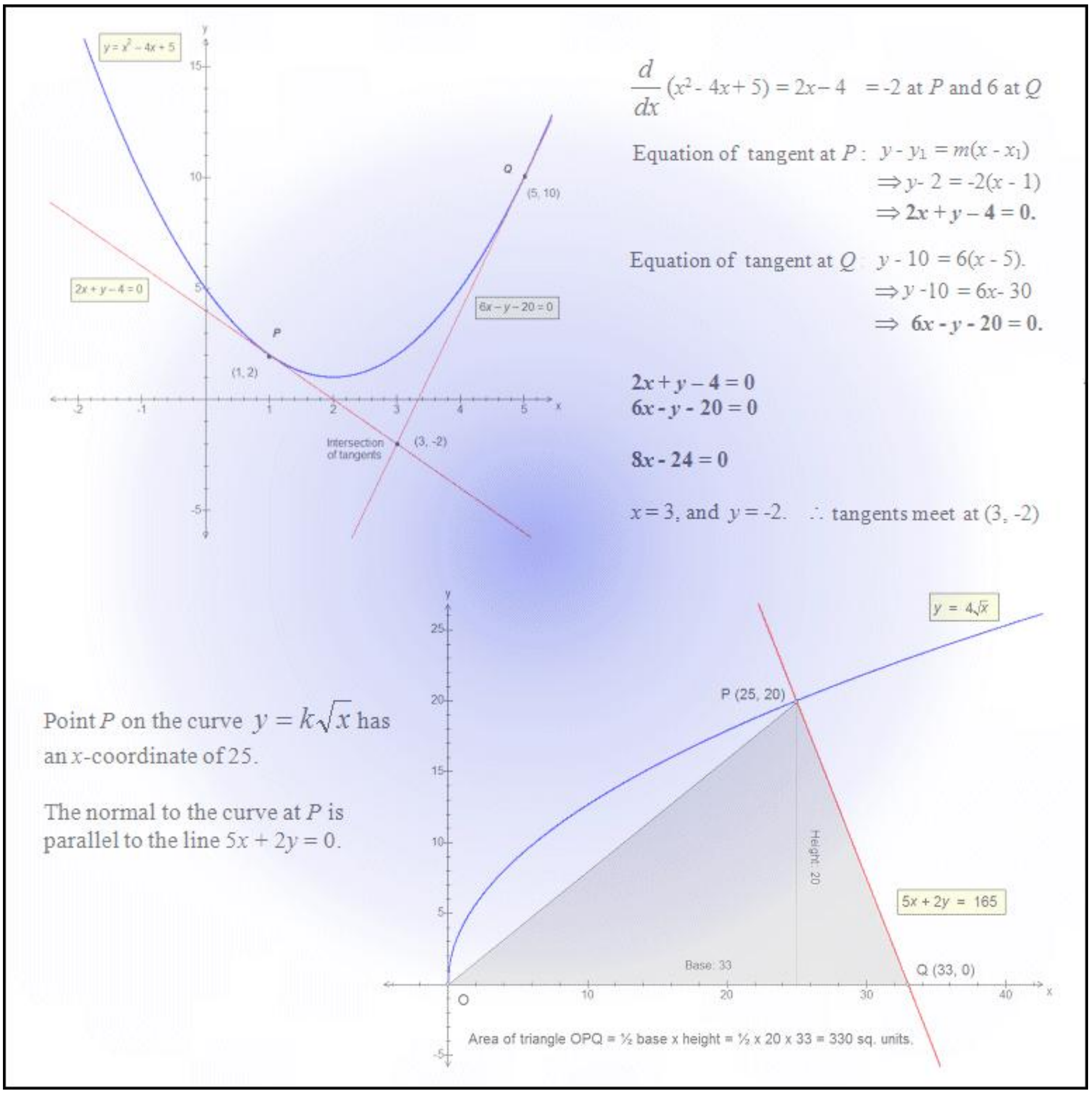
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 1 / AS

TANGENTS AND NORMALS



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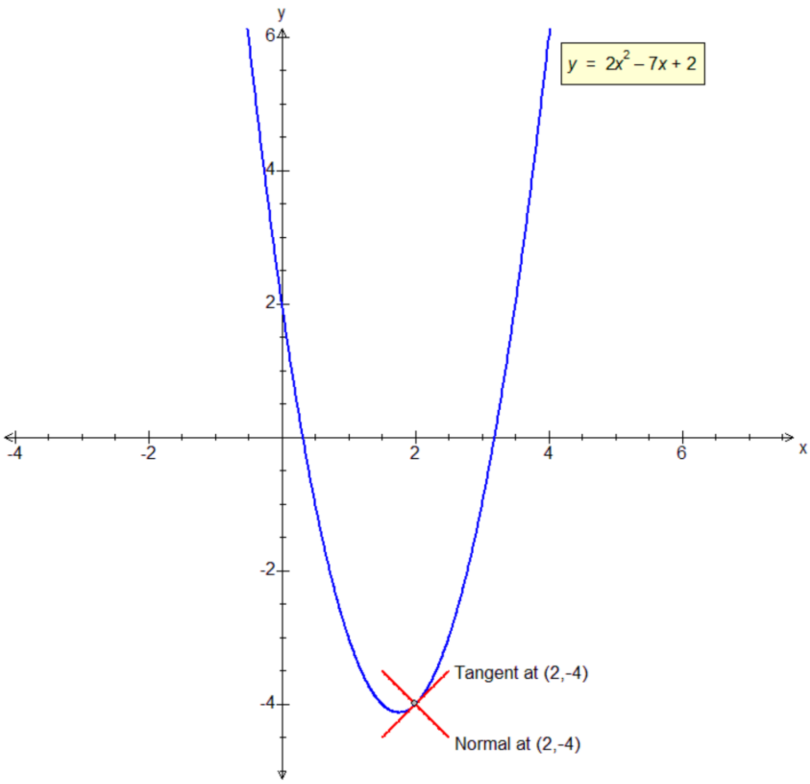
Tangents and Normals.

Differentiation helps to find the gradient of the **tangent** to a curve, but we can use ideas learnt in the section of straight lines to find the actual equation of the tangent at a given point. The **normal** at the same point is perpendicular to the tangent, therefore the product of their gradients is –1.



Example (1): Find the gradients, and hence the equations, of the tangent and the normal to the curve $2x^2 - 7x + 2$ at the point (2, -4).

The tangent and normal have been shown here for reference. (Note that the axes of the graph must be shown to uniform scales, or the normal and tangent might not appear at right angles)

The derivative of $2x^2 - 7x + 2$ is $4x - 7$; substituting for $x = 2$ gives a gradient of 1.

The equation of the tangent at (2, -4) is therefore $y - y_1 = m(x - x_1)$
$\Rightarrow y + 4 = 1(x - 2)$.

In gradient-intercept form ('$mx + c$'):
$y + 4 = x - 2$
$\Rightarrow y = x - 6$ $\therefore$ equation of tangent is $\boldsymbol{y = x - 6}$.

In '$ax + by + c = 0$' form :
$y + 4 = 1(x - 2)$ $\Rightarrow$ $y + 4 = x - 2$
$\Rightarrow$ $x - 2 - y - 4 = 0$ $\Rightarrow x - y - 6 = 0$
$\therefore$ equation of tangent is $\boldsymbol{x - y - 6 = 0}$.

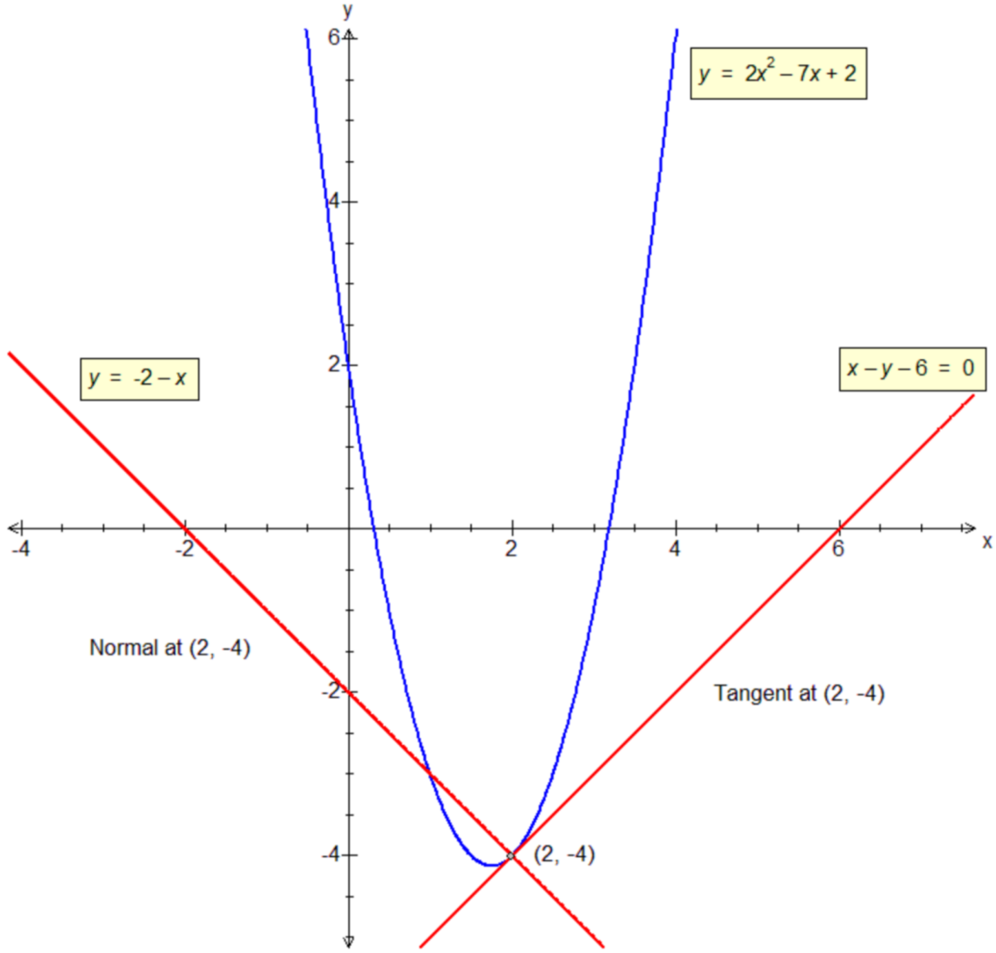
The product of the gradients of the tangent and the normal must be –1, and therefore the normal to the curve must have a gradient of –1.

The equation of the normal at (2, -4) is therefore $y + 4 = -1(x - 2)$.

In gradient-intercept form (‘$mx + c$’):
$y + 4 = -1(x - 2) \Rightarrow y + 4 = 2 - x \Rightarrow y = -2 - x$ $\therefore$ equation of normal is $\boldsymbol{y = -2 - x}$.

In ’$ax + by + c = 0$’ form :
$y + 4 = -1(x - 2) \Rightarrow y + 4 = 2 - x$
$\Rightarrow y + 4 - 2 + x = 0$
$\Rightarrow x + y + 2 = 0$ $\therefore$ equation of normal is $\boldsymbol{x + y + 2 = 0}$.

See diagram below.



Example (2): Find the equations of the tangent and normal to the curve $x^3 - 4x^2 + 2x$ at the point (2, -4). Give the equations in gradient-intercept ('$mx + c$') form, and hence show that the tangent passes through the origin.

The derivative of $x^3 - 4x^2 + 2x$ is $3x^2 - 8x + 2$, and thus its value at (2, -4) is $(3 \times 2^2) - (8 \times 2) + 2$, or -2 . The tangent to the curve therefore has a gradient of –2.

The equation of the tangent will be therefore $y - y_1 = m(x - x_1)$

$\Rightarrow$ $y + 4 = -2(x - 2)$.

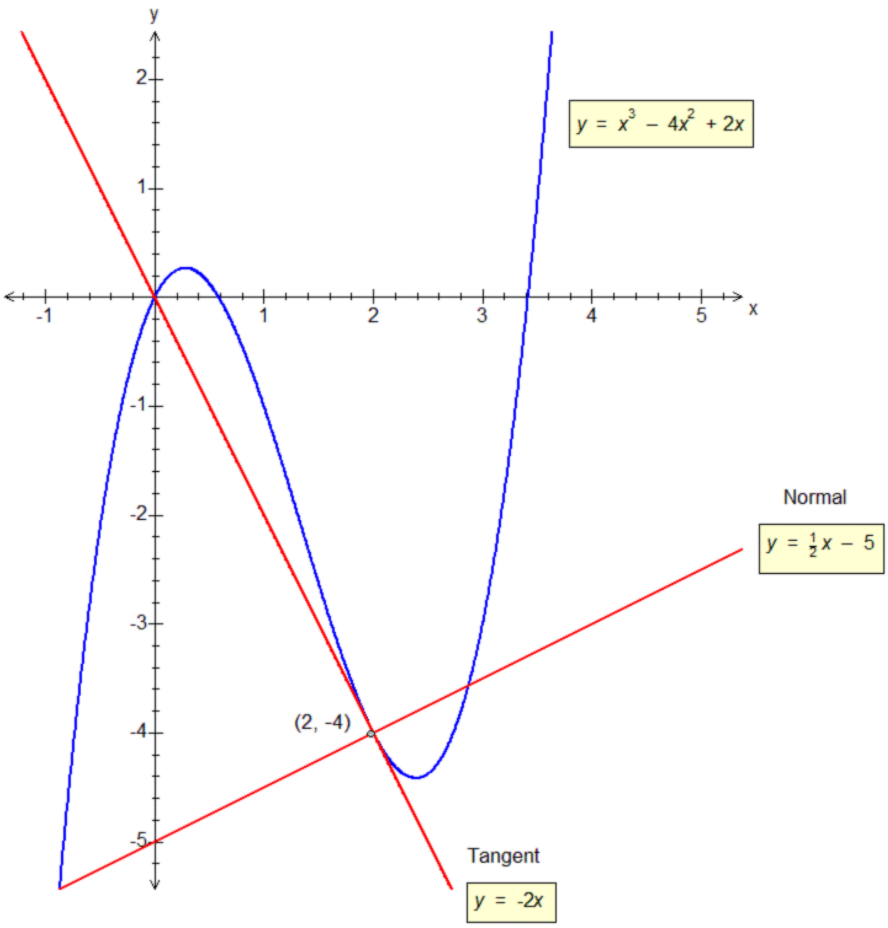
$\Rightarrow$ $y + 4 = -2x + 4$

$\Rightarrow$ $y = -2x$ $\therefore$ equation of tangent is $\boldsymbol{y = -2x}$. This line has a y-intercept at the origin.

The gradient of the normal must be $\frac{1}{2}$ (product of the gradients of tangent and normal must be –1).

The equation of the normal is therefore $y + 4 = \frac{1}{2}(x - 2)$ $\Rightarrow$ $y + 4 = ½x - 1$

$\Rightarrow$ $y = ½x - 5$ $\therefore$ equation of normal is $\boldsymbol{y = ½x - 5}$.



Example (3): i) Find the equations of the tangent and normal to the curve $\frac{4}{x^2}$ at $(4, \frac{1}{4})$.

Give the result in '$ax + by + c = 0$' form.

ii) Find the coordinates of the point where the tangent meets the curve again.

i) The tangent has a gradient of $\frac{-8}{x^3}$, so when $x = 4$, its gradient is $-\frac{1}{8}$,

and its equation is $y - y_1 = m(x - x_1)$ or $y - \frac{1}{4} = -\frac{1}{8}(x-4) \Rightarrow 8y - 2 = -(x-4)$

$\Rightarrow 8y - 2 + x - 4 = 0 \Rightarrow x + 8y - 6 = 0 \quad \therefore$ equation of tangent is $\mathbf{x + 8y - 6 = 0.}$

The gradient of the normal at $(4, \frac{1}{4})$ is thus 8 by the rule of the product of gradients.

Its equation is therefore $y - \frac{1}{4} = 8(x-4) \Rightarrow 4y - 1 = 32(x-4) \Rightarrow 32x - 128 - 4y + 1 = 0$

$\Rightarrow 32x - 4y - 127 = 0 \quad \therefore$ equation of normal is $\mathbf{32x - 4y - 127 = 0.}$

We solve the equations of the curve and tangent simultaneously by substituting for y;

$x + \frac{32}{x^2} - 6 = 0 \Rightarrow x^3 - 6x^2 + 32 = 0$ (multiplying by x^2 throughout).

One solution is $x = 4$, as the tangent meets the curve at $x = 4$. By the factor theorem, $(x-4)$ is a factor of the cubic, so we divide:

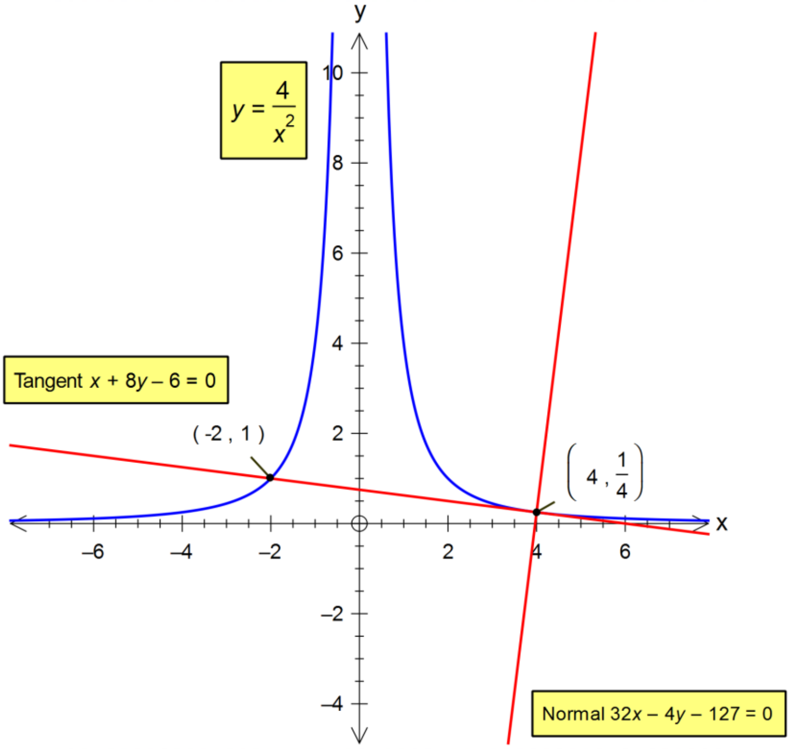
		x^2	$-2x$	-8
$x-4$	x^3	$-6x^2$	$+0x$	$+32$
	x^3	$-4x^2$		
		$-2x^2$	$+0x$	
		$-2x^2$	$+8x$	
			$-8x$	$+32$
			$-8x$	$+32$
				0

The quadratic factorises to $(x-4)(x+2)$, implying that $(x-4)$ is a repeated factor.

Hence the tangent meets the curve again when $x = -2$.

Substituting $x = -2$ into either equation gives $y = 1$.

$\therefore$ the tangent to the curve $y = \frac{4}{x^2}$ meets the curve again at $(-2, 1)$.



Example (4): A curve has equation $y = (x-4)^2(x+5)$.

i) Show that $\frac{dy}{dx} = 3x^2 - 6x - 24$.

ii) A tangent meets the curve at point P , whose x-coordinate is 3.
Find the equation of the tangent, in '$y = mx + c$' form.
iii) Show that the tangent intersects the curve at point Q where $x = -3$, and find the y–coordinate of Q..

Another tangent to the curve is parallel to the one at $x = 3$.

iv) Find the coordinates of the point R where this parallel tangent touches the curve.
v) Find the equation of this parallel tangent.
vi) Show that this parallel tangent meets the curve again at point S when $x = 5$, and find the y–coordinate of S.

i) Expanding, $(x-4)^2(x+5) = x^3 - 3x^2 - 24x + 80$, and so $\frac{dy}{dx} = 3x^2 - 6x - 24$.

ii) When $x = 3$, $y = (3-4)^2(3+5) = 8$, so the curve passes through (3, 8).

Also, when $x = 3$, $\frac{dy}{dx} = 27 - 18 + 24 = -15$.

$\therefore$ the equation of the tangent at (3, 8) is $y - 8 = -15(x - 3) \Rightarrow y - 8 = -15x + 45 \Rightarrow$ $\mathbf{y = 53 - 15x}$.

iii) Set the equations of the curve and line equal to each other ;
$x^3 - 3x^2 - 24x + 80 = 53 - 15x \Rightarrow x^3 - 3x^2 - 9x + 27 = 0$.

Substitute $x = -3$ into the cubic equation, and we have

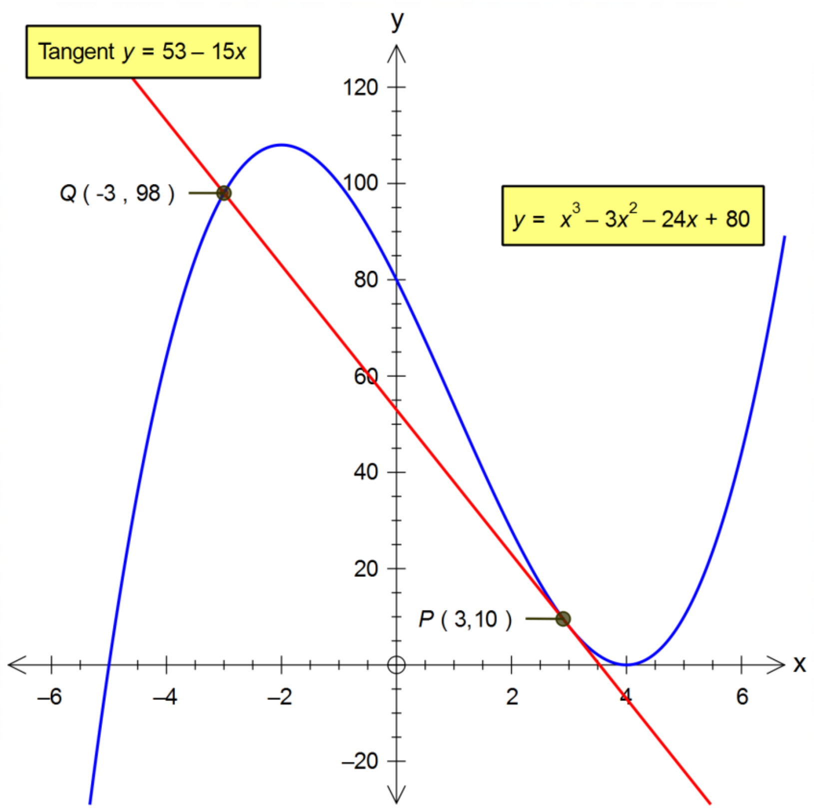
$(-3)^3 - 3(-3)^2 - 9(-3) + 27$

$= -27-27+27+27 = 0$,

so the x-coordinate of Q is -3.

Substituting $x = -3$ into the equation of the line $y = 53 - 15x$, we have the y-coordinate of Q, or 98.

The coordinates of Q are therefore (-3, 98).



iv) Since the gradient of the tangent in part i) is equal to -15, the gradient of the required parallel tangent will also be -15.

Hence $\frac{dy}{dx} = 3x^2 - 6x - 24 = -15$.

This quadratic derivative rearranges to $3x^2 - 6x - 9 = 0$ and factorises as $3(x - 3)(x + 1) = 0$.
The solutions of $3(x - 3)(x + 1) = 0$ are $x = 3$ (corresponding to the result from ii)) and $x = -1$.

$\therefore$ the x-coordinate of the contact point between the curve and this parallel tangent is -1.

v) When $x = -1$, $y = ((-1)-4)^2((-1)+5) = 100$ (substituting into the curve equation) ,
so the coordinates of R are (-1, 100).

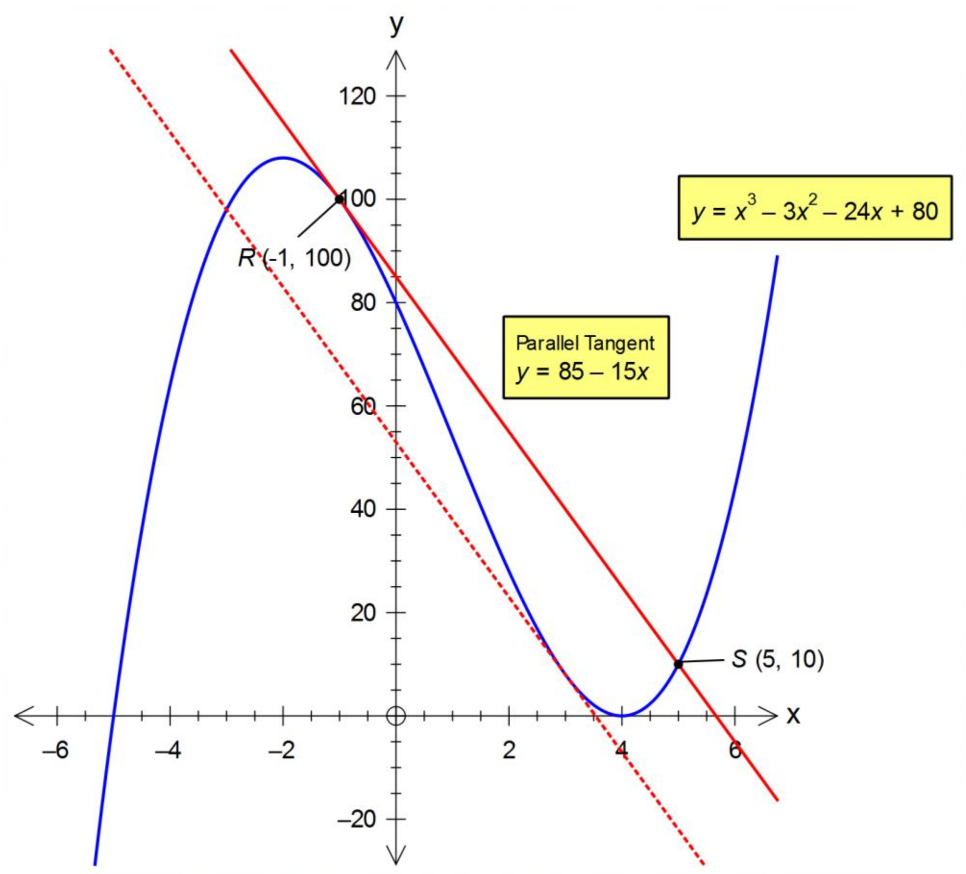
The equation of the tangent at (-1, 100) is $y - 100 = -15(x + 1) \Rightarrow y - 100 = -15x - 15$
$\Rightarrow$ $\mathbf{y = 85 - 15x}$.

vi) The tangent meets the curve again when $x^3 - 3x^2 - 24x + 80 = 85 - 15x$, or when
$x^3 - 3x^2 - 9x - 5 = 0$

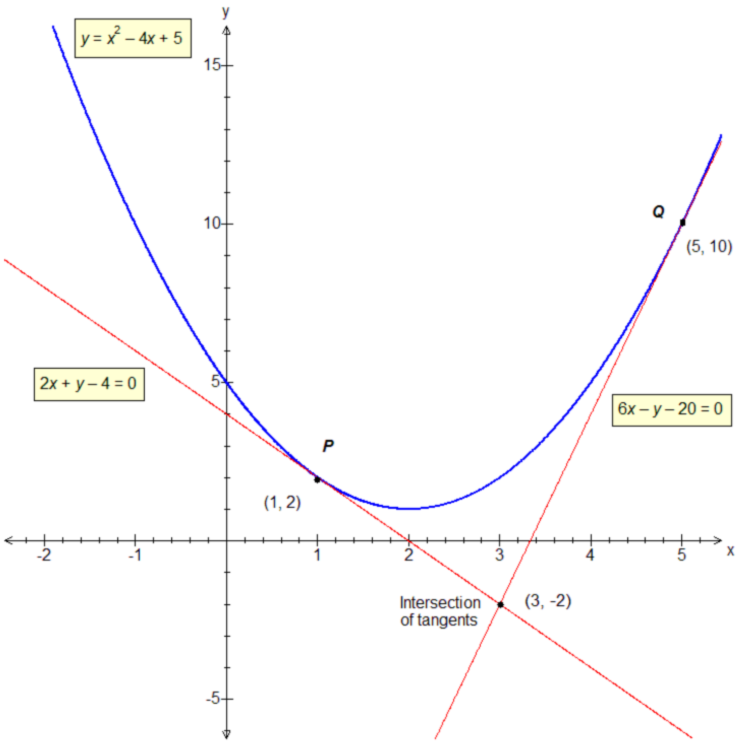
Substituting $x = 5$ into the equation we obtain $5^3 - 3(5)^2 - 9(5) - 5 = 125 - 75 - 45 - 5 = 0$.
Hence the x-coordinate of S is 5.

Substituting $x = 5$ into the equation of the line $y = 85 - 15x$, we have the y-coordinate of S, i.e. 10.

The coordinates of S are thus (5, 10).



Example (5): Two tangents are drawn to the curve $x^2 - 4x + 5$ at the points P (1, 2) and Q (5, 10). Give the coordinates of their point of intersection.



The derivative of $x^2 - 4x + 5$ is $2x - 4$, and thus its values at P and Q are –2 and 6 respectively.

The equation of the tangent at P is
$y - y_1 = m(x - x_1)$
$\Rightarrow y - 2 = -2(x - 1)$
$\Rightarrow y - 2 = -2x + 2$
$\Rightarrow y - 2 + 2x - 2 = 0.$
$\Rightarrow \mathbf{2x + y - 4 = 0.}$

Similarly the equation of the tangent at Q is
$y - 10 = 6(x - 5).$

$\Rightarrow y - 10 = 6x - 30$
$\Rightarrow 6x - 30 - y + 10 = 0.$
$\Rightarrow \mathbf{6x - y - 20 = 0.}$

We now have a pair of simultaneous linear equations which can be solved by elimination:

$2x + y - 4 = 0$ *A*
$6x - y - 20 = 0$ *B*

$8x - 24 = 0$ *A+B*

This gives $x = 3$, and substituting into either equation gives $y = -2$.

$\therefore$ the two tangents meet at (3, -2).

Example (6): The equation of a curve is given by $y = \frac{x^3}{3} - 16x$.

i) Find $\frac{dy}{dx}$ and hence the coordinates of the stationary points on the curve $y = \frac{x^3}{3} - 16x$.

ii) Distinguish between the maximum and minimum points obtained in part i).
iii) Given that the line 20*x* - *y* – 144 = 0 is the equation of the tangent to the curve at the point (*p*, *q*), find the values of *p* and *q*.

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i) If $y = \frac{x^3}{3} - 16x$, then $\frac{dy}{dx} = x^2 - 16 \Rightarrow \frac{dy}{dx} = (x+4)(x-4)$.

The *x*-coordinates of the stationary points are 4 and -4.

$\therefore$ The two stationary points are $\left(4, \frac{-128}{3}\right)$ and $\left(-4, \frac{128}{3}\right)$ after substituting in $y = \frac{x^3}{3} - 16x$.

ii) The second derivative, $\frac{d^2y}{dx^2} = 2x$.

At $\left(4, \frac{-128}{3}\right)$, $\frac{d^2y}{dx^2} = 8$ (i.e. > 0) , hence $\left(4, \frac{-128}{3}\right)$ is a local minimum.

On the other hand, at $\left(-4, \frac{128}{3}\right)$, $\frac{d^2y}{dx^2} = -8$ (i.e. < 0), hence $\left(-4, \frac{128}{3}\right)$ is a local maximum.

iii) We can find the gradient of the line 20*x* - *y* – 144 = 0 $\Rightarrow$ 20*x* = *y* + 144 $\Rightarrow$ *y* = 20*x* – 144.
$\therefore$ the gradient of the line 20*x* - *y* – 144 = 0 is 20.

(Or we could have used the fact that a line with equation *ax* + *by* + *c* = 0 has a gradient of $-\frac{a}{b}$).

Next, we must find the points on the curve where the gradient is also 20, i.e. we solve
$x^2 - 16 = 20 \Rightarrow x^2 - 36 = 0 \Rightarrow (x + 6)(x - 6) = 0$.

When *x* = 6, $\frac{x^3}{3} - 16x$ = 72 - 96 = -24; similarly when *x* = -6, $\frac{x^3}{3} - 16x$ = -72 + 96 = 24.
The two possible values for (*p, q*) are (6, -24) or (-6, 24).

We then substitute each pair of values into the expression 20*x* - *y* – 144; the correct pair should give a result of 0.

At (6, -24), 20*x* – *y* –144 = 0; at (-6, 24), 20*x* – *y* –144 = -288.

$\therefore$ the line 20*x* - *y* – 144 = 0 is a tangent to the curve $y = \frac{x^3}{3} - 16x$ at the point (6, -24).

Example (7):

Find the equation of the normal to the curve $y = x^2 - 3x + 2$ at the point P where $x = 3$.
The normal intersects the curve at another point Q.
Find the coordinates of Q, giving your results as rational numbers.

$y = x^2 - 3x + 2$, so $\frac{dy}{dx} = 2x - 3$, so when $x = 3$, $y = 2$ and $\frac{dy}{dx} = 3$.
i.e. the coordinates of P are (3, 2).

Since the tangent has a gradient of 3, the normal has a gradient of $-\frac{1}{3}$.

The equation of the normal is $y - 2 = -\frac{1}{3}(x - 3) \Rightarrow 3y - 6 = 3 - x \Rightarrow$ $\mathbf{3y + x - 9 = 0}$.

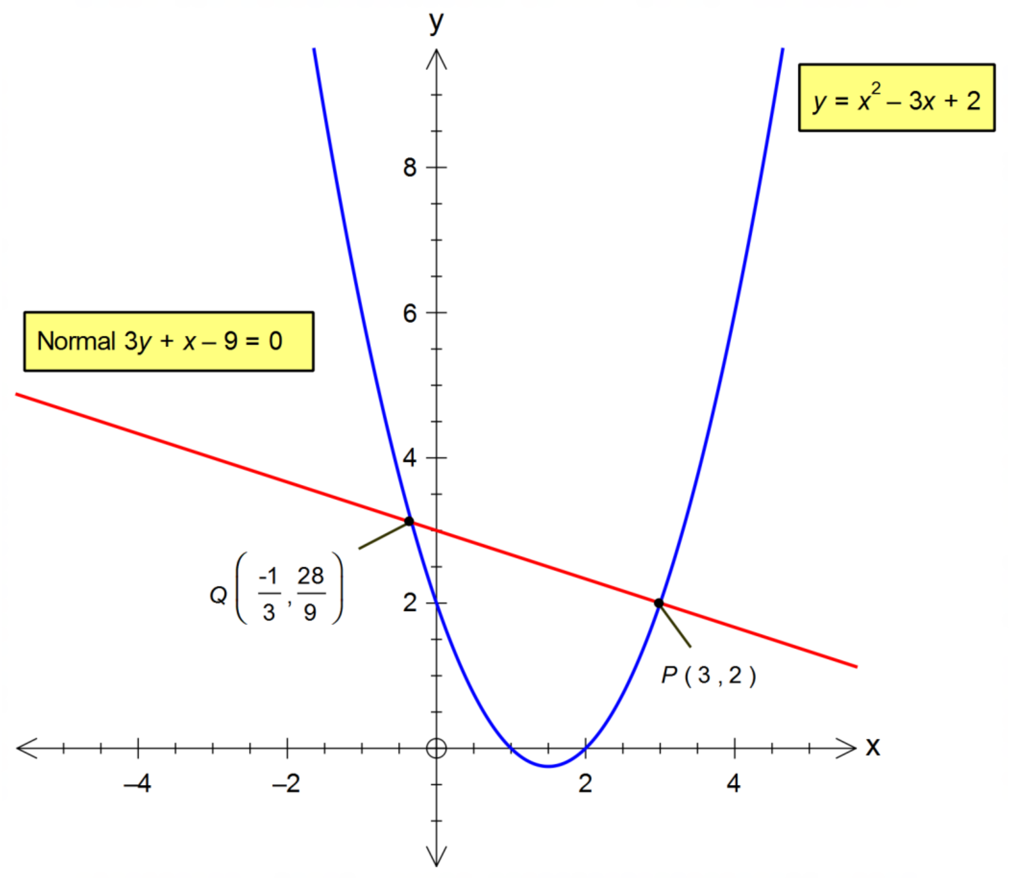
To find where the curve and the normal meet again, we solve the simultaneous equations by substituting for y into the quadratic:

$$3(x^2 - 3x + 2) + x - 9 = 0 \Rightarrow 3x^2 - 8x - 3 = 0 \Rightarrow (3x + 1)(x - 3) = 0.$$

The root of $x = 3$ corresponds to point P, so the x-coordinate of point Q is $-\frac{1}{3}$.

Substituting $x = -\frac{1}{3}$ into the equation of the curve $y = x^2 - 3x + 2$ gives the y-coordinate of Q as $\frac{1}{9} + 1 + 2 = \frac{28}{9}$.

Hence the coordinates of Q are $\left(-\frac{1}{3}, \frac{28}{9}\right)$.



Example (8):

i) Find the equation, in the form $y = mx + c$, of the normal to the curve $y = x^3 - x$ at point P , where the x-coordinate is = $\frac{1}{2}$.

ii) The normal intersects the curve at two other points, Q and R. Show that the x-coordinates of those points are the roots of the equation $4x^2 + 2x - 19 = 0$. (Hint: use the factor theorem.)

iii) Hence find the coordinates of Q and R to three decimal places.
(Assume that Q has a positive x-coordinate)

i) Differentiating, $\frac{dy}{dx} = 3x^2 - 1$, so when $x = \frac{1}{2}$, $\frac{dy}{dx} = -\frac{1}{4}$; also $y = -\frac{3}{8}$.

The gradient of the tangent at $P = -\frac{1}{4}$, so the gradient of the normal is 4.

The equation of the normal is therefore $y + \frac{3}{8} = 4(x - \frac{1}{2}) \Rightarrow y + \frac{3}{8} = 4x - 2 \Rightarrow y = 4x - \frac{19}{8}$.

ii) The curve and the normal intersect when the equations $y = x^3 - x$ and $y = 4x - \frac{19}{8}$ have a simultaneous solution, i.e. when $x^3 - x = 4x - \frac{19}{8} \Rightarrow x^3 - 5x + \frac{19}{8} = 0$.
Multiplying by 8, we have $8x^3 - 40x + 19 = 0$.

The normal meets the curve at $x = \frac{1}{2}$, so by the factor theorem, $2x - 1$ is a factor of $8x^3 - 40x + 19$.

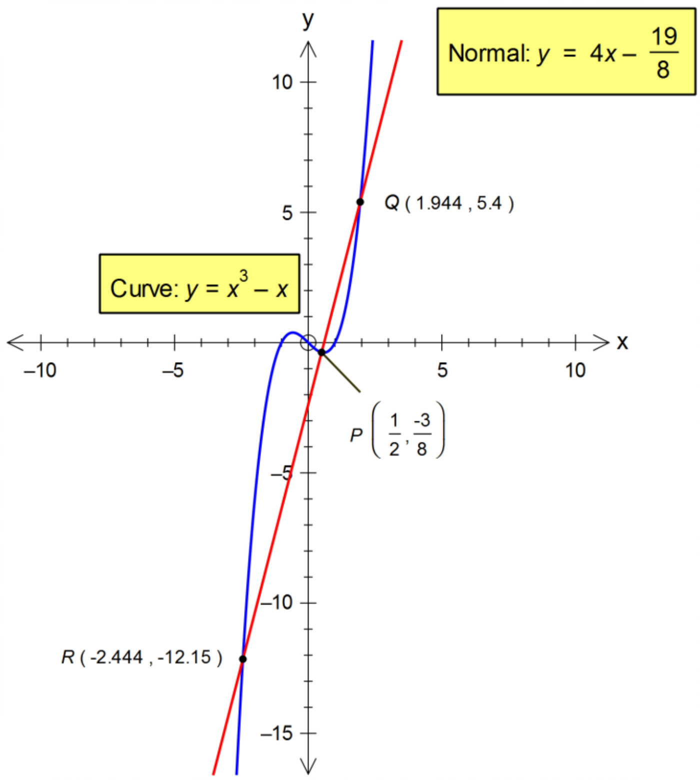
		$4x^2$	$+2x$	-19
$2x - 1$	$8x^3$	$+0x^2$	$-40x$	$+19$
	$8x^3$	$-4x^2$		
		$4x^2$	$-40x$	
		$4x^2$	$-2x$	
			$-38x$	$+19$
			$-38x$	$+19$
				0

By long division, the quotient is $4x^2 + 2x - 19$, which cannot be factorised, so we use the general quadratic formula to find the roots:

$$x = \frac{-2 \pm \sqrt{4 + 304}}{8} = \frac{-1 \pm \sqrt{77}}{4},$$

or 1.944 and -2.444 to 3 decimal places.

Substituting those calculated x-values into the equation of the normal $y = 4x - \frac{19}{8}$ will give the coordinates of the points where the curve intersects the normal as Q (1.944, 5.400) and R (-2.444, -12.150).



Example (9): The point P on the curve $y = k\sqrt{x}$ has an x-coordinate of 25. The normal to the curve at P is parallel to the line $5x + 2y = 0$.

i) Find the value of k and hence the coordinates of P.
ii) The normal to the curve meets the x-axis at the point Q. Calculate the area of the triangle OPQ where O is the origin.

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i) The gradient of the normal to the curve $y = k\sqrt{x}$ when $x = 25$ can be found by rearranging $5x + 2y = 0$ as $2y = -5x$ and $y = -\frac{5}{2}x$. $\therefore$ the gradient of the normal is $-\frac{5}{2}$ when $x = 25$.

(Or we could have used the fact that a line with equation $ax + by + c = 0$ has a gradient of $-\frac{a}{b}$).

Since a tangent and normal to a curve at a given point are perpendicular, the gradient of the tangent to the curve when $x = 25$ is $\frac{2}{5}$.

Differentiating the function $y = k\sqrt{x}$ gives $\frac{dy}{dx} = \frac{k}{2\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}k\left(x^{-\frac{1}{2}}\right)$.

The gradient of the tangent at $x = 25$ is $\frac{2}{5}$ i.e. $\frac{k}{2\sqrt{x}} = \frac{2}{5}$.

We then rearrange to find k ; $\frac{k}{2\sqrt{25}} = \frac{2}{5} \Rightarrow \frac{k}{10} = \frac{2}{5} \Rightarrow k = 4$.

Hence $y = 4\sqrt{x}$ and the coordinates of point P are (25, 20).

ii) We know that the normal at P is parallel to the line $5x + 2y = 0$, and also that it passes through the point (25, 20). Substituting for (x, y) gives the equation of the normal as $5x + 2y = 165$.

The x-coordinate of the x-intercept at Q satisfies $5x + 2y = 165$ for $y = 0$, so $5x = 165$ and $x = 33$.
The coordinates of Q are therefore (33,0).

See diagram for the calculation of the area of triangle OPQ.