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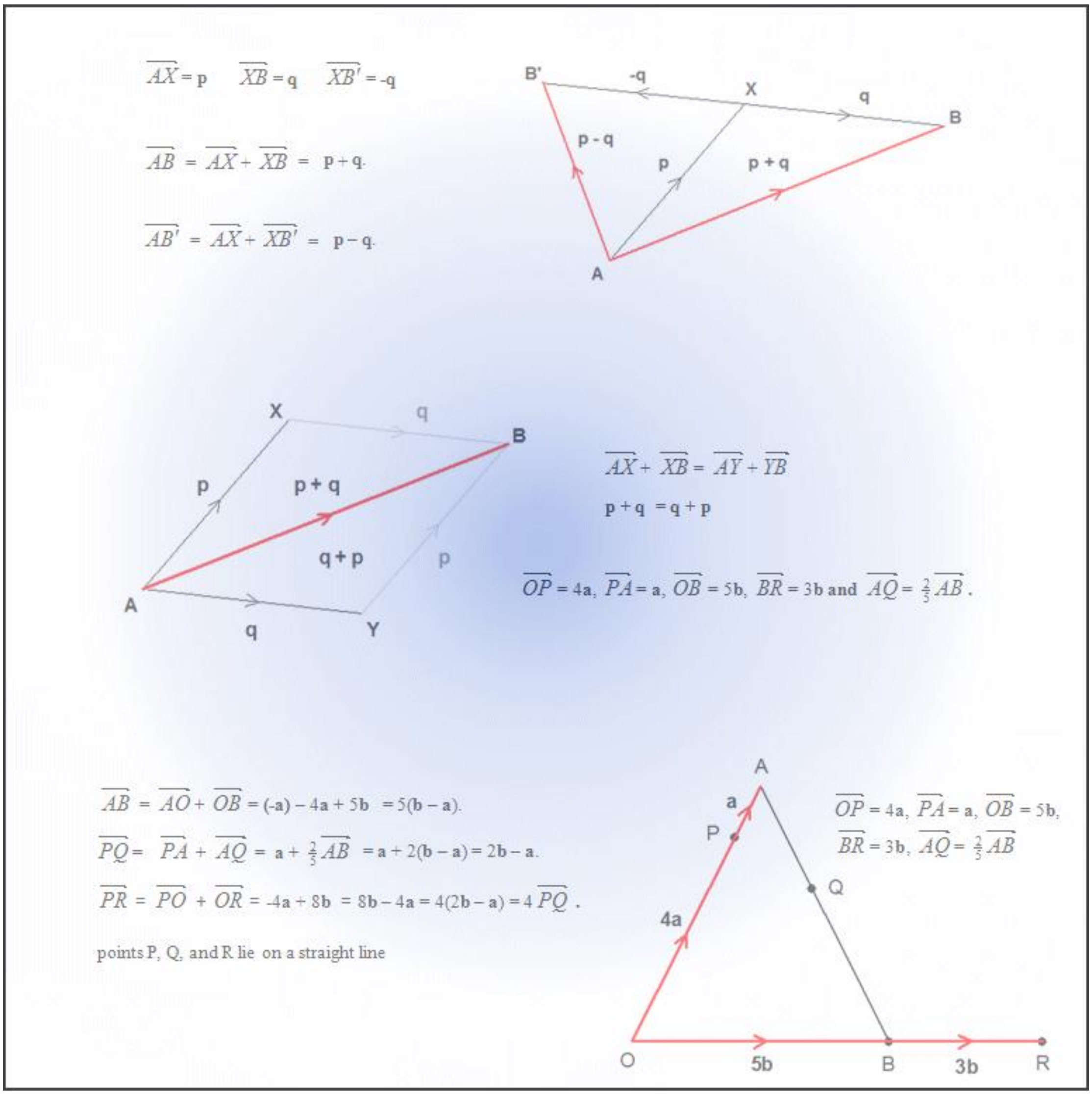
M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 1 / AS

VECTOR GEOMETRY



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VECTOR GEOMETRY

Many problems and theorems in geometry can be analysed and proven using vectors.

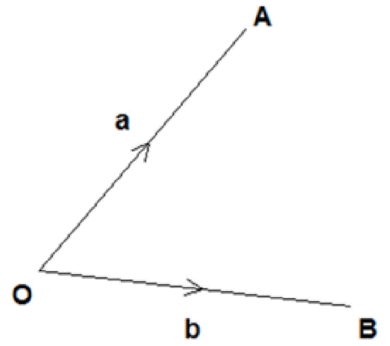
Vectors can be denoted by a single boldface letter, but another notation is to state their end points and write an arrow above them.

In the right-hand diagram, vector **a** joins points O and A and vector **b** joins point O and B.

Therefore $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

The direction of the arrow is important here; the vector $\overrightarrow{AO}$ goes in the opposite direction to $\overrightarrow{OA}$ although it has the same magnitude.

Hence $\overrightarrow{AO} = -\overrightarrow{OA} = -\mathbf{a}$.

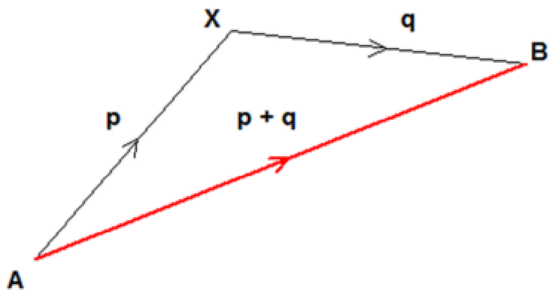


The Triangle Law.

(Recall) To add two vectors, apply the first, and then the second.

Thus $\overrightarrow{AB} = \overrightarrow{AX} + \overrightarrow{XB}$.

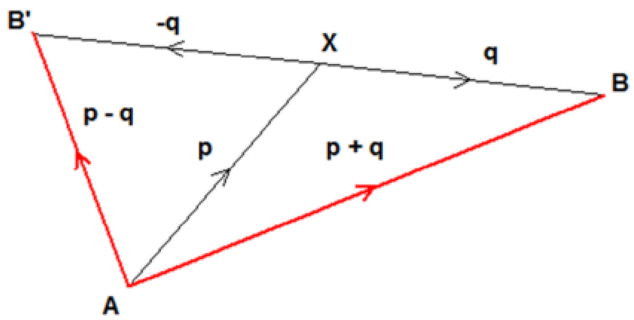
Here $\overrightarrow{AX} = \mathbf{p}$ and $\overrightarrow{XB} = \mathbf{q}$.



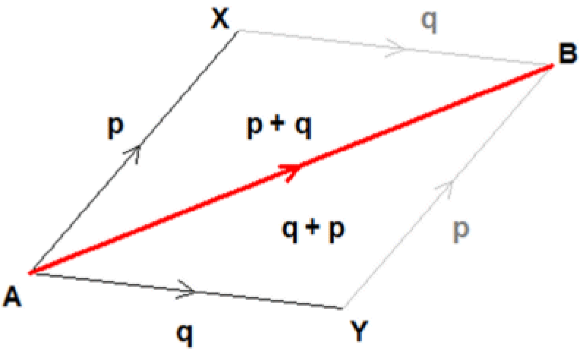
(Recall) Subtracting a vector is the same as adding its inverse, i.e. the parallel vector of the same magnitude but in the opposite direction.

Here, $\overrightarrow{XB'} = -\mathbf{q}$.

Thus $\overrightarrow{AB'} = \overrightarrow{AX} + \overrightarrow{XB'} = \mathbf{p} - \mathbf{q}$.



The Parallelogram Law.



Going from A to B via Y gives the same result as going from A to B via X.

Therefore $\overrightarrow{AX} + \overrightarrow{XB} = \overrightarrow{AY} + \overrightarrow{YB}$.

In other words, **p** + **q** = **q** + **p**.

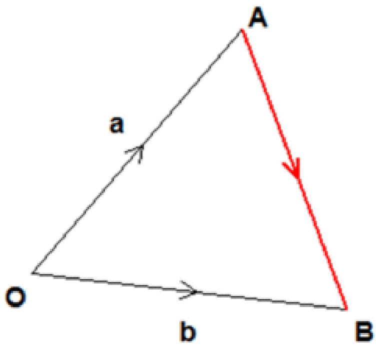
Notice as well that $\overrightarrow{AX} = \overrightarrow{YB}$ = **p** and $\overrightarrow{AY} = \overrightarrow{XB}$ = **q**.

Since the opposite pairs of sides of any parallelogram are equal and parallel, they can always be represented by the same vector provided their directions are equal, thus $\overrightarrow{BY}$ = **-p** here.

Geometrical Applications.

When asked to find an unknown vector between two points, just work it out as an alternative route made up of known segments, as per the parallelogram law.

Example (1): Express the vector $\overrightarrow{AB}$ in terms of **a** and **b**.



We want to go from A to B directly, but we do not have the vector for it.

We therefore go via O, as in $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$.

Now $\overrightarrow{AO}$ is the same as **a** but in the reverse direction, whilst $\overrightarrow{OB}$ = **b**.

Hence $\overrightarrow{AB}$ = **-a** + **b** or **b – a**.

Position vectors.

Up to now, we have treated vectors as directed line segments, but we can also say that any point on the plane has a **position vector** relative to the origin O.

In fact, any point (p, q) has a position vector of $p\mathbf{i} + q\mathbf{j}$ or $\begin{pmatrix} p \\ q \end{pmatrix}$ relative to the origin.

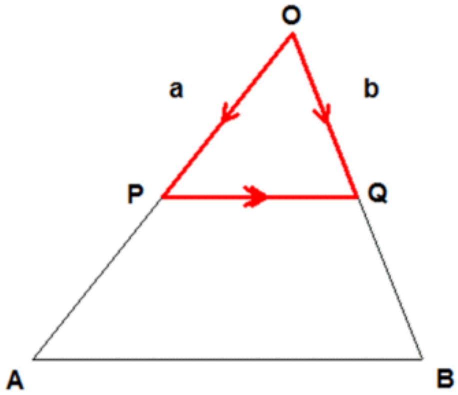
Example (1a): The coordinates of points A and B are (2, 4) and (6, 1) respectively. Find $\overrightarrow{AB}$ in column notation given that O is the origin.

Since O is the origin, vector $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$.

Hence $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

Example (2): In the triangle *OAB*, point *P* is the midpoint of *OA* and point *Q* is the midpoint of *OB*.

Show that *PQ* is parallel to *AB*, and also half its length.



$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\mathbf{a} + \mathbf{b}$ or $\mathbf{b} - \mathbf{a}$.

We are also told that

$\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OA}$, thus $\overrightarrow{OA} = 2\overrightarrow{OP} = 2\mathbf{a}$.

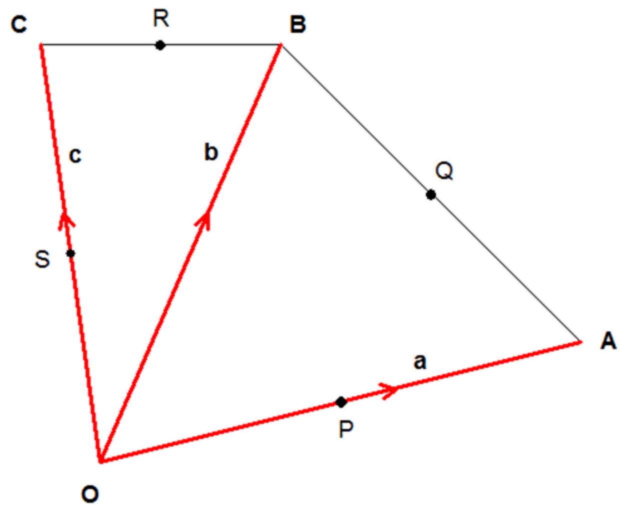
$\overrightarrow{OQ} = \frac{1}{2}\overrightarrow{OB}$, thus $\overrightarrow{OB} = 2\overrightarrow{OQ} = 2\mathbf{b}$.

Finally, $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -2\mathbf{a} + 2\mathbf{b} = 2(\mathbf{b} - \mathbf{a}) = 2\overrightarrow{PQ}$.

$\therefore$ *PQ* is parallel to *AB*, and half its length.
(Two vectors are parallel if either can be expressed as a scalar multiple of the other).

Example (3): *OABC* is a quadrilateral. $\overrightarrow{OA}$ = **a**, $\overrightarrow{OB}$ = **b** and $\overrightarrow{OC}$ = **c.**

Points *P*, *Q*, *R* and *S* are the midpoints of *OA*, *AB*, *BC* and *OC* respectively.



i) Find the following vectors in terms of **a**, **b** and **c**: $\overrightarrow{OP}$, $\overrightarrow{AB}$, $\overrightarrow{AQ}$, $\overrightarrow{PQ}$ **and** $\overrightarrow{SR}$.
ii) Prove that *PQ* is parallel to *SR*.
iii) What type of quadrilateral is *PQRS* ?

i) *P* is the midpoint of *OA*, so $\overrightarrow{OP} = \frac{1}{2}$ **a.**

($\overrightarrow{PA}$ **is also** $\frac{1}{2}$ **a**).

By going via *O*, $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ = **b – a.**

Since *Q* is the midpoint of *AB*, $\overrightarrow{AQ} = \frac{1}{2}$ **(b – a).**

By going via *A*, $\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ} = \frac{1}{2}$ **a** + $\frac{1}{2}$ (**b – a**) = $\frac{1}{2}$ **b.**

To find $\overrightarrow{SR}$, we must find $\overrightarrow{CB}$ first; it is (via *O*) **b** – **c**. Now $\overrightarrow{CR} = \frac{1}{2}$ (**b** – **c**).

Hence $\overrightarrow{SR} = \overrightarrow{SC} + \overrightarrow{CR} = \frac{1}{2}$ **c** + $\frac{1}{2}$ (**b** – **c**) = $\frac{1}{2}$ **b.**

ii) The vectors $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ are equal, so *PQ* is parallel to *SR* and also equal in length.

iii) Because *PQ* is parallel to *SR*, the quadrilateral *PQRS* must be at least a trapezium. However , *PQ* and *SR* are also equal, so *PQRS* is a parallelogram (sides equal and parallel).

Note : We can prove that *PS* = *QR*, and that *PQRS* is a parallelogram, as follows :

By going via *O*, $\overrightarrow{PS} = \overrightarrow{PO} + \overrightarrow{OS}$ = **c – a.**

We can find $\overrightarrow{QR}$ by going via P and S;

$\overrightarrow{QR} = \overrightarrow{QP} + \overrightarrow{PS} + \overrightarrow{SR}$ = -$\frac{1}{2}$ **b** + **c** – **a** + $\frac{1}{2}$ **b** = **c – a.**

The vectors $\overrightarrow{PS}$ and $\overrightarrow{QR}$ are equal, so *PQ* is equal and parallel to *SR*.

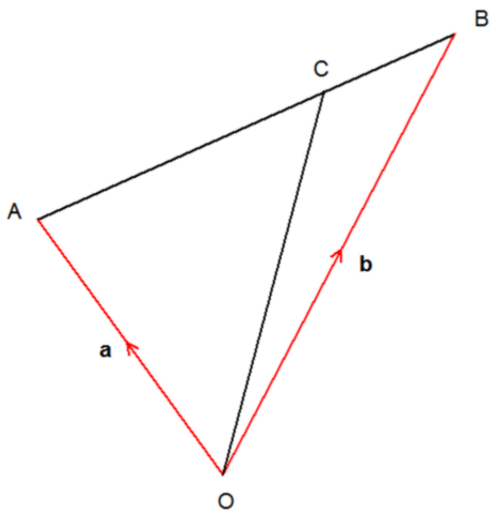
Example (4):

In the diagram, $\overrightarrow{OA}$ = **a**, $\overrightarrow{OB}$ = **b**.

Point *C* is on the line *AB* so that

$\overrightarrow{AC} = k\overrightarrow{AB}$. where $0 < k < 1$,

and $\overrightarrow{OC}$ = s**a** + t**b** where *s* and *t* are scalar multipliers.



i) Find *s* and *t* in terms of *k*.

ii) We are then told that point *C* is three-fifths along *AB*.
Using the result from i), find $\overrightarrow{OC}$.

i) Firstly, $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ = **b – a.**

Hence $\overrightarrow{AC} = k\overrightarrow{AB}$ = k(**b – a**).

Thus $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$, or

a + k(**b – a**) = **a** + k**b** – k**a**

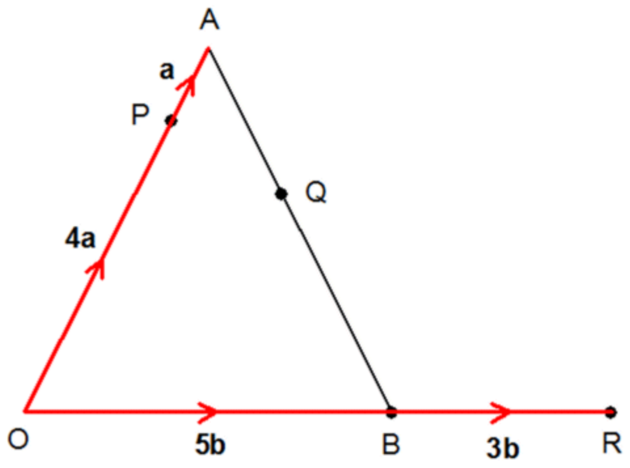
= $(1 – k)$**a** + k**b.**

Hence $s = (1-k)$ and $t = k$.

ii) Given that *C* is three-fifths of the distance along *AB*, $k = \frac{3}{5}$.

Hence in this case $\overrightarrow{OC} = \frac{2}{5}$ **a** + $\frac{3}{5}$ **b.**

Example (5): In the diagram, $\overrightarrow{OP} = 4\mathbf{a}$, $\overrightarrow{PA} = \mathbf{a}$, $\overrightarrow{OB} = 5\mathbf{b}$, $\overrightarrow{BR} = 3\mathbf{b}$ **and** $\overrightarrow{AQ} = \frac{2}{5}\overrightarrow{AB}$.



i) Find $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ in terms of **a** and **b**.

ii) Show clearly that points *P*, *Q*, and *R* lie on a straight line.

i) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = (-\mathbf{a}) - 4\mathbf{a} + 5\mathbf{b}$

$= 5(\mathbf{b} - \mathbf{a})$.

$\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ} = \mathbf{a} + \frac{2}{5}\overrightarrow{AB}$

$= \mathbf{a} + 2(\mathbf{b} - \mathbf{a}) = 2\mathbf{b} - \mathbf{a}$.

ii) We want to show that $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are scalar multiples of each other.

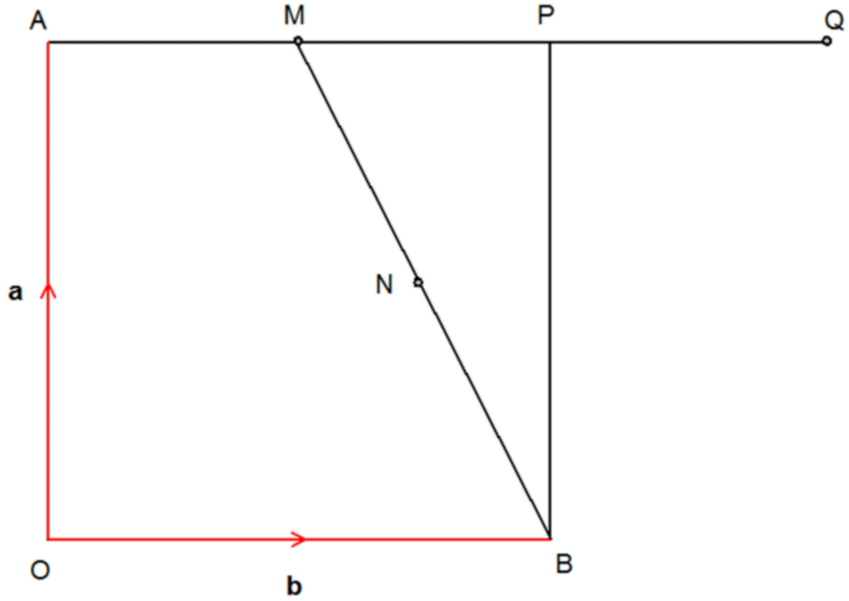
Now $\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} = -4\mathbf{a} + 8\mathbf{b} = 8\mathbf{b} - 4\mathbf{a} = 4(2\mathbf{b} - \mathbf{a}) = 4\overrightarrow{PQ}$.

Because $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are scalar multiples of each other and contain the point P in common, the points *P*, *Q*, and *R* lie on a straight line.

Point *Q* is one quarter of the way between *P* and *R*.

Example (6): The diagram shows a square *OAPB*.
M and *N* are the midpoints of *AP* and *BM* respectively.
The side *AP* is extended to point *Q* where *AQ* = 1½ *AP*.

$\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.



Write the following vectors in terms of **a** and **b** , giving your answers in the simplest form.

i) $\overrightarrow{OQ}$ ii) $\overrightarrow{BM}$ iii) $\overrightarrow{BN}$ iv) $\overrightarrow{ON}$

v) What can be deduced about points *O*, *N* and *Q* ? Justify your answer.

Start with the obvious: $\overrightarrow{AP} = \overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{BP} = \overrightarrow{OA} = \mathbf{a}$
since the opposite sides of a square are equal in length and parallel.

i) $\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ} = \mathbf{a} + \frac{3}{2}\mathbf{b}$.

ii) $\overrightarrow{BM} = \overrightarrow{BP} + \overrightarrow{PM} = \mathbf{a} - \frac{1}{2}\mathbf{b}$.

iii) $\overrightarrow{BN} = \frac{1}{2}\overrightarrow{BM} = \frac{1}{2}\mathbf{a} - \frac{1}{4}\mathbf{b}$.

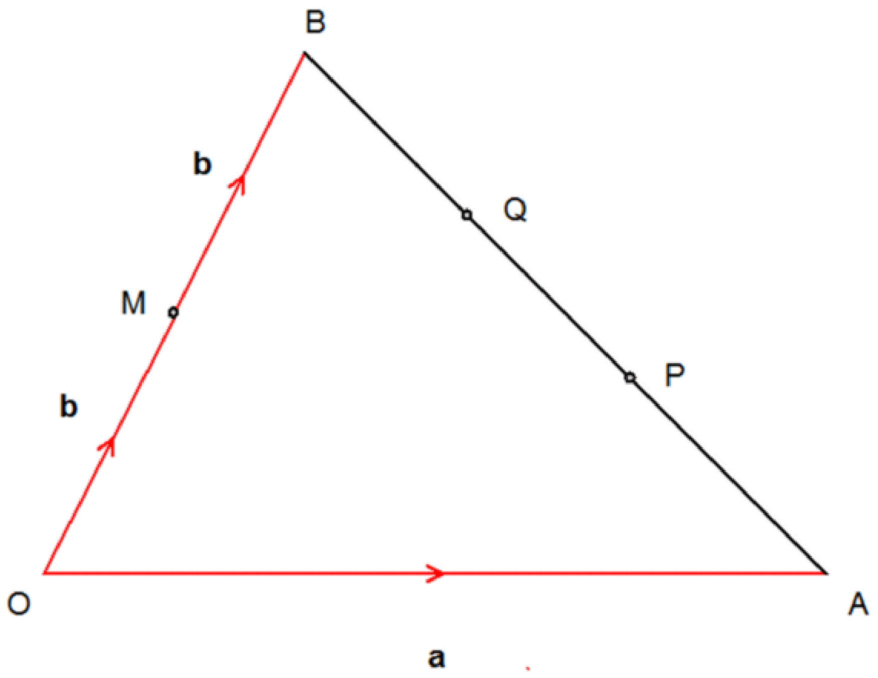
iv) $\overrightarrow{ON} = \overrightarrow{OB} + \overrightarrow{BN} = \mathbf{b} + \frac{1}{2}\mathbf{a} - \frac{1}{4}\mathbf{b} = \frac{1}{2}\mathbf{a} + \frac{3}{4}\mathbf{b}$.

v) The results from i) and iv) show that $\overrightarrow{ON} = \frac{1}{2}\overrightarrow{OQ}$.

Because $\overrightarrow{ON}$ and $\overrightarrow{OQ}$ are scalar multiples of each other and contain the point O in common, the points *O*, *N* and *Q* are therefore collinear.

In addition, *N* is the midpoint of *OQ*.

Example (7):



OAB is a triangle where *M* is the midpoint of *OB*.
P and *Q* are points on *AB* such that *AP* = *PQ* = *QB*.

$\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = 2\mathbf{b}$.

Find expressions for the following in terms of **a** and **b**:

i) $\overrightarrow{BA}$; ii) $\overrightarrow{MQ}$; iii) $\overrightarrow{OP}$

iv) What can you deduce about the quadrilateral *OMQP* ? Justify your answer.

i) $\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = -2\mathbf{b} + \mathbf{a}$

$= \mathbf{a} - 2\mathbf{b}$.

ii) $\overrightarrow{MQ} = \overrightarrow{MB} + \overrightarrow{BQ} = \mathbf{b} + \frac{1}{3}\overrightarrow{BA} = \mathbf{b} + \frac{1}{3}(\mathbf{a} - 2\mathbf{b}) = \mathbf{b} + \frac{1}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}$

$= \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$.

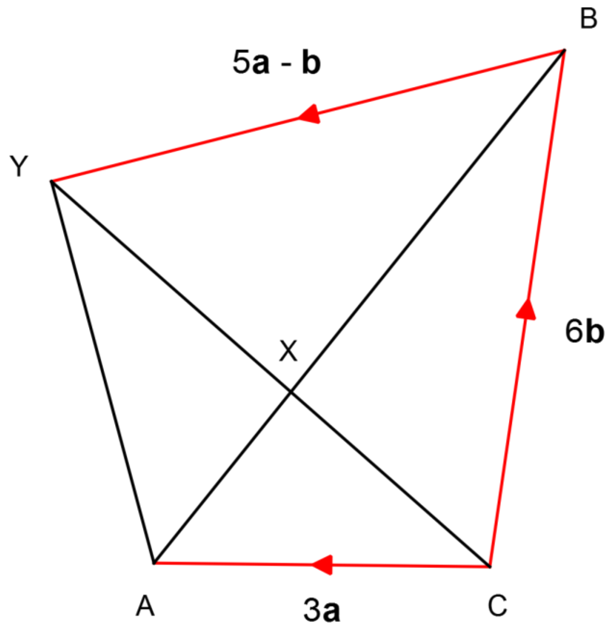
iii) $\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP} = 2\mathbf{b} + \frac{2}{3}\overrightarrow{BA} = 2\mathbf{b} + \frac{2}{3}\mathbf{a} - \frac{4}{3}\mathbf{b}$

$= \frac{2}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$.

iv) The quadrilateral *OMQP* is a trapezium because $\overrightarrow{OP} = 2\overrightarrow{MQ}$.

If one vector is a scalar multiple of another, then the two vectors are parallel.

Example (8):



ACBY is a quadrilateral, with the diagonals *AB* and *CY* intersecting at point *X*.
The point *X* divides the line *AB* in the ratio 1 : 2.

$\overrightarrow{CA} = 3\mathbf{a}$, $\overrightarrow{CB} = 6\mathbf{b}$ and $\overrightarrow{BY} = 5\mathbf{a} - \mathbf{b}$.

Prove that *X* divides the line *CY* in the ratio 2 : 3 .

$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB} = -3\mathbf{a} + 6\mathbf{b}$, or $6\mathbf{b} - 3\mathbf{a}$.

Point *X* divides *AB* in the ratio 1 : 2, so it lies one-third of the way along *AB*.

Hence $\overrightarrow{AX} = \frac{1}{3}\overrightarrow{AB} = 2\mathbf{b} - \mathbf{a}$.

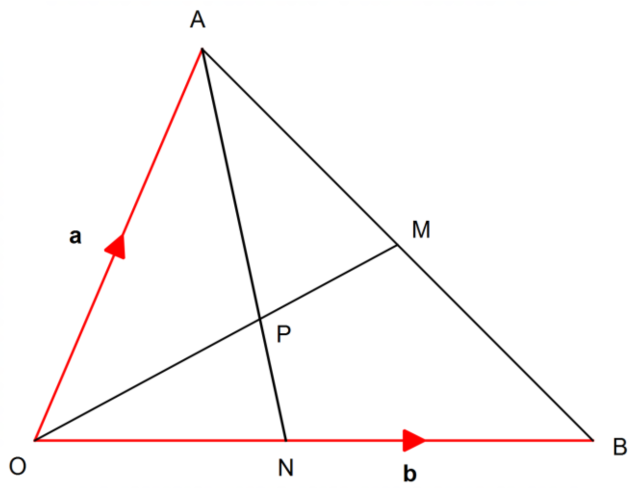
We then find vectors $\overrightarrow{CX}$ and $\overrightarrow{CY}$:

$\overrightarrow{CX} = \overrightarrow{CA} + \overrightarrow{AX} = 3\mathbf{a} + 2\mathbf{b} - \mathbf{a} = 2\mathbf{a} + 2\mathbf{b}$.

$\overrightarrow{CY} = \overrightarrow{CB} + \overrightarrow{BY} = 6\mathbf{b} + 5\mathbf{a} - \mathbf{b} = 5\mathbf{a} + 5\mathbf{b}$.

The length of *CX* is evidently two-fifths that of *CY*, so point *X* does indeed divide the diagonal *CY* in the ratio 2 : 3.

Example (8a):



OAB is a triangle where *M* is the midpoint of *AB*.
$\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$.

The point *P* divides the line *OM* in the ratio 3 : 2.
The line *AN* also passes through point *P*.
Find the ratio *ON* : *NB* in its simplest form.

The point *N* lies on *OB*, so we can say that
$\overrightarrow{ON} = s\mathbf{b}$ where s is a constant and $0 < s < 1$.

We can also say that $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$.

The trickiest part is to find a vector equation for $\overrightarrow{AN}$, but we can begin with
$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$.

Since *M* is the midpoint of *AB*, we have $\overrightarrow{AM} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$ and
$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \mathbf{a} + \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$.

Next, we find $\overrightarrow{OP}$ and $\overrightarrow{AP}$.

Since *P* divides *OM* in the ratio 3 : 2, $\overrightarrow{OP} = \frac{3}{5}\overrightarrow{OM} = \frac{3}{10}\mathbf{a} + \frac{3}{10}\mathbf{b}$.
Next, $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = -\mathbf{a} + \frac{3}{10}\mathbf{a} + \frac{3}{10}\mathbf{b} = \frac{3}{10}\mathbf{b} - \frac{7}{10}\mathbf{a}$.

Since *A*, *P* and *N* lie on a straight line, we can also say that $\overrightarrow{AN}$ is a scalar multiple of $\overrightarrow{AP}$,
or $\overrightarrow{AN} = k(\frac{3}{10}\mathbf{b} - \frac{7}{10}\mathbf{a})$ where k is another constant.

We can get rid of the fractions by bringing out a factor of $\frac{1}{10}$ outside the brackets to obtain
$\overrightarrow{AN} = \frac{1}{10}k(3\mathbf{b} - 7\mathbf{a})$, and use a new constant t to replace $\frac{1}{10}k$. Hence $\overrightarrow{AN} = t(3\mathbf{b} - 7\mathbf{a})$,

Also, since $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$, we now have $\overrightarrow{ON} = \mathbf{a} + t(3\mathbf{b} - 7\mathbf{a}) = (1-7t)\,\mathbf{a} + (3t)\,\mathbf{b}$.

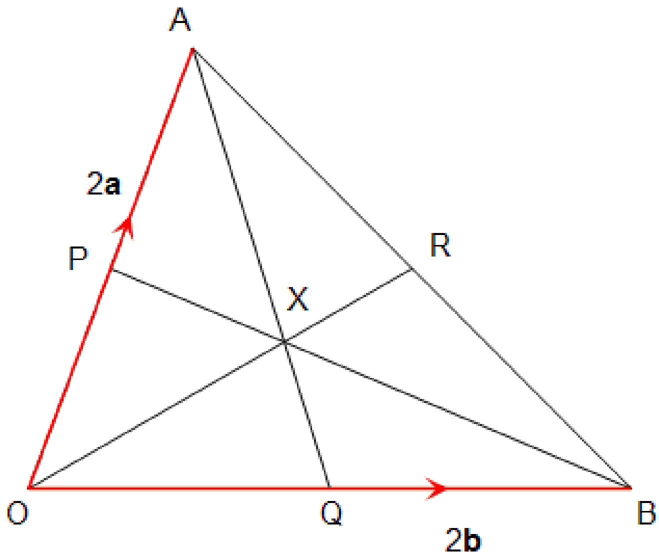
We now have two vector equations for $\overrightarrow{ON}$:
$\overrightarrow{ON} = s\mathbf{b}$ and $\overrightarrow{ON} = (1-7t)\,\mathbf{a} + (3t)\,\mathbf{b}$.

The final stage is to equate the **a-** and **b-** components - in other words, to compare them.

As $\overrightarrow{ON}$ has an **a**-component of zero, $1-7t = 0$, and so $t = \frac{1}{7}$.
Equating the **b**-components, we have $s = 3t$, hence $s = \frac{3}{7}$.

Hence $\overrightarrow{ON} = \frac{3}{7}\overrightarrow{OB}$ and $\overrightarrow{NB} = \frac{4}{7}\overrightarrow{OB}$, with *N* dividing *OB* in the ratio **3 : 4.**

Example (9): *OAB* is a triangle where *P*, *Q* and *R* are the midpoints of *OA*, *OB* and *AB* respectively, and *X* is the point at which all three intersect.



$\overrightarrow{OA} = 2\mathbf{a}$, $\overrightarrow{OB} = 2\mathbf{b}$.

As a matter of interest, the lines *AQ*, *BP* and *OR* are known as the **medians** of the triangle, and point *X* is the **centroid** and also the triangle's centre of gravity..

Find expressions for the following in terms of **a** and **b**:

i) $\overrightarrow{AB}$; ii) $\overrightarrow{OR}$; iii) $\overrightarrow{AQ}$; iv) $\overrightarrow{BP}$

The ratios *OX* : *OR* , *AX* : *AQ* and *BX* : *BP* are all equal to *k* : 1 where *k* is a fractional constant.

v) Express $\overrightarrow{OX}$ in terms of a) $\overrightarrow{OR}$; b) **a** and $\overrightarrow{AQ}$; b) **b** and $\overrightarrow{BP}$.

vi) Hence solve the vector equations in v) and thus find the value of *k*.

i) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -2\mathbf{a} + 2\mathbf{b} = 2\mathbf{b} - 2\mathbf{a}.$

ii) $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR} = 2\mathbf{a} + \frac{1}{2}\overrightarrow{AB} = 2\mathbf{a} + \mathbf{b} - \mathbf{a} = \mathbf{a} + \mathbf{b}.$

iii) $\overrightarrow{AQ} = \overrightarrow{AO} + \overrightarrow{OQ} = -2\mathbf{a} + \mathbf{b} = \mathbf{b} - 2\mathbf{a}.$

iv) $\overrightarrow{BP} = \overrightarrow{BO} + \overrightarrow{OP} = -2\mathbf{b} + \mathbf{a} = \mathbf{a} - 2\mathbf{b}.$

v) a) $\overrightarrow{OX} = k\overrightarrow{OR} = k\mathbf{a} + k\mathbf{b}.$

v) b) $\overrightarrow{OX} = 2\mathbf{a} + k\overrightarrow{AQ} = 2\mathbf{a} + k(\mathbf{b} - 2\mathbf{a}) = 2\mathbf{a} + k\mathbf{b} - 2k\mathbf{a} = (2-2k)\mathbf{a} + k\mathbf{b}.$

v) c) $\overrightarrow{OX} = 2\mathbf{b} + k\overrightarrow{BP} = 2\mathbf{b} + k(\mathbf{a} - 2\mathbf{b}) = 2\mathbf{b} + k\mathbf{a} - 2k\mathbf{b} = (2-2k)\mathbf{b} + k\mathbf{a}.$

vi) Equating the results in v), we have $k = 2 - 2k \rightarrow 3k = 2$
and thus $k = \frac{2}{3}$.

Hence the centroid *X* is two-thirds of the way along all three medians *AQ*, *OR* and *BP*.

Coordinate geometry with vectors.

This section complements the other examples on coordinate geometry, but with a greater emphasis on vectors, even though there will be a considerable amount of Pythagoras involved.

(The following examples use both **i** - **j** and column notations interchangeably for practice).

There are methods for finding angles between vectors, and testing for right angles, but they now come under the remit of Further Maths.

Vectors in Plane Figures.

Recall the following facts concerning side and diagonal lengths of triangles and quadrilaterals:
(Angle properties not stated)

Two vectors **a** and **b** are parallel if one is a scalar multiple of the other.

An equilateral triangle has all three sides equal.
An isosceles triangle has two sides of equal length, with the third one different.

A trapezium has one pair of parallel sides.
A kite has two pairs of adjacent sides of equal length.
A parallelogram has two pairs of parallel opposite sides, of equal length.
A rhombus has all of the properties of a parallelogram, plus having all sides equal in length.
A rectangle has all of the properties of a parallelogram, plus having its diagonals equal in length.
A square has all of the properties of a rectangle and rhombus combined.

Example (10): *ABC* is a triangle such that the position vector of A is $\begin{pmatrix}0\\1\end{pmatrix}$ and $\overrightarrow{AB} = \begin{pmatrix}7\\6\end{pmatrix}$.

Given that $\overrightarrow{BC} = \begin{pmatrix}-2\\-9\end{pmatrix}$,

i) find the coordinates of *B* and *C*,
ii) show that triangle *ABC* is isosceles.
iii) show that cos *ABC* = $\frac{4}{5}$.

iv) show that the area of the triangle is exactly 25.5 square units..

i) To find the coordinates of *B*, we add the position vector of *A* , i.e. $\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}7\\6\end{pmatrix}=\begin{pmatrix}7\\7\end{pmatrix}$

Hence *A* = (0, 1) and *B* = (7, 7).
The coordinates of *C* are found in the same way: $\begin{pmatrix}7\\7\end{pmatrix}+\begin{pmatrix}-2\\-9\end{pmatrix}=\begin{pmatrix}5\\-2\end{pmatrix}$, so *C*= (5, -2)

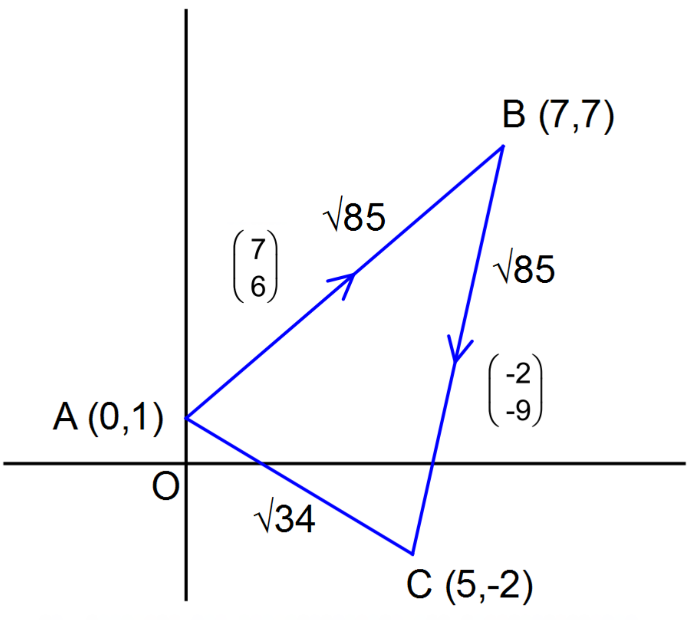
ii) The length of *AB*, i.e. $|\overrightarrow{AB}|$, = $\sqrt{7^2+6^2} = \sqrt{85}$.

In the same way, $|\overrightarrow{BC}|$, = $\sqrt{(-2)^2+(-9)^2} = \sqrt{85}$.

Hence the lengths *AB* and *BC* are equal.

The distance between *A* (0, 1) and *C* (5, -2) can be found similarly by Pythagoras:

$|\overrightarrow{AC}|$, = $\sqrt{(5-0)^2+(-2-1)^2} = \sqrt{34}$.



The two sides *AB* and *BC* are equal in length, but the length of *AC* is different. Hence triangle *ABC* is isosceles.

iii) By the cosine rule,

$$\cos ABC = \frac{(AB)^2+(BC)^2-(AC)^2}{2(AB)(BC)}.$$

$$= \frac{85+85-34}{2\sqrt{85}\sqrt{85}} = \frac{136}{170} = \frac{4}{5}.$$

iv) If cos *ABC* = $\frac{4}{5}$, then

sin *ABC* = $\sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{3}{5}$.

(using $\cos^2 ABC + \sin^2 ABC = 1$).

The area of the triangle *ABC* is therefore ½ (*AB*)(*BC*) sin *ABC* =

$$\frac{1}{2}\left(\sqrt{85}\right)\left(\sqrt{85}\right)\times\frac{3}{5} = 85\times\frac{3}{10} = 25.5$$ square units.

Example (11): The vertices of a quadrilateral *OABC* have the following position vectors:

O = **0**; $A = 2\mathbf{i} + 9\mathbf{j}$; $B = 8\mathbf{i} + 2\mathbf{j}$; $C = 6\mathbf{i} - 7\mathbf{j}$.

i) Show that *OABC* is a rhombus. ii) Show that *OABC* is not a square.

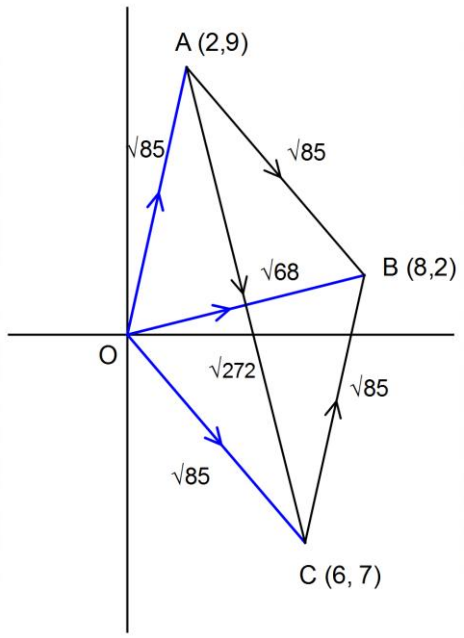
i) Because the position vector of *O* is the zero vector, we can immediately say that
$\overrightarrow{OA} = 2\mathbf{i} + 9\mathbf{j}$ and $\overrightarrow{OC} = 6\mathbf{i} - 7\mathbf{j}$. These vectors represent the adjacent sides of the quadrilateral.

The length of *OA*, i.e. $|\overrightarrow{OA}|$, = $\sqrt{2^2 + 9^2} = \sqrt{85}$; that of *OC*, or. $|\overrightarrow{OC}|$, = $\sqrt{6^2 + (-7)^2} = \sqrt{85}$.

Hence the adjacent sides *OA* and *OC* are equal.

To prove that *OABC* is a rhombus, we could either calculate the lengths of *OB* and *BC* and show that all four sides are equal, or we could show that *OA* and *CB* are equal and parallel, as are *OC* and *AB*.

Since $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, its vector is $(8\mathbf{i} + 2\mathbf{j}) - (2\mathbf{i} + 9\mathbf{j}) = 6\mathbf{i} - 7\mathbf{j}$. Hence $\overrightarrow{AB} = \overrightarrow{OC}$, so its length is also $\sqrt{6^2 + (-7)^2} = \sqrt{85}$.



Also, $\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC}$, so its vector is

$(8\mathbf{i} + 2\mathbf{j}) - (6\mathbf{i} - 7\mathbf{j}) = 2\mathbf{i} + 9\mathbf{j}$.
Thus $\overrightarrow{CB} = \overrightarrow{OA}$, so its length is $\sqrt{2^2 + 9^2} = \sqrt{85}$.

All the sides of *OABC* are equal, and both pairs of opposite sides are parallel, so *OABC* is a rhombus**.**

ii) The diagonals of a square are equal in length, but those of a rhombus are not.

The length of the diagonal *OB* = $\sqrt{8^2 + 2^2} = \sqrt{68}$.

To find the length of the other diagonal *AC*, we reckon

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$

and its vector is $(6\mathbf{i} - 7\mathbf{j}) - (2\mathbf{i} + 9\mathbf{j}) = 4\mathbf{i} - 16\mathbf{j}$.
The length is $\sqrt{4^2 + (-16)^2} = \sqrt{272}$.
The diagonals *OB* and *DC* are unequal, so *OABC* is not a square.

Example (12): The vertices of a quadrilateral *ABCD* have the following position vectors:

$A = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$; $B = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$; $C = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$; $D = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$.

i) Show that *ABCD* is a rectangle, but not a square.
ii) Find the area of the rectangle.

i) A sketch would show that *AB* and *AD* are two adjacent sides, and that *AC* is a diagonal.

Since point *A* has a position vector of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, we can also say that $\overrightarrow{OA} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, with the same applying to the other three points.

So $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$; $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 8 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$;

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} 9 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$.

We then work out *BC* and *DC* :

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$, so sides *BC* and *AD* are equal and parallel.

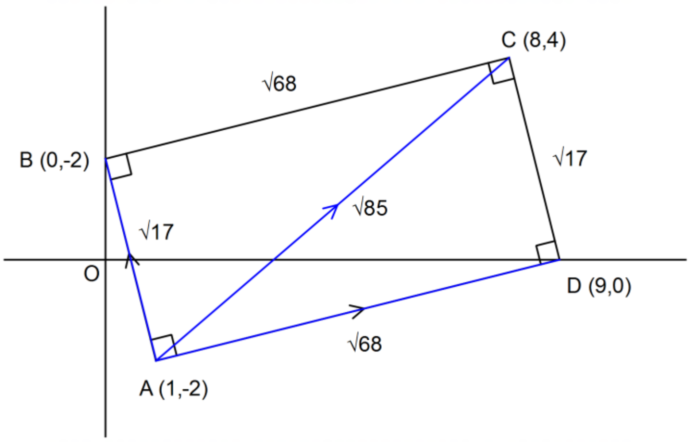
$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} - \begin{pmatrix} 9 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, so sides *DC* and *AB* are equal and parallel.

ABCD is therefore at least a parallelogram, so we apply Pythagoras in reverse to show that the triangle *ABD* is right-angled, i.e.

$|\overrightarrow{AC}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2$.

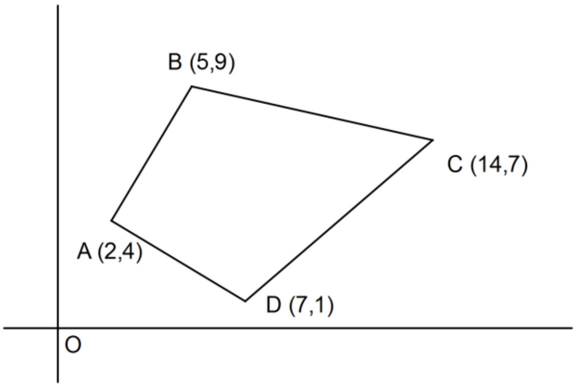
Now $|\overrightarrow{AC}|^2 = 7^2 + 6^2 = 85$,
$|\overrightarrow{AB}|^2 = (-1)^2 + 4^2 = 17$,
and $|\overrightarrow{BC}|^2 = 8^2 + 2^2 = 68$.
Hence $|\overrightarrow{AC}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2$,
angle *ABC* = 90°, and *ABCD* is a rectangle.

Because the squares of the lengths of *AB* and *BC* are different, *ABCD* cannot be a square.



ii) The area of the rectangle is √17 × √68, or 34, square units.

Example (13): The diagram below shows a quadrilateral *ABCD*.



i) Show, using Pythagoras, that angle *BAD* is a right angle.

ii) Show. using Pythagoras, that *ABCD* is a kite.

iii) Find the area of the kite.

iv) The point *C* is moved to *C'* such that *ABC'D* is a square.

a) Find the position vector of *C'*.
b) Find the area of the resulting square.

v) The point *A* is moved to *A'* such that *A'BCD* is a rhombus.

a)Find the position vector of *A'*. (*C* is the point (14, 7) again).
b) Find the area of the rhombus.

i) Using Pythagoras,

$(AB)^2 = (5\text{-}2)^2 + (9\text{-}4)^2 = 3^2 + 5^2 = 34$
$(AD)^2 = (7\text{-}2)^2 + (1\text{-}4)^2 = 5^2 + (-3)^2 = 34$
$(BD)^2 = (5\text{-}7)^2 + (9\text{-}1)^2 = (-2)^2 + 8^2 = 68$

Since $(BD)^2 = (AB)^2 + (AD)^2$, the triangle *BAD* is right-angled.

ii) A kite has two adjacent pairs of sides equal, and from the last part, $AB = AD = \sqrt{34}$.
We work out the lengths of *BC* and *DC* in the same way:

$(BC)^2 = (14\text{-}5)^2 + (7\text{-}9)^2 = 9^2 + (-2)^2 = 85$
$(DC)^2 = (14\text{-}7)^2 + (7\text{-}1)^2 = 7^2 + 6^2 = 85$

$BC = DC = \sqrt{85}$, so both pairs of adjacent sides of *ABCD* are equal, therefore *ABCD* is a kite.

The area of a kite (like that of a rhombus) is half the product of the diagonals, so we need to find the length of *AC*

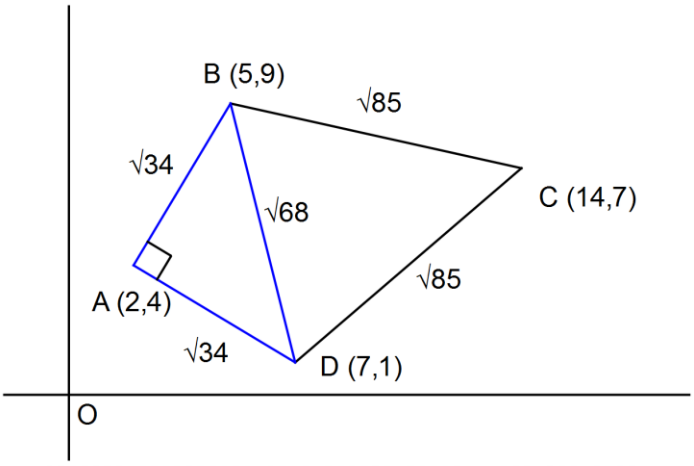
By Pythagoras, $(AC)^2 =$

$(14\text{-}2)^2 + (7\text{-}4)^2 = 12^2 + 3^2 = 153$, so $AC = \sqrt{153}$ or $3\sqrt{17}$.

As $(BD)^2 = 68$, $BD = \sqrt{68}$ or $2\sqrt{17}$.

The area of the kite is therefore

$\frac{1}{2}\left(3\sqrt{17}\right)\left(2\sqrt{17}\right) = 51$ square units.



iv) a) We have established that angle *BAD* is a right angle, and that lengths *AB* and *AD* are equal. Therefore for *ABC'D* to be a square, the vectors *BC'* and *AD* must be equal and parallel, as must *AB* and *DC'*.

Let the position vector of *C*' be $\begin{pmatrix} x \\ y \end{pmatrix}$.

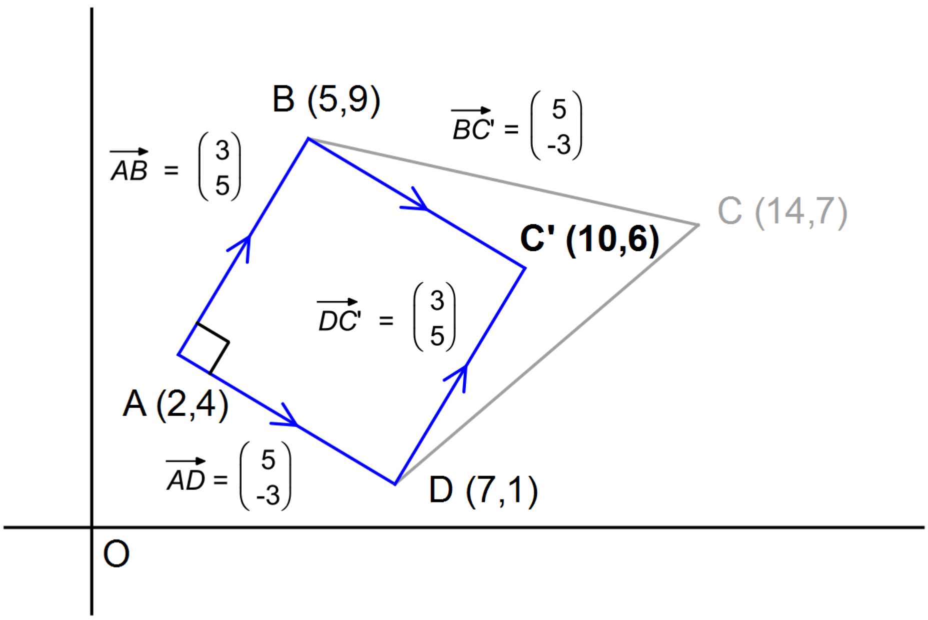
Now $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} 7 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$, and $\overrightarrow{BC'} = \overrightarrow{OC'} - \overrightarrow{OB} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 5 \\ 9 \end{pmatrix} = \begin{pmatrix} x-5 \\ y-9 \end{pmatrix}$.

Thus $\begin{pmatrix} x-5 \\ y-9 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$, hence the position vector of *C*' , $\begin{pmatrix} x \\ y \end{pmatrix}$, $= \begin{pmatrix} 10 \\ 6 \end{pmatrix}$ for *ABC'D* to be a square.

Using *AB* and *DC'* would lead to the same result, since if three sides of a quadrilateral are equal and two of them form a parallel pair, then the fourth side is equal to the other three, as well as parallel to the" unmatched "side.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 5 \\ 9 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, and $\overrightarrow{DC'} = \overrightarrow{OC'} - \overrightarrow{OD} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 7 \\ 1 \end{pmatrix} = \begin{pmatrix} x-7 \\ y-1 \end{pmatrix}$.

Thus $\begin{pmatrix} x-7 \\ y-1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, hence the position vector of *C*' , $\begin{pmatrix} x \\ y \end{pmatrix}$, $= \begin{pmatrix} 10 \\ 6 \end{pmatrix}$ for *ABC'D* to be a square.



b) We have worked out in part i) that $(AB)^2 = 34$, so the area of the square is 34 square units.

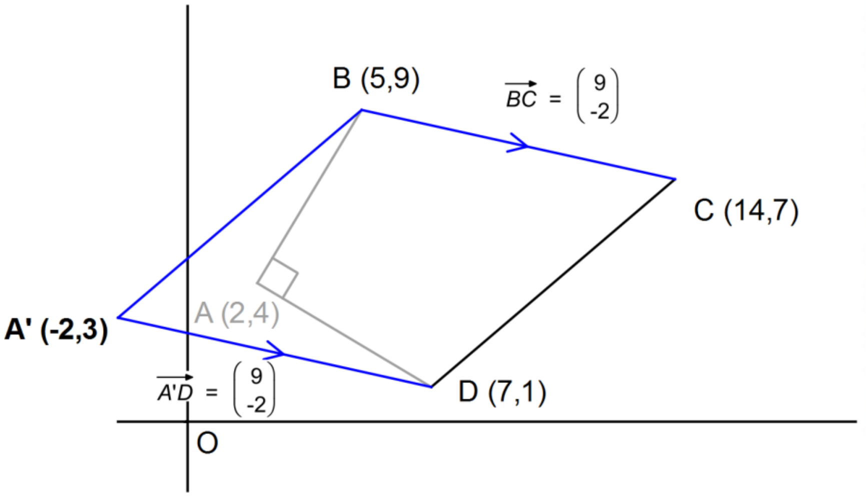
v) Unlike angle *BAD* , *BCD* is not a right angle, but the process of finding the direction vector of *A'* is identical.

The lengths of *AB* and *AD* are equal, so for *A'BCD* to be a rhombus, the vectors *BC* and *A'D* must be equal and parallel, as must *A'B* and *DC*.

Again, let the position vector of *A*' be $\begin{pmatrix} x \\ y \end{pmatrix}$, and choose to work with vectors $\overrightarrow{A'D}$ and $\overrightarrow{BC}$.

Now $\overrightarrow{A'D} = \overrightarrow{OD} - \overrightarrow{OA'} = \begin{pmatrix} 7 \\ 1 \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7-x \\ 1-y \end{pmatrix}$, and $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 14 \\ 7 \end{pmatrix} - \begin{pmatrix} 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \end{pmatrix}$.

Thus $\begin{pmatrix} 7-x \\ 1-y \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \end{pmatrix}$, hence the position vector of *A* , $\begin{pmatrix} x \\ y \end{pmatrix}$, $= \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ for *A'BCD* to be a rhombus.



The area of a rhombus is half the product of the diagonals, so we need to find the length of *A'C*

By Pythagoras, $(A'C)^2 = (14-(2))^2 + (7-3)^2 = 16^2 + 4^2 = 272$, so $A'C = \sqrt{272}$ or $4\sqrt{17}$.

As $(BD)^2 = 68$, $BD = \sqrt{68}$ or $2\sqrt{17}$.

Hence the area of the rhombus is $\frac{1}{2}\left(4\sqrt{17}\right)\left(2\sqrt{17}\right) = 68$ square units.