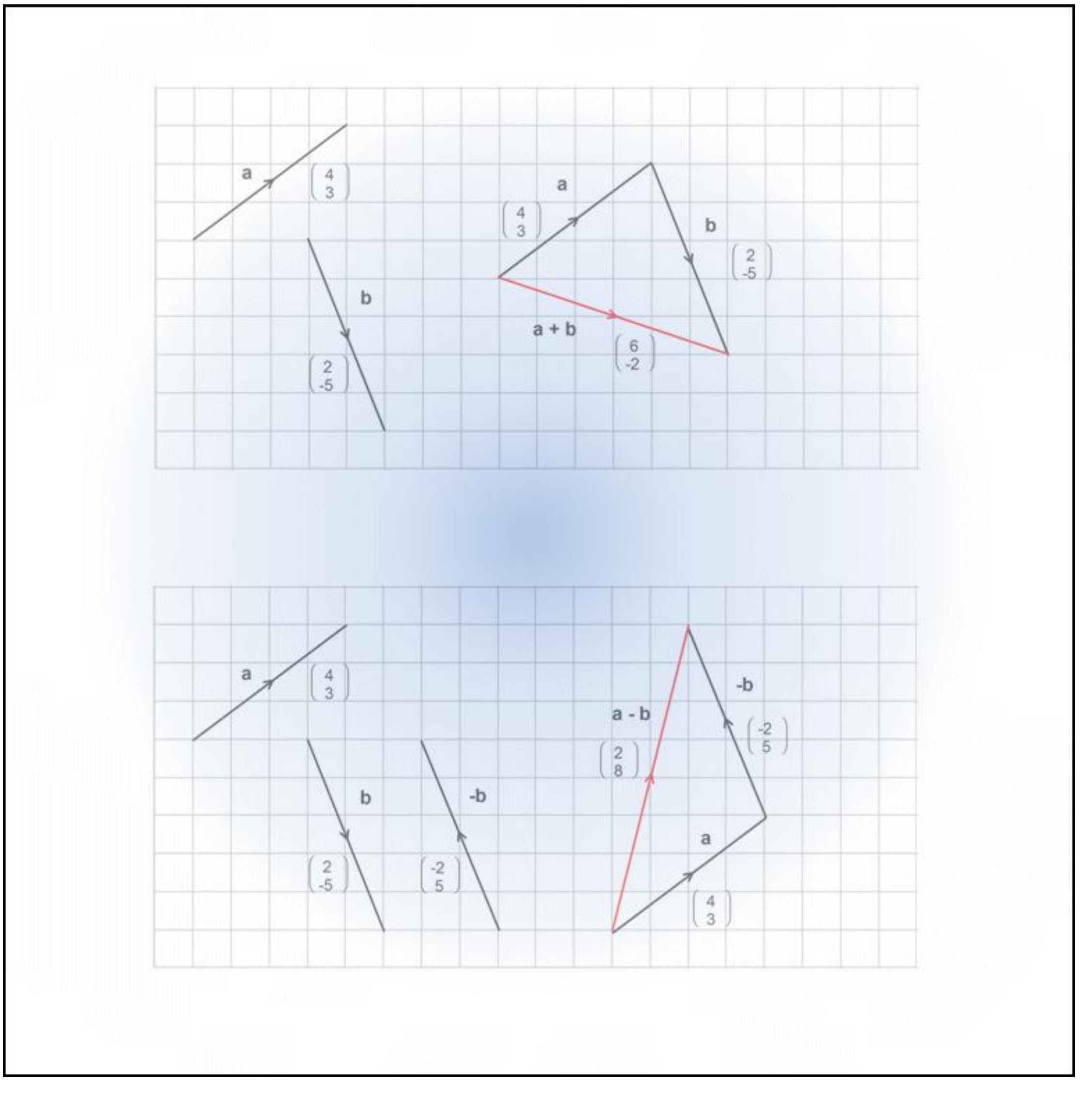
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 1 / AS

VECTORS



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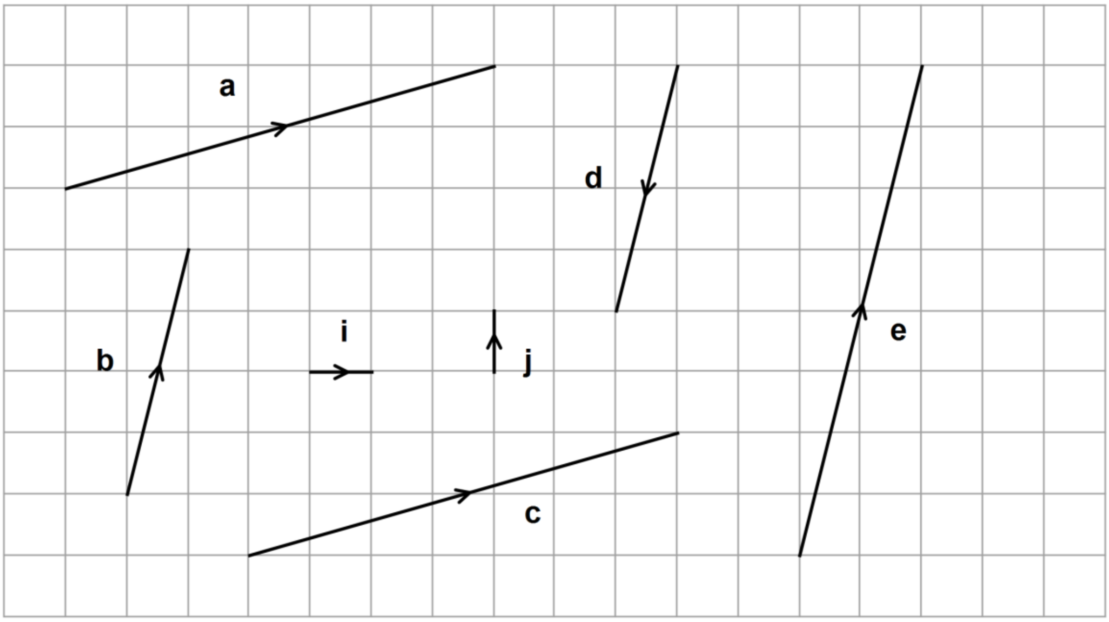
VECTORS (Vector geometry is now in a separate document)

Vectors are used in mathematics to illustrate quantities that have size (magnitude) and direction.
Quantities like mass and length have magnitude only, and are called **scalars**.
Velocity and force, on the other hand, have direction as well as size and can be expressed as vectors.

There are various ways of denoting vectors: typed documents use boldface, but written work uses underlining. Thus **a** and a are the same vector.



Example (1): The diagram above shows a collection of vectors in the plane.

Describe the relationships between the following vector pairs :
i) **a** and **c** ; ii) **b** and **d** ; iii) **b** and **e ;** iv) **i** and **j**.

i) Vectors **a** and **c** are equal here; hence **a** = **c**.
Two vectors are equal if they have the same size and the same direction.

The fact that **a** and **c** have different start and end points is irrelevant.

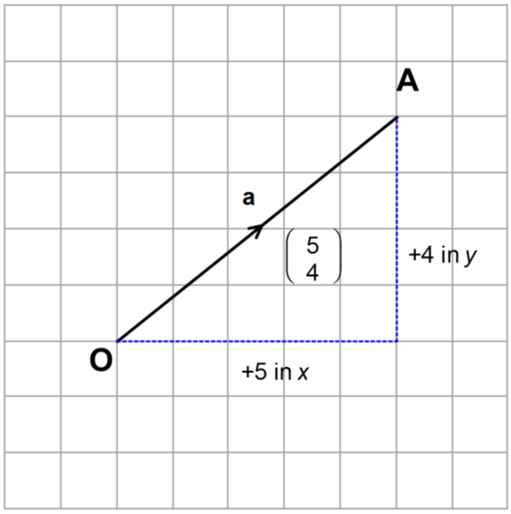
ii) Vectors **b** and **d** have the same size, but opposite directions, therefore **d** = **-b**.
Two vectors are inverses of each other if they have the same size, but opposite directions.

iii) Vector **e** is exactly twice as long as vector **b**, so **e** = 2**b**.
The 2 is what is known as a **scalar** multiplier.
(A scalar multiplier of -1 signifies an inverse vector.)

iv) Vectors **i** and **j** are perpendicular to each other.

We have come across vectors in the section on "Transformations", when describing a translation.

Example (1a): In the diagram on the right, the point *O* is transformed to point *A* by a translation of +5 units in the *x*-direction and +4 units in the *y*-direction.

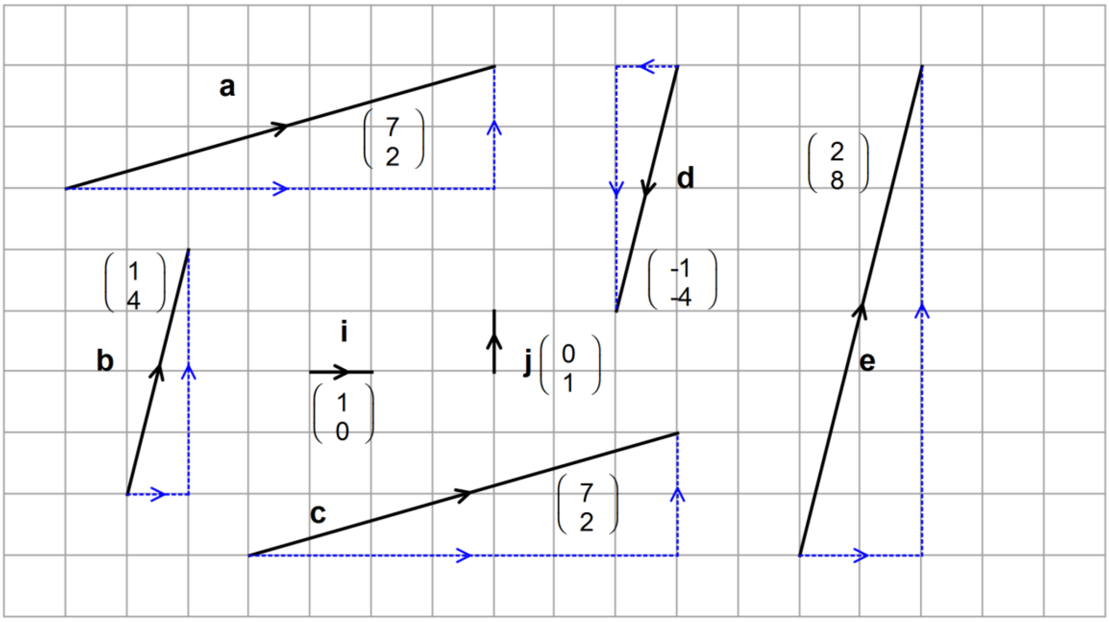


We can denote this translation as the vector

a = $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$, termed **column notation**.

Also, written work uses underlining where typing uses boldface, so we print **a** but write a̲.

Example (2): Express the seven vectors in the last example in column notation.



A movement in the direction of vector **a** corresponds to 7 units horizontally and 2 units vertically, as does that in the direction of vector **c**, given that the two vectors are equal .

Hence **a** = **c** = $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$

For vector **b**, the values are 1 horizontally and 4 vertically. Vector **d** is the inverse of vector **b**, so the movement is -1 unit horizontally and -4 units vertically.

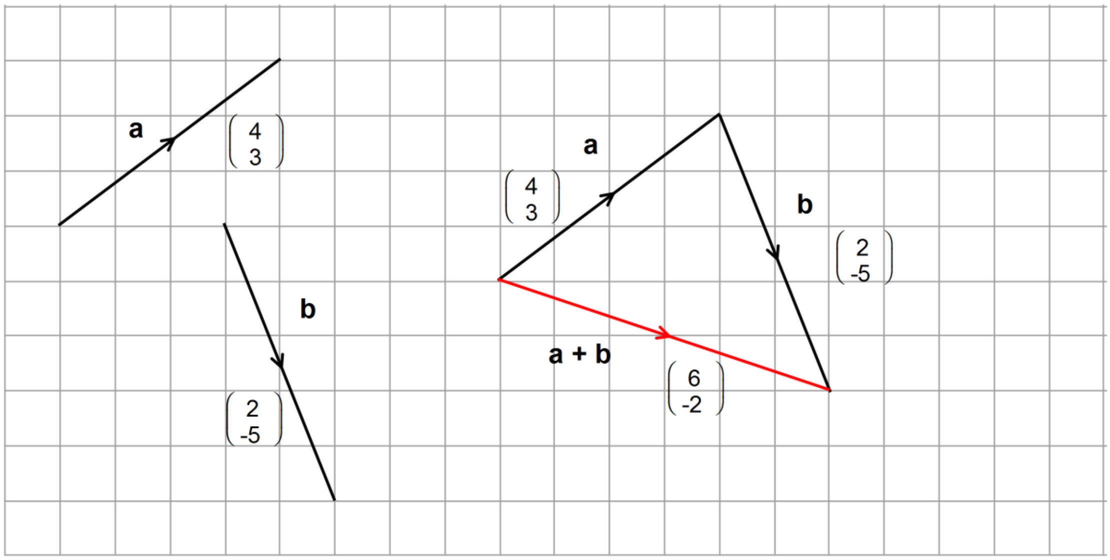
Hence **b** = $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and **d** = - $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ = $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$.

Vector **e** is twice vector **b** , so **e** = $2\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ = $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$.

Finally the vector **i** = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the vector **j** = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Addition of vectors.

Example (3):



To add two vectors, join them "nose to tail" as in the diagram.

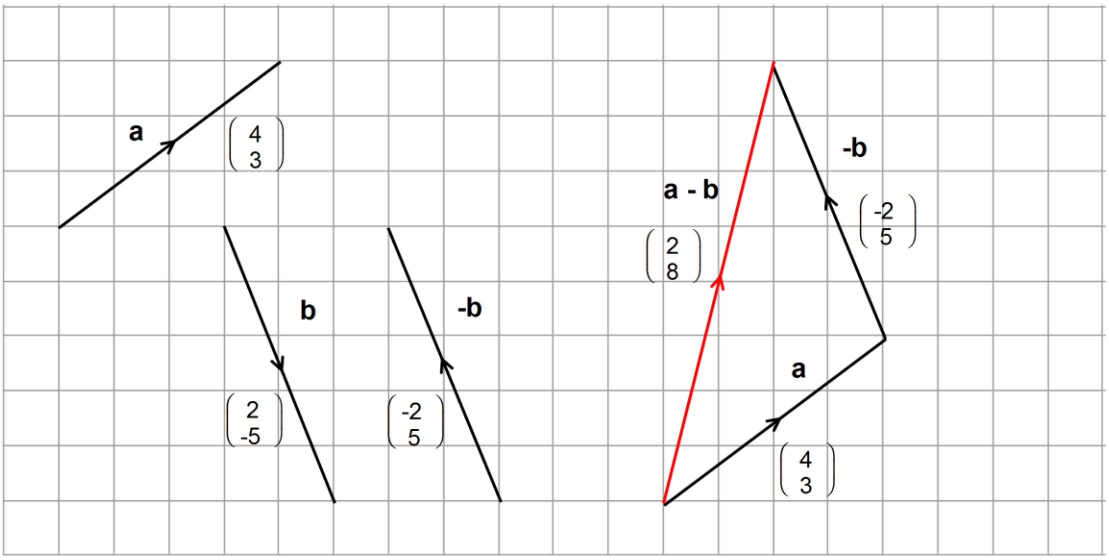
In column notation, **a** = $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, **b** = $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ and **a+b =** $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$.

This result could also have been obtained without drawing the diagram.

a+b = $\begin{pmatrix} 4+2 \\ 3-5 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$.

Subtraction of vectors.

Example (4):



Subtracting vector **b** from **a** is identical to adding the inverse of vector **b** to **a**.

This time, we join **–b** to **a** "nose to tail".

In column notation, **a** = $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, **b** = $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$, **-b** = $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ and **a-b =** **a** + **(-b)** = $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$.

This result could again have been obtained without drawing the diagram.

a-b = $\begin{pmatrix} 4-2 \\ 3+5 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$.

Addition or subtraction of vectors in column form is very easy - just add or subtract the components !

Another special case is **a-a =** $\begin{pmatrix} 4-4 \\ 3-3 \end{pmatrix}$, where for example, **a** = $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

The result here is the **zero vector**, **0**. This is not the same as the number 0, which is a scalar.

Standard Unit Vectors.

In Example (2), we came across two special vectors in the two-dimensional *x*-*y* plane :

i = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$; **j =** $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. These are termed the **standard unit vectors.**

Vector **i** is parallel to the *x*-axis and vector **j** is parallel to the *y*-axis.

All two-dimensional vectors can also be expressed as combinations of **i** and **j**.
(This is also known as component form.)

The vector **a** = $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ from Example (1a) can thus be expressed as 5**i** + 4**j**.

Similarly vector **b** = $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ from Example (4) can be expressed as 2**i** - 5**j,**

(The document uses both notations interchangeably for practice).

Example (5): Let vector **r** = 3**i - j** and **s** = **i** + 4**j**.

Express the following in column form:

i) **r + 3s**; ii) 2**s - r**; iii) **r + i - j**.

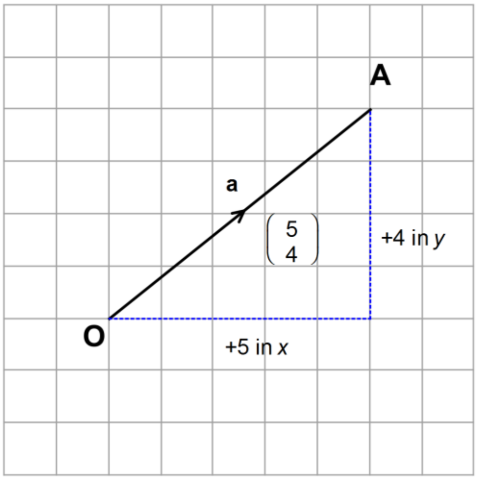
i) **r + 3s =** $\begin{pmatrix} 3 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3+3 \\ -1+12 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \end{pmatrix}$

ii) 2**s - r =** $2\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2-3 \\ 8+1 \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \end{pmatrix}$

iii) **r** + **i** - **j =** $\begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

The Magnitude of a Vector.

An important property of a vector is its **magnitude**, and it can be determined very easily by Pythagoras.



Since the vector **a** from Example (1a) can be visualised as the hypotenuse of a right-angled triangle with a base of 5 units and a height of 4 units, its magnitude is simply

$\sqrt{5^2+4^2} = \sqrt{41}$ units.

In general, the magnitude of any vector $\begin{pmatrix}a\\b\end{pmatrix}$ is $\sqrt{a^2+b^2}$.

(Recall the method used to find the distance between two given points in "Straight Line Graphs".)

Example (6): Find the magnitudes of the following vectors:

i) $\begin{pmatrix}3\\4\end{pmatrix}$; ii) $\begin{pmatrix}0.28\\0.96\end{pmatrix}$; iii) $\begin{pmatrix}0\\5\end{pmatrix}$

i) The magnitude of the vector $\begin{pmatrix}3\\4\end{pmatrix}$ is $\sqrt{3^2+4^2} = 5$ units.

ii) The vector $\begin{pmatrix}0.28\\0.96\end{pmatrix}$ has a magnitude of $\sqrt{0.28^2+0.96^2} = 1$ unit.

iii) Because the **i**- component of the vector is zero, its magnitude is $\sqrt{0^2+5^2} = 5$.

Finding Unit Vectors.

In Example 6(ii) we encountered a unit vector distinct from the standard ones of **i** and **j**.

Any vector can be re-expressed as scalar multiple of a unit vector in the same direction, as the next example shows.

Example (7): Find the unit vectors having the same direction as i) $\begin{pmatrix}3\\4\end{pmatrix}$; ii) $\begin{pmatrix}2\\-1\end{pmatrix}$; iii) $\begin{pmatrix}7\\0\end{pmatrix}$

i) The magnitude of the vector $\begin{pmatrix}3\\4\end{pmatrix}$ is $\sqrt{3^2+4^2} = 5$, so we divide the components by 5 to obtain the unit vector in the same direction; it is $\frac{1}{5}\begin{pmatrix}3\\4\end{pmatrix}$ = $\begin{pmatrix}3/5\\4/5\end{pmatrix}$. This example has rational components, but this is rarely the case.

ii) Vector $\begin{pmatrix}2\\-1\end{pmatrix}$ has a magnitude $\sqrt{2^2+(-1)^2} = \sqrt{5}$, so we divide the components by √5 to obtain the corresponding unit vector of $\frac{1}{\sqrt{5}}\begin{pmatrix}2\\-1\end{pmatrix}$ or $\begin{pmatrix}2/\sqrt{5}\\-1/\sqrt{5}\end{pmatrix}$.

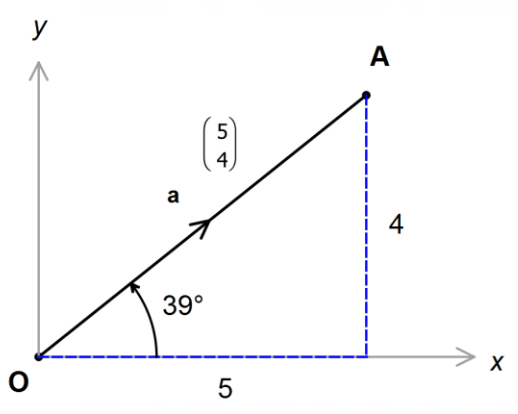
iii) Since vector $\begin{pmatrix}7\\0\end{pmatrix}$ has a zero **j**-component, the unit vector with the same direction is simply $\begin{pmatrix}1\\0\end{pmatrix}$.

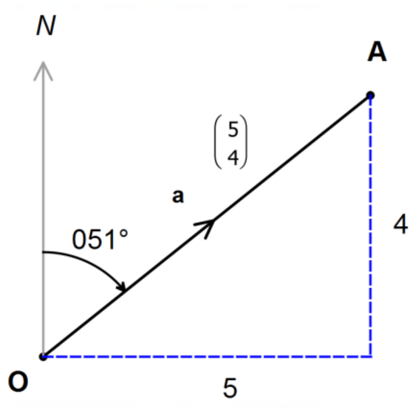
The Direction of a Vector.

The vector from Example (1), **a** = $\begin{pmatrix}5\\4\end{pmatrix}$ also makes an angle with the axes, and the direction of this angle can be expressed in two different ways.

From the left-hand diagram, it can be seen that vector **a** makes an angle of $\tan^{-1}\left(\frac{4}{5}\right)$, or 39°, anticlockwise with the positive *x*-axis. This is known as the Cartesian form.

We could use bearing notation instead and work out the angle clockwise from the northline *N*, which coincides with the positive *y*-axis, as per the right-hand diagram Thus vector **a** is on a bearing of 051°.





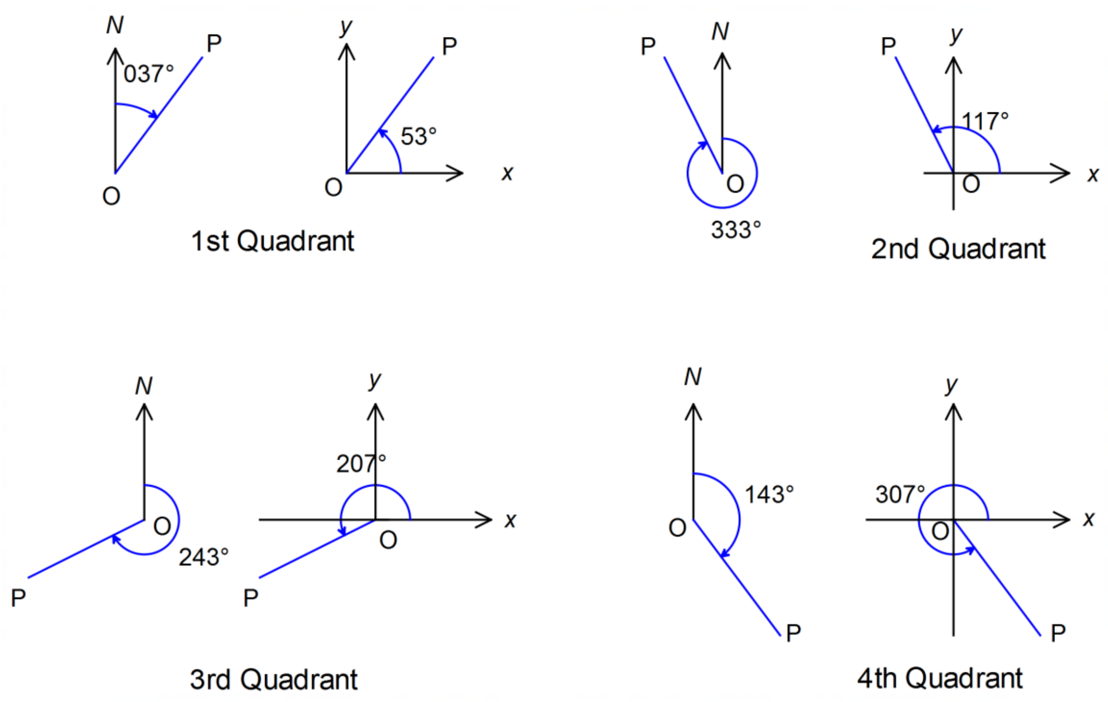
The direction of the last vector was easy enough to work out, as both its components were positive and it was just a matter of pressing calculator buttons.

More often than not, though, the answer given on the calculator will not be the required angle, so we need to be aware of the trigonometric ratios of angles outside the first quadrant.

A sketch would be particularly useful in such cases, as in the examples below.

The diagrams below show a line *OP* originating from an origin *O* such that the angles formed are in the four different quadrants.

Bearings are shown on the left of each example; Cartesian angles on the right.



In the first quadrant, *OP* forms an acute angle of 37° clockwise with the northline i.e. a bearing of 037°, but an angle of 53° anticlockwise with the positive *x*-axis. Note how the two different measures add to 90°.

Moving to the second quadrant, the anticlockwise angle between *OP* and the positive *x*-axis is now obtuse at 117°, but the clockwise bearing is now 333°. Note how the acute angle in the bearing diagram is (360-333)° or 27°, which is the same as the angle between the *y*-axis and line *OP* in the Cartesian diagram. Note how 117° + 333° = 450°.

In the third quadrant, the anticlockwise angle between *OP* and the positive *x*-axis has increased to 207°, with the clockwise bearing down to 243°. Note how the obtuse angle in the bearing diagram is (360-243)° or 117°, which is the same as the angle between the *y*-axis and line *OP* in the Cartesian diagram. Again, note how 243° + 207° = 450°.

Finally in the fourth quadrant, the anticlockwise angle between *OP* and the positive *x*-axis has increased to 307°, with the clockwise bearing now down to 143°. Note how the acute angle between the *x*-axis and *OP* is (360-307)° = 53°, which, when added to the right angle between the positive axes, is also 143° Again, note how 143° + 307° = 450°.

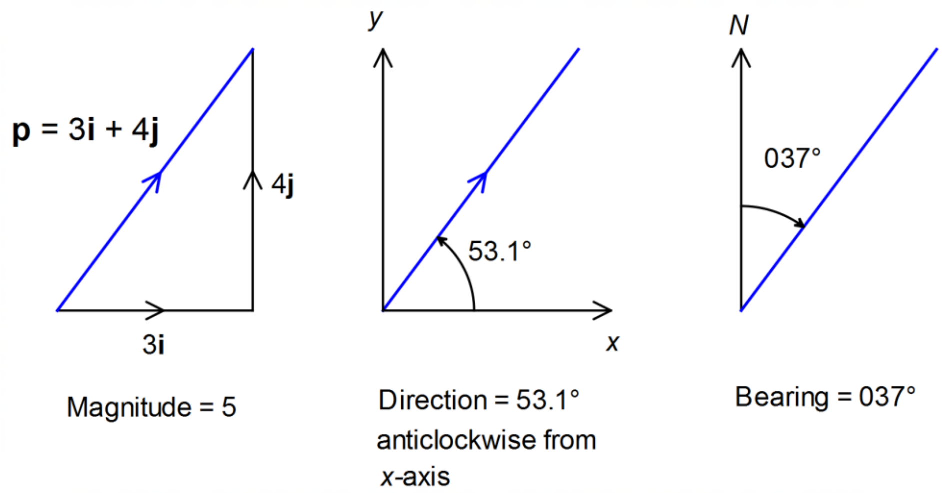
Hence, to convert an angle from bearing to Cartesian notation, we can use this rule:
If the angle is in the first quadrant, subtract from 90°; in all other quadrants, subtract from 450°.
It is still preferable to use diagrams for practice.

Examples (8): Find the magnitude and direction of the following vectors:

i) **p** = 3**i** + 4**j**; ii) **q** = 2**i** –2**j**; iii) **r** = -12**i** + 5**j**; iv) **s** = -2**i** – 3**j**; v) **t** = -**i**; vi) **u** = 3**j** .

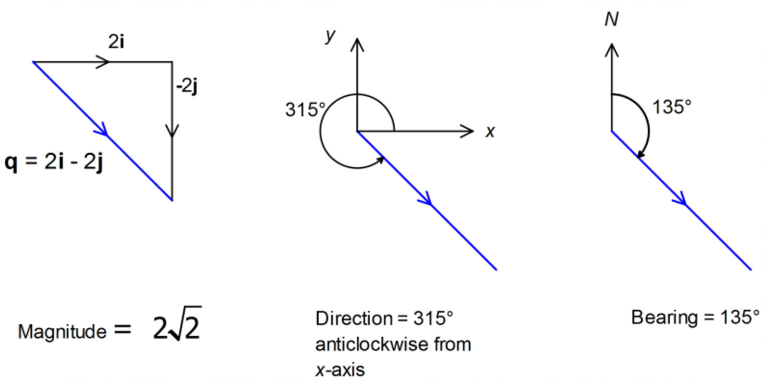
i) The magnitude of **p** is $\sqrt{3^2+4^2} = 5$ and its direction is $\tan^{-1}\left(\frac{4}{3}\right)$ or 53.1° to 1d.p.

In bearing notation, the angle is 037° to the nearest degree. Note how 53° + 37° = 90°
Any angle in the first quadrant can be converted into bearing notation by subtracting from 90°, and the same rule applies in reverse – thus, a bearing of 073° corresponds to an angle of 17° with the positive *x*-axis.



ii) Similarly, the magnitude of **q** = $\sqrt{2^2+(-2)^2} = 2\sqrt{2}$, and it makes an angle $\theta = \tan^{-1}\left(\frac{-2}{2}\right)$ with the positive *x*-axis. The calculator will give a result of -45°, but a sketch will show that **q** is in the **fourth** quadrant (270° to 360°). The true direction of the vector is therefore (-45 + 360)° or 315° anticlockwise from the *x*-axis.

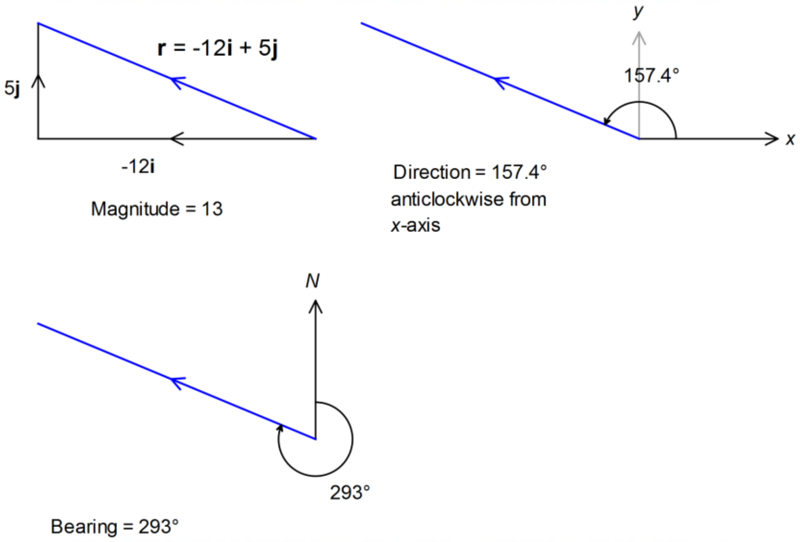
In bearing notation, the direction of the vector is 135° - we add the right angle between the axes to the acute angle between the *x*-axis and the vector, namely (360-315)° or 45° to give the bearing of 135°.



iii) The magnitude of **r** is $= \sqrt{(-12)^2+5^2} = 13$; the angle θ = $\tan^{-1}\left(\frac{5}{-12}\right)$.

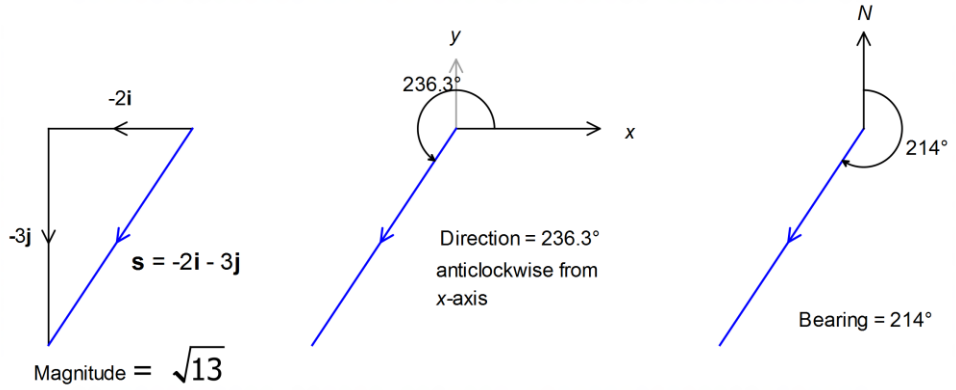
The calculator will state the angle as –22.6°, but a sketch shows that the vector is in the **second** quadrant (90° to 180°), and hence we must add 180° to obtain the true direction, namely (–22.6 + 180)° or 157.4° anticlockwise from the *x*-axis.

Alternatively, the bearing of **r** is 293°- obtained by taking the angle of 67° between the *y*-axis and the vector and then subtracting from 360°.



iv) The magnitude of vector **s** is $=\sqrt{(-2)^2+(-3)^2}=13$; the angle θ = $\tan^{-1}\left(\frac{-3}{-2}\right)$.
The calculator will display the angle as 56.3°, but a sketch would show the vector to lie in the **third** quadrant (180° to 270°), and hence we must add 180° to obtain the true direction, i.e. (56.3 + 180)° or 236.3° anticlockwise from the *x*-axis.

As a bearing, the direction of **s** is 214° ; the obtuse angle between the vector and the *y*-axis is (236-90)° = 146°, and we subtract 146 from 360 to find the bearing of 214°.



v) Vector **t** is a scalar multiple of **i** , here by a factor of –1. As there is no **j**-component, its length is 1 and its anticlockwise direction relative to **i** is 180°. (It is merely a horizontal unit vector in the negative *x*-direction). In bearing notation, its direction is west, or 270°.

vi) Vector **u** is a scalar multiple of **j**, here by a factor of 3. There is no **i**-component, and so its length is 3 and its anticlockwise direction relative to **i** is 90°. (It is a vector of length 3 in the positive *y*-direction.). Its bearing is north, or 000°

Resolving a two-dimensional vector.

The last examples asked us to find the magnitude and direction of a vector given its **i-** and **j-** components.

Now, if we were given the magnitude and direction of a vector, we can use trigonometry to resolve it into its **i-** and **j**- components.

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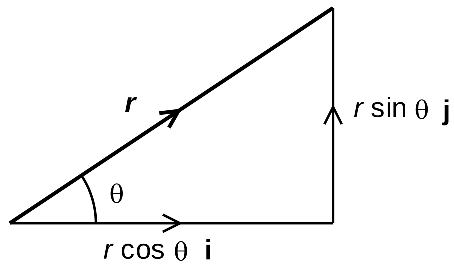


If a vector in two dimensions has magnitude r and makes an angle θ with the positive x-axis, then this same vector can be expressed as r cos θ **i** + r sin θ **j**.

If the direction is given as a bearing, it would need converting into Cartesian form, usually with the aid of a sketch.

Examples (9): Resolve the following vectors into **i** and **j** components, giving answers to 2 d.p.
(Angles all anticlockwise from the x-axis, i.e from **i**.)

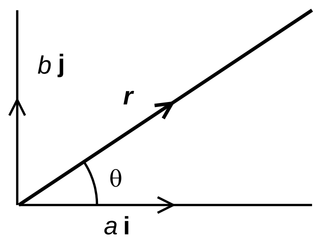
i) **p**, magnitude 5, angle 300°;
ii) **q**, magnitude 10, angle 32°;
iii) **r** , magnitude 8, angle 225°;
iv) **s,** magnitude 4, angle 156°
v) **t**, magnitude 6, angle 90°;
vi) **u** , magnitude 7, angle 180°

The angles are all stated in Cartesian form, so the components are easy to find.

i) **p** = (5 cos 300°) **i** + (5 sin 300°) **j =** 2.5**i** – 4.33**j**.
ii) **q** = (10 cos 32°) **i** + (10 sin 32°) **j =** 8.48**i** + 5.30**j**.
iii) **r** = (8 cos 225°) **i** + (8 sin 225°) **j =** –5.66**i** - 5.66**j**. .
iv) **s** = (4 cos 156°) **i** + (4 sin 156°) **j =** –3.65**i** + 1.63**j**.
v) **t** = (6 cos 90°) **i** + (6 sin 90°) **j =** 6**j**. (vertical, in positive direction)
vi) **u** = (7 cos 180°) **i** + (7 sin 180°) **j =** -7**i**. (horizontal, in negative direction.

When the angles are quoted as bearings, we need to convert them into Cartesian notation first.
For that, we can either sketch the vectors as in Examples (16), or subtract from 450° (or 90°).
The last two results can be quoted as:

Any vector parallel to the *x*-axis has an i-component only.
Any vector parallel to the *y*-axis has a j-component only.



Sometimes we might need to find the magnitude and direction of a vector, given its resolved form.

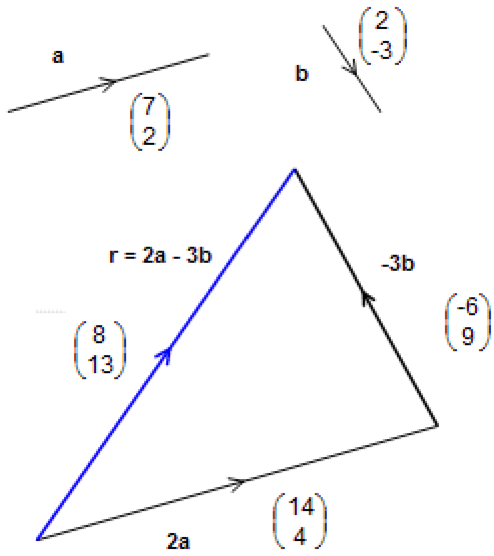
The magnitude of a two-dimensional vector **r** = a**i** + b**j** is given by applying Pythagoras' theorem.

|**r**| = $\sqrt{a^2+b^2}$.

The direction of the vector r is given by $\theta = \tan^{-1}\left(\frac{b}{a}\right)$, measured anticlockwise from the unit vector **i**.

Simultaneous Vector Equations.

The diagram on the right shows how scalar multiples of vectors **a** and **b** can be combined to form vector **r = 2a – 3b.**



r = 2a – 3b = $2\begin{pmatrix}7\\2\end{pmatrix}-3\begin{pmatrix}2\\-3\end{pmatrix}=\begin{pmatrix}8\\13\end{pmatrix}$

Question: How can we express **r** in terms of **a** and **b** without using a diagram ?

In other words, we want to express **r** in the form **r** = *s***a** + *t***b** where *s* and *t* are constants to be determined.

Example (10):

We therefore set up a vector equation as follows:

r = *s***a** + *t***b** → $s\begin{pmatrix}7\\2\end{pmatrix}+t\begin{pmatrix}2\\-3\end{pmatrix}=\begin{pmatrix}8\\13\end{pmatrix}$

From this vector equation, we can set up a pair of linear simultaneous equations by reading the column entries from left to right;

$7s + 2t = 8$ *A*
$2s – 3t = 13$ *B*

$21s + 6t = 24$ 3*A*
$4s - 6t = 26$ 2*B*

$25s = 50$ 3*A* + 2*B* ∴ ***s* = 2**

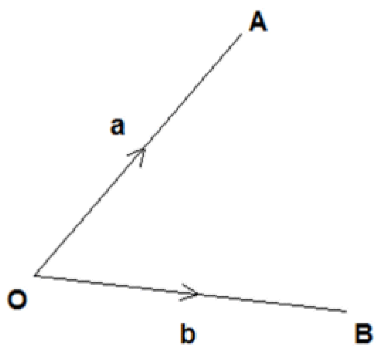
Substituting in equation *A* gives $14 + 2t = 8$, so $t = -3$.

Hence **r** = $2\begin{pmatrix}7\\2\end{pmatrix}-3\begin{pmatrix}2\\-3\end{pmatrix}$.

"Double-letter and arrow" notation.

Vectors can be denoted by a single boldface letter, but another notation is to state their end points and write an arrow above them.

In the right-hand diagram, vector **a** joins points *O* and *A* and vector **b** joins point *O* and *B*.



Therefore $\overrightarrow{OA}$ = **a** and $\overrightarrow{OB}$ = **b**.

The direction of the arrow is important here;
the vector $\overrightarrow{AO}$ goes in the opposite direction to $\overrightarrow{OA}$ although it has the same magnitude.

Hence $\overrightarrow{AO}$ = - $\overrightarrow{OA}$ = -**a**.

Angles between vectors.

To find the angle between two vectors, we calculate their lengths using Pythagoras and then use the cosine rule.

Examples (11): Find the angle between the vector pairs:

i) $\overrightarrow{OR}$ = 3**i** + **j** and $\overrightarrow{OS}$ = 2**i** + 2**j**
ii) $\overrightarrow{OR}$ = 3**i** + **j** and $\overrightarrow{OS}$ = **i** + **j**
iii) $\overrightarrow{OR}$ = 4**i** + **j** and $\overrightarrow{OS}$ = -**i** + 4**j**
iv) $\overrightarrow{OR}$ = **i** + 2**j** and $\overrightarrow{OS}$ = 2**i** - 3**j.**

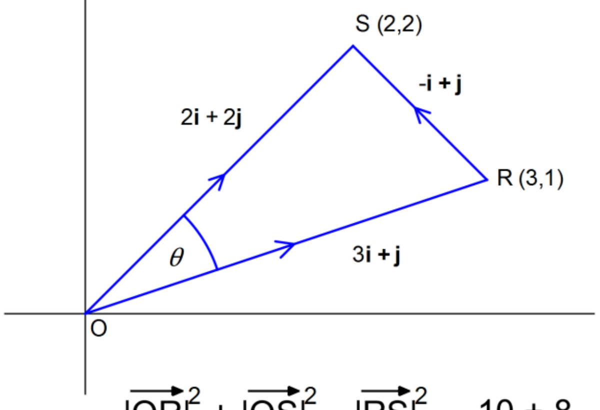
i) $\overrightarrow{OR}$ = 3**i** + **j** and $\overrightarrow{OS}$ = 2**i** + 2**j**

The easiest way to visualise the problem is to show both vectors starting away from the origin and drawing a sketch.

The calculations are shown on the diagram and begin with finding the magnitudes of the vectors $\overrightarrow{OR}$ and $\overrightarrow{OS}$, followed by completing the triangle *ORS* and ending by using the cosine rule to find the angle θ between the vectors.

$$|\overrightarrow{OR}| = \sqrt{3^2 + 1^2} = \sqrt{10}$$
$$|\overrightarrow{OS}| = \sqrt{2^2 + 2^2} = \sqrt{8}$$

$$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR}$$
$$|\overrightarrow{RS}| = \sqrt{(2-3)^2 + (2-1)^2} = \sqrt{2}$$



$$\cos\theta = \frac{|\overrightarrow{OR}|^2 + |\overrightarrow{OS}|^2 - |\overrightarrow{RS}|^2}{2|\overrightarrow{OR}||\overrightarrow{OS}|} = \frac{10 + 8 - 2}{2\sqrt{10}\sqrt{8}} = \frac{2}{\sqrt{5}}$$

$$\theta = 26.6°$$

ii) $\overrightarrow{OR} = 3\mathbf{i} + \mathbf{j}$ and $\overrightarrow{OS} = \mathbf{i} + \mathbf{j}$

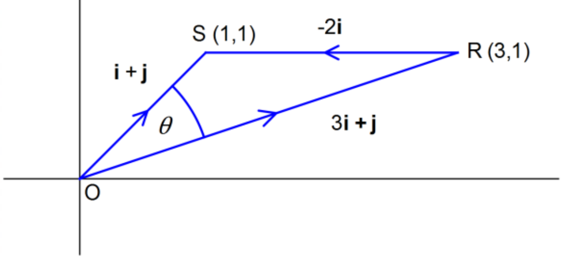
Notice how the vector $\overrightarrow{OS}$ still has the same direction as in part (i), but its magnitude has been halved. We could have used any scalar multiple of either vector and still ended up with the same value of the angle θ between the vectors.

$$|\overrightarrow{OR}| = \sqrt{3^2 + 1^2} = \sqrt{10}$$

$$|\overrightarrow{OS}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR}$$

$$|\overrightarrow{RS}| = \sqrt{(1-3)^2 + (1-1)^2} = 2$$



$$\cos\theta = \frac{|\overrightarrow{OR}|^2 + |\overrightarrow{OS}|^2 - |\overrightarrow{RS}|^2}{2|\overrightarrow{OR}||\overrightarrow{OS}|} = \frac{10 + 2 - 4}{2\sqrt{10}\sqrt{2}} = \frac{2}{\sqrt{5}}$$

$$\theta = 26.6°$$

iii) $\overrightarrow{OR} = 4\mathbf{i} + \mathbf{j}$ and $\overrightarrow{OS} = -\mathbf{i} + 4\mathbf{j}$

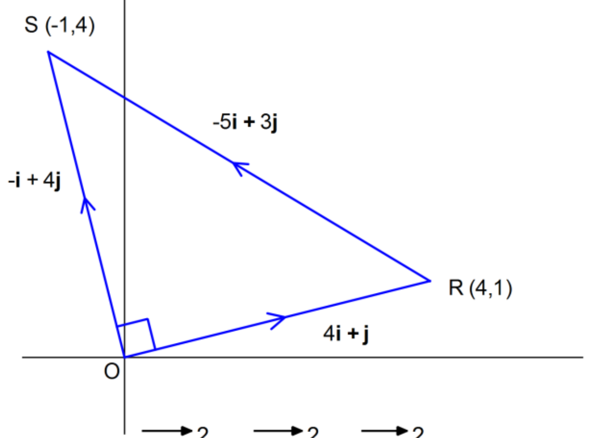
In this example, the vectors $\overrightarrow{OR}$ and $\overrightarrow{OS}$ happen to be perpendicular.

$$|\overrightarrow{OR}| = \sqrt{4^2 + 1^2} = \sqrt{17}$$

$$|\overrightarrow{OS}| = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$$

$$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR}$$

$$|\overrightarrow{RS}| = \sqrt{(-1-4)^2 + (4-1)^2} = \sqrt{34}$$



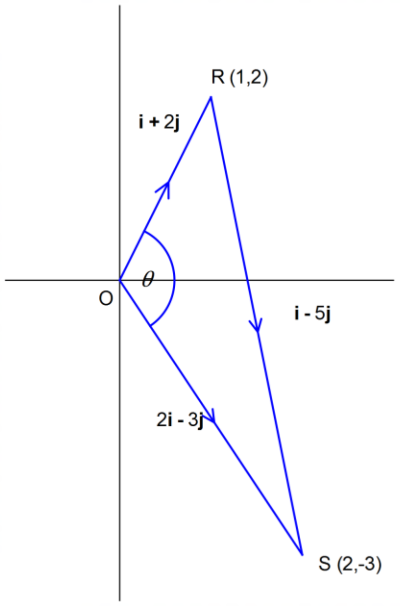
$$\cos\theta = \frac{|\overrightarrow{OR}|^2 + |\overrightarrow{OS}|^2 - |\overrightarrow{RS}|^2}{2|\overrightarrow{OR}||\overrightarrow{OS}|} = \frac{17 + 17 - 34}{2\sqrt{17}\sqrt{17}} = 0$$

$$\theta = 90°$$

We could also deduce that the angle *SOR* is a right angle because the square of the length of $\overrightarrow{RS}$ is the sum of the squares of the lengths of $\overrightarrow{OR}$ and $\overrightarrow{OS}$.

iv) $\overrightarrow{OR} = \mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{OS} = 2\mathbf{i} - 3\mathbf{j}$.

Here, the angle between the vectors is obtuse – note the negative value of cos θ.



$$|\overrightarrow{OR}| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$|\overrightarrow{OS}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$$

$$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR}$$

$$|\overrightarrow{RS}| = \sqrt{(2-1)^2 + (-3-2)^2} = \sqrt{26}$$

$$\cos\theta = \frac{|\overrightarrow{OR}|^2 + |\overrightarrow{OS}|^2 - |\overrightarrow{RS}|^2}{2|\overrightarrow{OR}||\overrightarrow{OS}|} = \frac{5 + 13 - 26}{2\sqrt{5}\sqrt{13}} = \frac{-4}{\sqrt{65}}$$

$$\theta = 119.7°$$