(x)$.

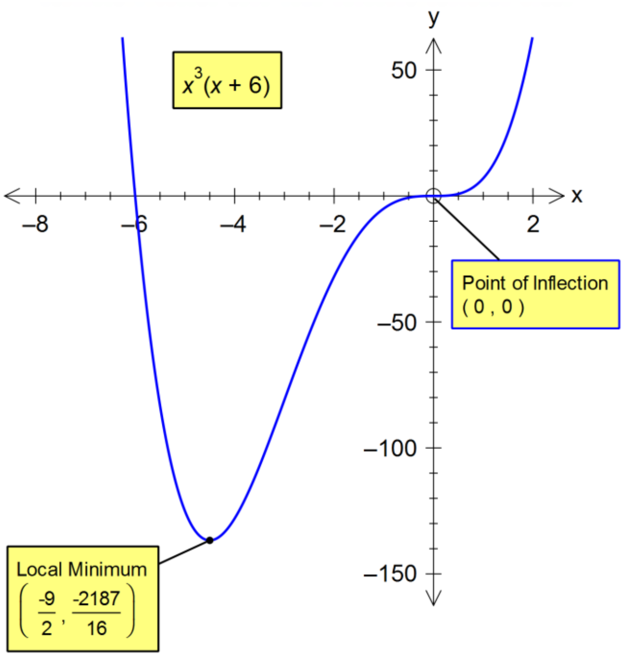
i) A translation by the vector $\begin{pmatrix}-4\\0\end{pmatrix}$ transforrns $f(x)$ into $f(x + 4)$, so $g(x) = ((x+4) + 2)((x - 4)+4)^3$ or $(x + 6)(x^3)$, or $x^3(x + 6)$.

ii) Expanding, $g(x) = x^3(x + 6) = x^4 + 6x^3$, and differentiation gives $g'(x) = 4x^3 + 18x^2 = x^2(4x + 18)$.

Therefore $g'(x) = 0$ when $x = 0$ or $x = -4\frac{1}{2}$, and the turning points of g are (0,0) and $\left(-\frac{9}{2}, \frac{-2187}{16}\right)$.

The second derivative, $g''(x) = 12x^2 + 36x$, so $g''(0) = 0$ and $g''(-4\frac{1}{2}) = 81$.

The point $\left(-\frac{9}{2}, \frac{-2187}{16}\right)$ is therefore a minimum point.



The point (0,0) is a stationary point, but its second derivative is also zero, so we select two values of x on either side of 0 and find the values of the derivative $g'(x)$. Two suitable values are -1 and 1.
Now $g'(-1) = 14$ and $g'(1) = 22$.

As x increases from -1 through zero to 1, the derivative changes from 14 (positive) through zero and then 22 (positive) , without changing sign in the process. The origin (0,0) is therefore a stationary point of inflection.

iii) Since $g(x)$ is a translation of $g(x)$ through the vector $\begin{pmatrix}-4\\0\end{pmatrix}$, we apply the inverse vector of $\begin{pmatrix}4\\0\end{pmatrix}$ to transform $g(x)$ back to $f(x)$. As a result, the turning points of (0,0) and $\left(-\frac{9}{2}, \frac{-2187}{16}\right)$ are similarly translated to (4, 0) and $\left(-\frac{1}{2}, \frac{-2187}{16}\right)$ on the graph of $f(x)$.

Example (7): The equation of a quartic curve is $f(x) = (x - 7)(x - 1)(x + 1)(x + 7)$.

i) Show that $f(x) = x^4 - 50x^2 + 49$.
ii) Find the turning points of $f(x)$ using calculus, and determine their nature.

i) There are two differences of squares in here; $(x - 7)(x + 7) = x^2 - 49$ and $(x - 1)(x + 1) = x^2 - 1$.
Multiplying, $(x^2 - 49)(x^2 - 1) = x^4 - 50x^2 + 49$.

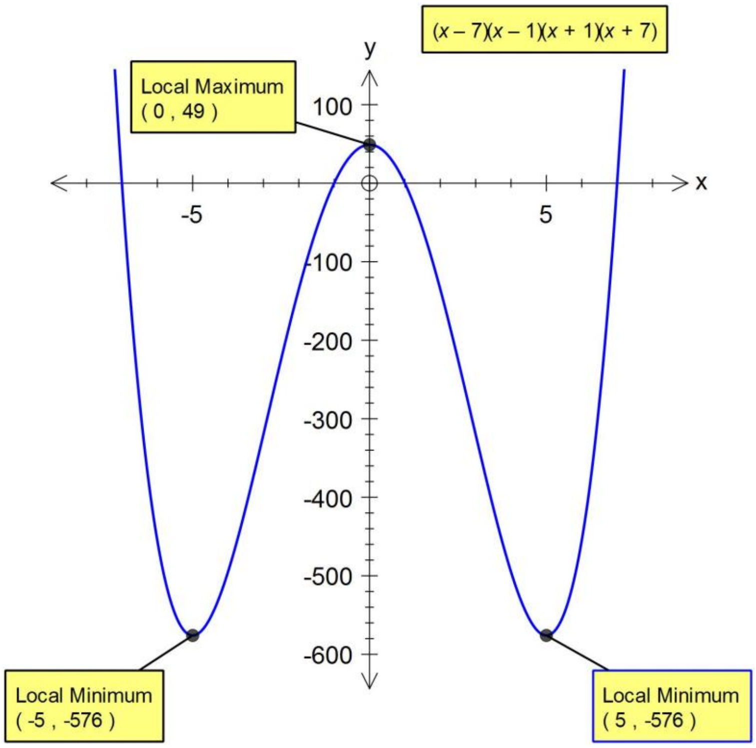
ii) Differentiation gives $f'(x) = 4x^3 - 100x$.

This readily factorises into $4x(x^2-25)$ and finally $4x(x-5)(x+5)$.

The x-coordinates of the stationary points are therefore 0, 5 and -5.
Substituting into the equation for $f(x)$, the stationary points are (0, 49), (5, -576) and (-5, -576).

The second derivative of $f(x)$ is $f''(x) = 12x^2 - 100$.

$f''(0) = -100$, so (0, 49) is a local **maximum.**
$f''(5) = 200$, so (5, -576) is a local **minimum**.
$f''(-5) = 200$, so (-5, -576) is a local **minimum**.



Example (8): Find the stationary points, if any, of the curve $f(x) = 1 - x^4$. Which, if any, is a maximum or a minimum ?

Differentiating, we have $f'(x) = -4x^3$.
Differentiating again, we have $f''(x) = -12x^2$.

At a stationary point, $f'(x) = 0$, so we solve $-4x^3 = 0$. The only solution is $x = 0$, so the only stationary point of the curve $f(x) = 1 - x^4$ is (0, 1).

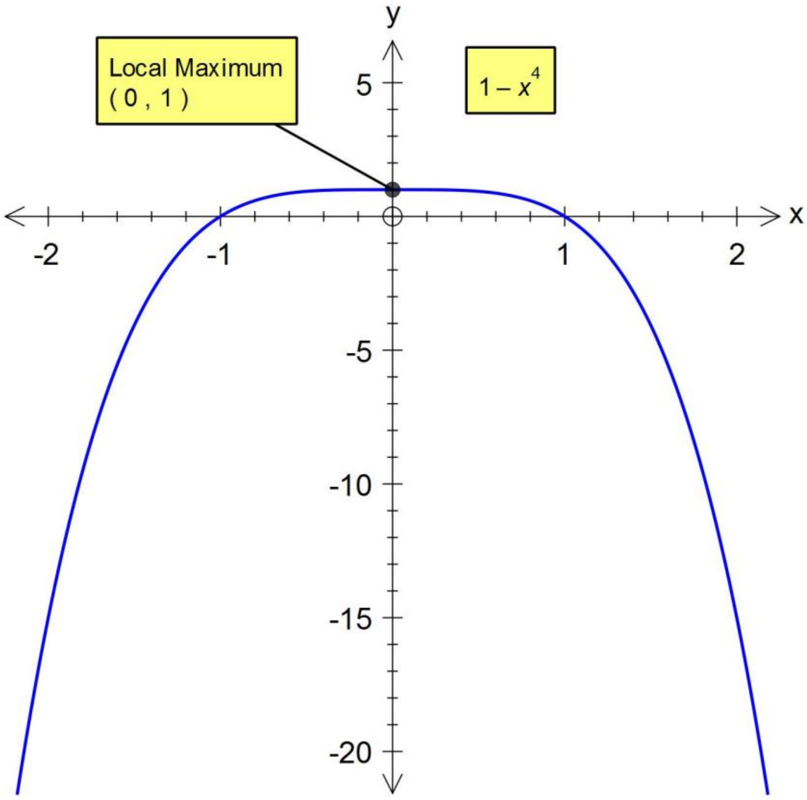
To determine if (0, 1) is a maximum or a minimum, we take the second derivative, $f''(x) = -12x^2$. Unfortunately, this second derivative is also zero, so the test does not help.

We therefore need to find the gradient of the curve at two other points, one on each side of the stationary point. Two suitable values are at $x = -1$ and $x = 1$, i.e. at points (-1,0) and (1,0).

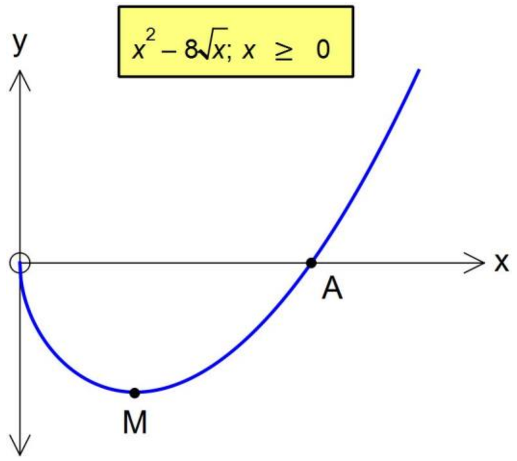
The gradient at (-1,0) is $f'(-1) = 4$; the corresponding gradient at (1, 0) is $f'(1) = -4$.

Therefore, as *x increases* from -1 through 0 to 1, the gradient changes sign from positive (4), through 0 to negative (-4) on either side of the stationary point at (0, 1).

Therefore the point (0, 1) is a local maximum.



Example (9): A curve is defined as $y = x^2 - 8\sqrt{x}, x \geq 0$. (Revise your indices !)



i) Find the coordinates of the *x*-intercept at *A*.

ii) Find the *x*-coordinate of the minimum point *M* , giving your answer in the form 2^k where *k* is a rational constant .

iii) Hence find the *y*-coordinate of *M*, giving the result in the form $a(2^k)$ where *a* is an integer and *k* a rational constant (not necessarily the same as *k* in part ii)).

i) We solve $x^2 - 8\sqrt{x} = 0 \Rightarrow x^4 - 64x = 0$ (squaring both sides) $\Rightarrow x(x^3 - 64) = 0$.

Hence at the *x*-intercept at *A*, $x^3 = 64$, so $x = 4$ and the coordinates of *A* are (4, 0).

ii) Differentiating, $\frac{dy}{dx} = 2x - \frac{4}{\sqrt{x}}$.

We therefore solve $2x - \frac{4}{\sqrt{x}} = 0 \Rightarrow 2x = \frac{4}{\sqrt{x}} \Rightarrow 4x^2 = \frac{16}{x}$ (squaring both sides)

$$\Rightarrow \quad 4x^3 = 16 \quad \Rightarrow \quad x^3 = 4 \quad \Rightarrow \quad x^3 = 2^2$$

$$\Rightarrow \quad x = 2^{\frac{2}{3}}.$$

iii) Substituting $x = 2^{\frac{2}{3}}$ in the original, we have $y = 2^{\frac{4}{3}} - 8\left(2^{\frac{1}{3}}\right) \Rightarrow y = 2^{\frac{4}{3}} - 2^3\left(2^{\frac{1}{3}}\right)$

$$\Rightarrow y = 2^{\frac{4}{3}} - 2^{\frac{10}{3}} \Rightarrow y = 2^{\frac{4}{3}}(1 - 2^2) \quad \Rightarrow y = -3\left(2^{\frac{4}{3}}\right)$$

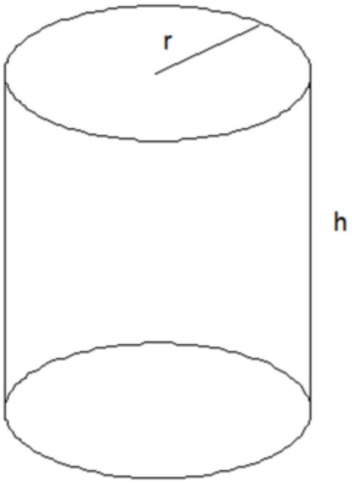
The coordinates of the minimum point *M* are therefore $\left(2^{\frac{2}{3}}, -3\left(2^{\frac{4}{3}}\right)\right)$.

Practical Applications of Maxima and Minima.

The idea of stationary points can often be used in real-life practical situations.

Example (10): A company manufactures cylindrical soup cans to hold a volume of 128π cm^3. What should be the dimensions of a can such that the surface area of metal used takes a minimum value ?

(Copyright OUP, *Understanding Pure Mathematics*, Sadler & Thorning, ISBN 9780199142590, Chapter 10,Example 11)



Each can will have a base radius r and a perpendicular height h. We require the surface area A to take a minimum value, so we need an expression for A.

The formula for the area of the curved surface of a cylinder is $2\pi rh$,but we must also include the areas of the two circular ends. Their combined area is $2\pi r^2$.

$\therefore$ $A = 2\pi r^2 + 2\pi rh$, but the expression cannot be differentiated in this form because there are *two* variables, r and h, in it.

The volume of the can is however $V = \pi r^2 h$. Given that $V = 128\pi$, we can eliminate h:

$\pi r^2 h = 128\pi \Rightarrow h = \dfrac{128}{r^2}$.

The area can therefore be redefined as $A = 2\pi r^2 + (2\pi r)\dfrac{128}{r^2} \Rightarrow A = 2\pi r^2 + \dfrac{256\pi}{r}$

To find when A is at a turning point, we differentiate and solve $\dfrac{dA}{dr} = 0$.

$$\frac{dA}{dr} = 4\pi r - \frac{256\pi}{r^2} \quad \therefore \frac{dA}{dr} = 0 \text{ when } 4\pi r = \frac{256\pi}{r^2} \Rightarrow 4\pi r^3 = 256\pi .$$

This gives $r^3 = 64$ and $r = 4$ cm, and thus $h = \dfrac{128}{16}$ cm = 8 cm.

This turning point is a minimum, since $\dfrac{d^2A}{dr^2} = 4\pi + \dfrac{512\pi}{r^3}$, which is positive when $r = 4$.
(Positive second derivative denotes a minimum point).

$\therefore$ Each can should have a base radius of 4 cm and a height of 8 cm to minimise the amount of metal used in its manufacture.

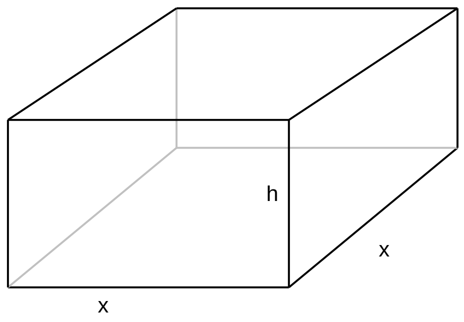
Example (11): A rectangular block has a base x centimetres square, and its total area is 150cm^2.

i) Prove that the volume of the block is ½(75x – x^3) cm^3.

ii) Calculate the dimensions of the block when its volume is at a maximum.

iii) Give the maximum volume, and show that it is indeed a maximum.

(Copyright OUP, *Understanding Pure Mathematics*, Sadler & Thorning, ISBN 9780199142590, Exercise 10G, Q.9)



i) The total area of the block is $A = 2x^2 + 4xh$, and its volume is $V = x^2h$.

Given that $2x^2 + 4xh = 150$, we have $4xh = 150-2x^2$ and therefore $h = \frac{150-2x^2}{4x}$.

The volume V is thus x^2h = $x^2\left(\frac{150-2x^2}{4x}\right) = \left(\frac{150x^2-2x^4}{4x}\right) = \left(\frac{75x-x^3}{2}\right)$.

ii) Differentiating V we have $\frac{dV}{dx} = \frac{75-3x^2}{2}$.

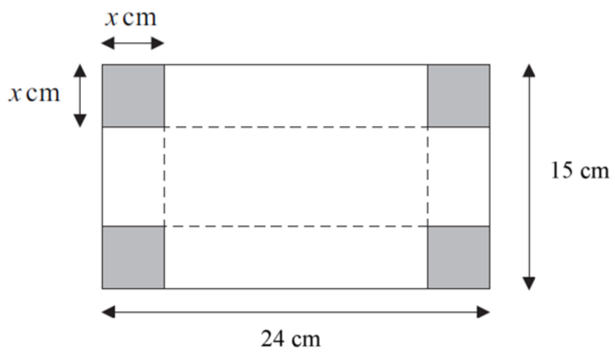
The value of x which makes V a maximum satisfies $\frac{dV}{dx} = 0$,
which corresponds to $75 - 3x^2 = 0$, i.e. $x = 5$.
To find h, we substitute $x = 5$ in $h = \frac{150-2x^2}{4x} = \frac{150-50}{20} = 5$.

iii) The maximum volume occurs when $x = 5$ and $h = 5$, i.e. when the block is a cube of side 5 cm. Its volume is therefore 5^3 or 125 cm^3.

The second derivative is $\frac{d^2V}{dx^2} = \frac{-6x}{2} = -3x$. When $x = 5$, the second derivative is -15.
The negative second derivative denotes a maximum point rather than a minimum.

Example (12):

The diagram on the right shows a rectangular sheet of metal measuring
24 cm × 15 cm.



A square of side x cm is cut from each corner and the metal then folded along the broken lines to make an open-topped box with a rectangular base and x cm tall.

i) Show that the volume, V cm^3, of liquid the box can hold is given by $V = 4x^3 - 78x^2 + 360x$.

ii) Find $\frac{dV}{dx}$.

iii) Show that any stationary values for V must occur when $x^2 - 13x + 30 = 0$, and solve the equation, explaining why only one value for x is feasible in this case.

iv) Work out the stationary value for V.

v) Find $\frac{d^2V}{dx^2}$ and hence determine if this stationary value is a maximum or a minimum.

(Copyright AQA, GCE Mathematics Paper MPC1, Jan. 2005, Q.6) (altered)

i) After folding, the base of the box formed has an area of $(24 - 2x)(15 - 2x)$ cm^2 and a height of x cm.

Its volume V is therefore $x(24 - 2x)(15 - 2x)$ cm^3 = $x(360 - 78x + 4x^2)$ cm^3.

Multiplying out, $V = 4x^3 - 78x^2 + 360x$.

ii) $\frac{dV}{dx} = 12x^2 - 156x + 360$.

iii) The volume V takes a stationary values when $12x^2 - 156x + 360 = 0$.

Dividing throughout by a common factor of 12 gives the equation $x^2 - 13x + 30 = 0$.

This factorises to $(x - 3)(x - 10) = 0$, suggesting solutions of $x = 3$ and $x = 10$.

The value of $x = 10$ is inadmissible here as it would make the short side of the box, $15 - 2x$, negative.

iv) With x taking the only permitted value of 3, the volume V of the box is $x(24 - 2x)(15 - 2x)$ cm^3
or $3 \times 18 \times 9$ cm^3, or 486 cm^3.

v) $\frac{d^2V}{dx^2} = 24x - 156$; when $x = 2$, this second derivative takes a value of 48-156 = -108.
This negative value confirms that this stationary value is indeed a maximum.

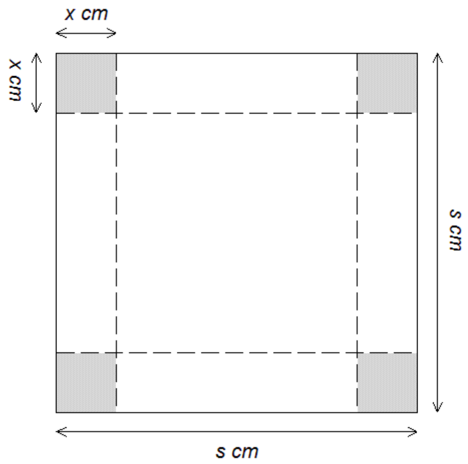
Example (13):

This is similar to the previous example, except that the original sheet of metal used is now a square of side s cm.

A square of side x cm is cut from each corner and the metal then folded along the broken lines to make an open-topped box with a square base and x cm tall.

i) Show that the volume, V cm^3, of liquid the box can hold is given by $V = 4x^3 - 4sx^2 + s^2x$.

ii) Using calculus and the general quadratic formula, prove that the height of the box must be one-sixth the side length of the original square for the volume V to take a maximum value.



i) After folding, the base of the box formed has an area of $(s - 2x)^2$ cm^2 and a height of x cm.

Its volume V is therefore $x(s - 2x)^2$ cm^3 or $x(s^2 - 4sx + 4x^2)$ cm^3.

Multiplying out, $V = 4x^3 - 4sx^2 + s^2x$.

ii) Firstly, we need to find $\frac{dV}{dx}$; here it is $12x^2 - 8sx + s^2$.

We then substitute into the general quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

where $a = 12$, $b = -8s$ and $c = s^2$, to solve $12x^2 - 8sx + s^2 = 0$.

Hence $x = \frac{8s \pm \sqrt{64s^2 - 48s^2}}{24} \Rightarrow x = \frac{8s \pm \sqrt{16s^2}}{24} \Rightarrow x = \frac{8s \pm 4s}{24}$

The roots of this equation are $x = \frac{12s}{24} = \frac{s}{2}$ and $x = \frac{4s}{24} = \frac{s}{6}$.

The first value for x, i.e. $\frac{s}{2}$, cannot give a maximum value for V, as then the base of the box would have sides $s - 2\left(\frac{s}{2}\right)$ in length, i.e. zero length.

$\therefore$ the volume V takes its maximum value when $x = \frac{s}{6}$, i.e. when the height of the box is one-sixth the side length of the original square.

Finally, $\frac{d^2V}{dx^2} = 24x - 8s$ and since $x = \frac{s}{6}$, $\frac{d^2V}{dx^2} = 4s - 8s = -4s$.

As s is always positive in the context of the problem, the second derivative is negative here, and hence V takes a maximum value when $x = \frac{s}{6}$.

Example (14): A particle moves in a straight line which passes through the fixed point *O*.

The particle's displacement, *s*, from *O* is given by $s = 12t^2 - 2t^3$
where *t* is the time in seconds and $0 \le t \le 6$.

i) Find an expression for the velocity of the particle in metres per second at time *t* seconds.

ii) Find the particle's displacement when $t = 4$, and show that this value is a maximum.

iii) At what time does the particle have zero acceleration ?

i) We differentiate *s* to find the velocity ; $v = \frac{ds}{dt} = 24t - 6t^2$.

ii) When $t = 4$, $s = (12 \times 16) - (2 \times 64) = 64$, i.e. the particle is 64 metres from *O*.

Also, $v = (24 \times 4) - (6 \times 16) = 96 - 96 = 0$, so the particle's velocity is zero at $t = 4$.
Differentiating again, $a = \frac{d^2s}{dt^2} = 24 - 12t$, and when $t = 4$, $a = -24$.

This second derivative is negative, so the particle's displacement takes a maximum value at 4 seconds.

iii) Since $a = 24 - 12t$ from ii), we solve $24 - 12t = 0$, giving $t = 2$.
$\therefore$ The particle has zero acceleration after 2 seconds.