

M.K. HOME TUITION

Mathematics Revision Guides

Level: A-Level Year 1 / AS

VECTOR GEOMETRY

$\vec{AX} = p$ $\vec{XB} = q$ $\vec{XB'} = -q$
 $\vec{AB} = \vec{AX} + \vec{XB} = p + q$
 $\vec{AB'} = \vec{AX} + \vec{XB'} = p - q$

$\vec{AX} + \vec{XB} = \vec{AY} + \vec{YB}$
 $p + q = q + p$

$\vec{OP} = 4a$, $\vec{PA} = a$, $\vec{OB} = 5b$, $\vec{BR} = 3b$ and $\vec{AQ} = \frac{2}{3}\vec{AB}$.

$\vec{AB} = \vec{AO} + \vec{OB} = (-a) - 4a + 5b = 5(b - a)$
 $\vec{PQ} = \vec{PA} + \vec{AQ} = a + \frac{2}{3}\vec{AB} = a + 2(b - a) = 2b - a$
 $\vec{PR} = \vec{PO} + \vec{OR} = -4a + 8b = 8b - 4a = 4(2b - a) = 4\vec{PQ}$

points P, Q, and R lie on a straight line

VECTOR GEOMETRY

Many problems and theorems in geometry can be analysed and proven using vectors.

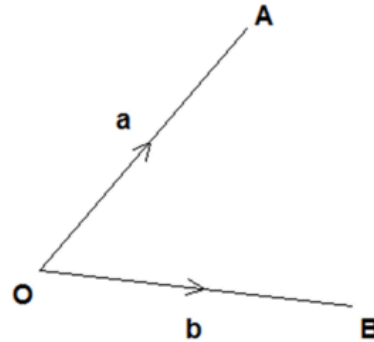
Vectors can be denoted by a single boldface letter, but another notation is to state their end points and write an arrow above them.

In the right-hand diagram, vector **a** joins points *O* and *A* and vector **b** joins point *O* and *B*.

Therefore $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

The direction of the arrow is important here; the vector \overrightarrow{AO} goes in the opposite direction to \overrightarrow{OA} although it has the same magnitude.

Hence $\overrightarrow{AO} = -\overrightarrow{OA} = -\mathbf{a}$.

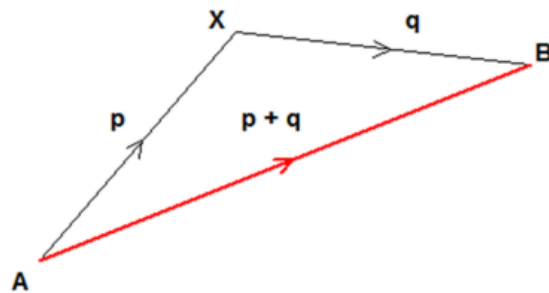


The Triangle Law.

(Recall) To add two vectors, apply the first, and then the second.

Thus $\overrightarrow{AB} = \overrightarrow{AX} + \overrightarrow{XB}$.

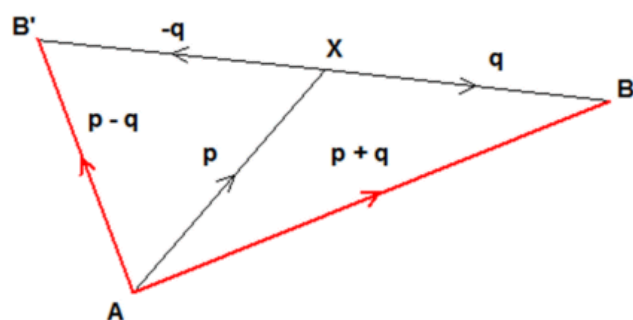
Here $\overrightarrow{AX} = \mathbf{p}$ and $\overrightarrow{XB} = \mathbf{q}$.



(Recall) Subtracting a vector is the same as adding its inverse, i.e. the parallel vector of the same magnitude but in the opposite direction.

Here, $\overrightarrow{XB'} = -\mathbf{q}$.

Thus $\overrightarrow{AB'} = \overrightarrow{AX} + \overrightarrow{XB'} = \mathbf{p} - \mathbf{q}$.

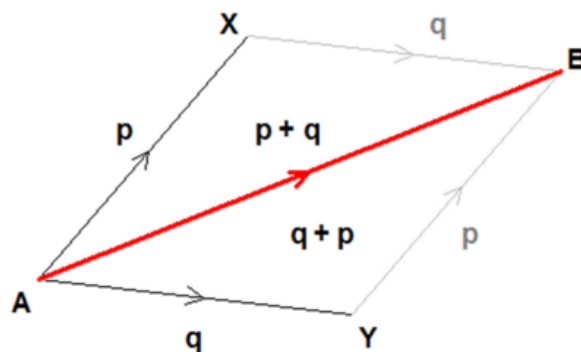


The Parallelogram Law.

Going from A to B via Y gives the same result as going from A to B via X .

Therefore $\vec{AX} + \vec{XB} = \vec{AY} + \vec{YB}$.

In other words, $\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p}$.



Notice as well that $\vec{AX} = \vec{YB} = \mathbf{p}$ and $\vec{AY} = \vec{XB} = \mathbf{q}$.

Since the opposite pairs of sides of any parallelogram are equal and parallel, they can always be represented by the same vector provided their directions are equal, thus $\vec{BY} = -\mathbf{p}$ here.

Geometrical Applications.

When asked to find an unknown vector between two points, just work it out as an alternative route made up of known segments, as per the parallelogram law.

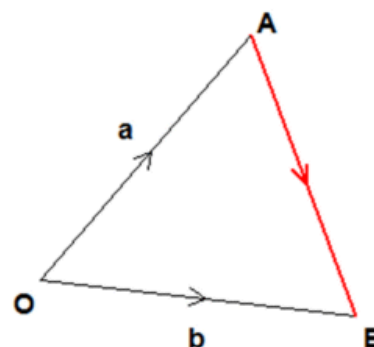
Example (1): Express the vector \vec{AB} in terms of \mathbf{a} and \mathbf{b} .

We want to go from A to B directly, but we do not have the vector for it.

We therefore go via O , as in $\vec{AB} = \vec{AO} + \vec{OB}$.

Now \vec{AO} is the same as \mathbf{a} but in the reverse direction, whilst $\vec{OB} = \mathbf{b}$.

Hence $\vec{AB} = -\mathbf{a} + \mathbf{b}$ or $\mathbf{b} - \mathbf{a}$.



Position vectors.

Up to now, we have treated vectors as directed line segments, but we can also say that any point on the plane has a **position vector** relative to the origin O .

In fact, any point (p, q) has a position vector of $p\mathbf{i} + q\mathbf{j}$ or $\begin{pmatrix} p \\ q \end{pmatrix}$ relative to the origin.

Example (1a): The coordinates of points A and B are $(2, 4)$ and $(6, 1)$ respectively. Find \vec{AB} in column notation given that O is the origin.

Since O is the origin, vector $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$.

Hence $\vec{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

Example (2): In the triangle OAB , point P is the midpoint of OA and point Q is the midpoint of OB .

Show that PQ is parallel to AB , and also half its length.

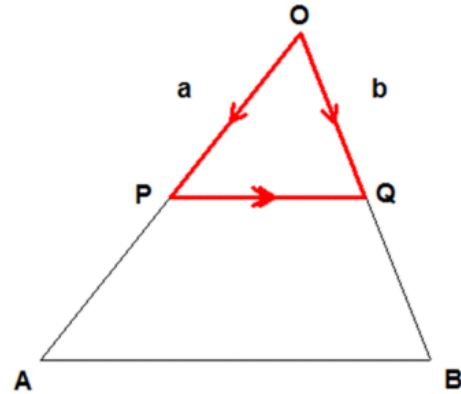
$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\mathbf{a} + \mathbf{b} \text{ or } \mathbf{b} - \mathbf{a}.$$

We are also told that

$$\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OA}, \text{ thus } \overrightarrow{OA} = 2\overrightarrow{OP} = 2\mathbf{a}.$$

$$\overrightarrow{OQ} = \frac{1}{2}\overrightarrow{OB}, \text{ thus } \overrightarrow{OB} = 2\overrightarrow{OQ} = 2\mathbf{b}.$$

$$\text{Finally, } \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -2\mathbf{a} + 2\mathbf{b} = 2(\mathbf{b} - \mathbf{a}) = 2\overrightarrow{PQ}.$$

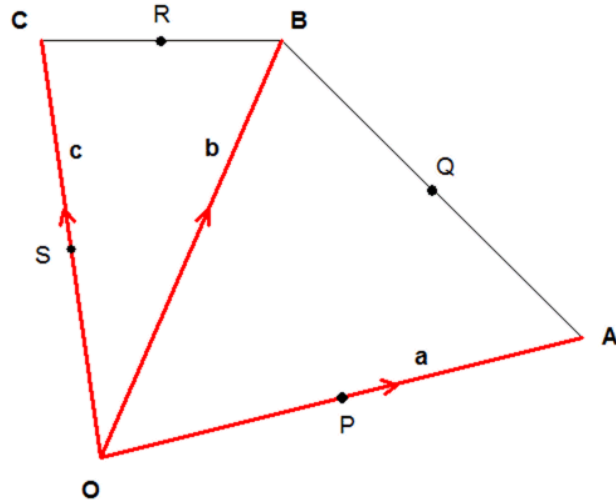


$\therefore PQ$ is parallel to AB , and half its length.

(Two vectors are parallel if either can be expressed as a scalar multiple of the other).

Example (3): $OABC$ is a quadrilateral. $\vec{OA} = \mathbf{a}$, $\vec{OB} = \mathbf{b}$ and $\vec{OC} = \mathbf{c}$.

Points P , Q , R and S are the midpoints of OA , AB , BC and OC respectively.



- i) Find the following vectors in terms of \mathbf{a} , \mathbf{b} and \mathbf{c} : \vec{OP} , \vec{AB} , \vec{AQ} , \vec{PQ} and \vec{SR} .
- ii) Prove that PQ is parallel to SR .
- iii) What type of quadrilateral is $PQRS$?

i) P is the midpoint of OA , so $\vec{OP} = \frac{1}{2} \mathbf{a}$.
 (\vec{PA} is also $\frac{1}{2} \mathbf{a}$).

By going via O , $\vec{AB} = \vec{AO} + \vec{OB} = \mathbf{b} - \mathbf{a}$.

Since Q is the midpoint of AB , $\vec{AQ} = \frac{1}{2} (\mathbf{b} - \mathbf{a})$.

By going via A , $\vec{PQ} = \vec{PA} + \vec{AQ} = \frac{1}{2} \mathbf{a} + \frac{1}{2} (\mathbf{b} - \mathbf{a}) = \frac{1}{2} \mathbf{b}$.

To find \vec{SR} , we must find \vec{CB} first; it is (via O) $\mathbf{b} - \mathbf{c}$. Now $\vec{CR} = \frac{1}{2} (\mathbf{b} - \mathbf{c})$.

Hence $\vec{SR} = \vec{SC} + \vec{CR} = \frac{1}{2} \mathbf{c} + \frac{1}{2} (\mathbf{b} - \mathbf{c}) = \frac{1}{2} \mathbf{b}$.

ii) The vectors \vec{PQ} and \vec{SR} are equal, so PQ is parallel to SR and also equal in length.

iii) Because PQ is parallel to SR , the quadrilateral $PQRS$ must be at least a trapezium. However, PQ and SR are also equal, so $PQRS$ is a parallelogram (sides equal and parallel).

Note : We can prove that $PS = QR$, and that $PQRS$ is a parallelogram, as follows :

By going via O , $\vec{PS} = \vec{PO} + \vec{OS} = \mathbf{c} - \mathbf{a}$.

We can find \vec{QR} by going via P and S ;

$$\vec{QR} = \vec{QP} + \vec{PS} + \vec{SR} = -\frac{1}{2} \mathbf{b} + \mathbf{c} - \mathbf{a} + \frac{1}{2} \mathbf{b} = \mathbf{c} - \mathbf{a}.$$

The vectors \vec{PS} and \vec{QR} are equal, so PQ is equal and parallel to SR .

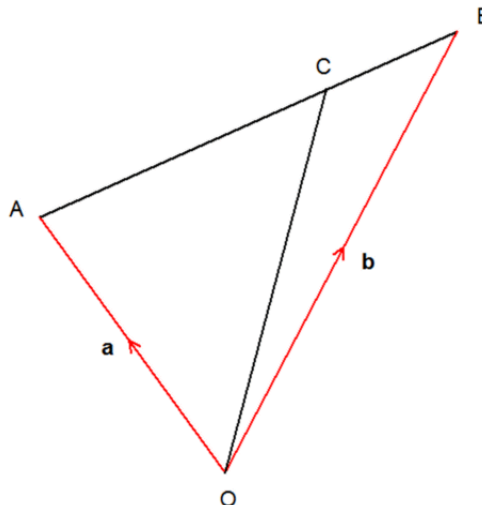
Example (4):

In the diagram, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$.

Point C is on the line AB so that

$$\overrightarrow{AC} = k \overrightarrow{AB}, \text{ where } 0 < k < 1,$$

and $\overrightarrow{OC} = s\mathbf{a} + t\mathbf{b}$ where s and t are scalar multipliers.



i) Find s and t in terms of k .

ii) We are then told that point C is three-fifths along AB .

Using the result from i), find \overrightarrow{OC} .

i) Firstly, $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \mathbf{b} - \mathbf{a}$.

$$\text{Hence } \overrightarrow{AC} = k \overrightarrow{AB} = k(\mathbf{b} - \mathbf{a}).$$

Thus $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$, or

$$\mathbf{a} + k(\mathbf{b} - \mathbf{a}) = \mathbf{a} + k\mathbf{b} - k\mathbf{a}$$

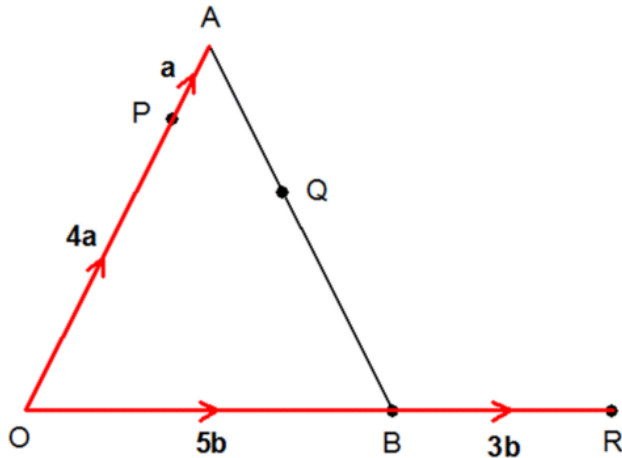
$$= (1 - k)\mathbf{a} + k\mathbf{b}.$$

Hence $s = (1-k)$ and $t = k$.

ii) Given that C is three-fifths of the distance along AB , $k = \frac{3}{5}$.

$$\text{Hence in this case } \overrightarrow{OC} = \frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}.$$

Example (5): In the diagram, $\vec{OP} = 4\mathbf{a}$, $\vec{PA} = \mathbf{a}$, $\vec{OB} = 5\mathbf{b}$, $\vec{BR} = 3\mathbf{b}$ and $\vec{AQ} = \frac{2}{5}\vec{AB}$.



- i) Find \vec{AB} and \vec{PQ} in terms of \mathbf{a} and \mathbf{b} .
- ii) Show clearly that points P , Q , and R lie on a straight line.

$$\begin{aligned} \text{i) } \vec{AB} &= \vec{AO} + \vec{OB} = (-\mathbf{a}) - 4\mathbf{a} + 5\mathbf{b} \\ &= 5(\mathbf{b} - \mathbf{a}). \end{aligned}$$

$$\begin{aligned} \vec{PQ} &= \vec{PA} + \vec{AQ} = \mathbf{a} + \frac{2}{5}\vec{AB} \\ &= \mathbf{a} + 2(\mathbf{b} - \mathbf{a}) = 2\mathbf{b} - \mathbf{a}. \end{aligned}$$

- ii) We want to show that \vec{PQ} and \vec{PR} are scalar multiples of each other.

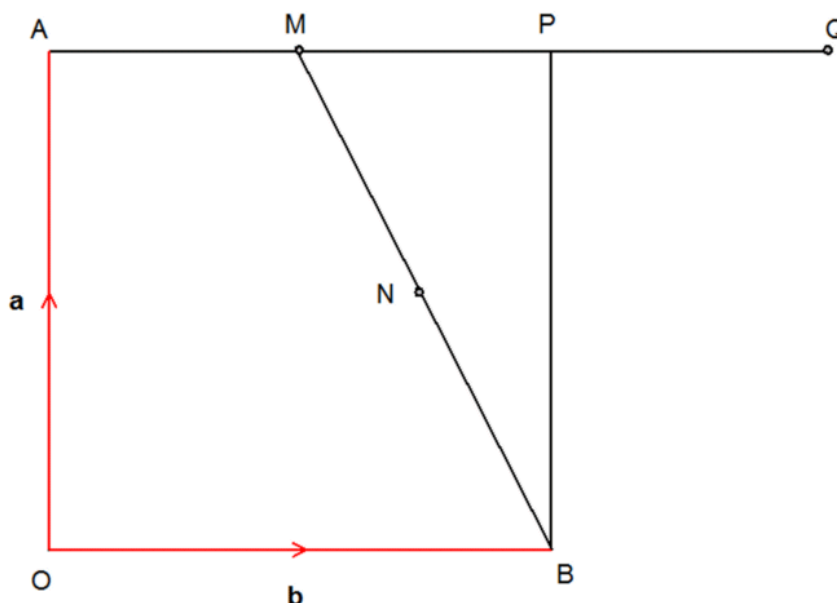
$$\text{Now } \vec{PR} = \vec{PO} + \vec{OR} = -4\mathbf{a} + 8\mathbf{b} = 8\mathbf{b} - 4\mathbf{a} = 4(2\mathbf{b} - \mathbf{a}) = 4\vec{PQ}.$$

Because \vec{PQ} and \vec{PR} are scalar multiples of each other and contain the point P in common, the points P , Q , and R lie on a straight line.

Point Q is one quarter of the way between P and R .

Example (6): The diagram shows a square $OAPB$.
 M and N are the midpoints of AP and BM respectively.
 The side AP is extended to point Q where $AQ = 1\frac{1}{2} AP$.

$\vec{OA} = \mathbf{a}$ and $\vec{OB} = \mathbf{b}$.



Write the following vectors in terms of \mathbf{a} and \mathbf{b} , giving your answers in the simplest form.

- i) \vec{OQ} ii) \vec{BM} iii) \vec{BN} iv) \vec{ON}

v) What can be deduced about points O , N and Q ? Justify your answer.

Start with the obvious: $\vec{AP} = \vec{OB} = \mathbf{b}$ and $\vec{BP} = \vec{OA} = \mathbf{a}$
 since the opposite sides of a square are equal in length and parallel.

i) $\vec{OQ} = \vec{OA} + \vec{AQ} = \mathbf{a} + \frac{3}{2}\mathbf{b}$.

ii) $\vec{BM} = \vec{BP} + \vec{PM} = \mathbf{a} - \frac{1}{2}\mathbf{b}$.

iii) $\vec{BN} = \frac{1}{2}\vec{BM} = \frac{1}{2}\mathbf{a} - \frac{1}{4}\mathbf{b}$.

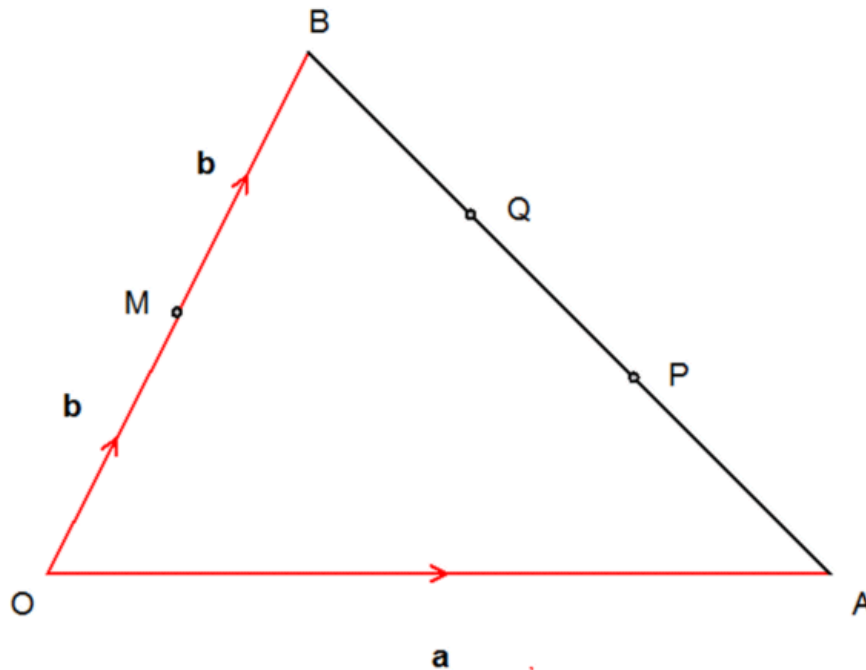
iv) $\vec{ON} = \vec{OB} + \vec{BN} = \mathbf{b} + \frac{1}{2}\mathbf{a} - \frac{1}{4}\mathbf{b} = \frac{1}{2}\mathbf{a} + \frac{3}{4}\mathbf{b}$.

v) The results from i) and iv) show that $\vec{ON} = \frac{1}{2}\vec{OQ}$.

Because \vec{ON} and \vec{OQ} are scalar multiples of each other and contain the point O in common, the points O , N and Q are therefore collinear.

In addition, N is the midpoint of OQ .

Example (7):



OAB is a triangle where M is the midpoint of OB .
 P and Q are points on AB such that $AP = PQ = QB$.

$$\vec{OA} = \mathbf{a}, \quad \vec{OB} = 2\mathbf{b}.$$

Find expressions for the following in terms of \mathbf{a} and \mathbf{b} :

i) \vec{BA} ; ii) \vec{MQ} ; iii) \vec{OP}

iv) What can you deduce about the quadrilateral $OMQP$? Justify your answer.

$$\begin{aligned} \text{i) } \vec{BA} &= \vec{BO} + \vec{OA} = -2\mathbf{b} + \mathbf{a} \\ &= \mathbf{a} - 2\mathbf{b}. \end{aligned}$$

$$\begin{aligned} \text{ii) } \vec{MQ} &= \vec{MB} + \vec{BQ} = \mathbf{b} + \frac{1}{3}\vec{BA} = \mathbf{b} + \frac{1}{3}(\mathbf{a} - 2\mathbf{b}) = \mathbf{b} + \frac{1}{3}\mathbf{a} - \frac{2}{3}\mathbf{b} \\ &= \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}. \end{aligned}$$

$$\begin{aligned} \text{iii) } \vec{OP} &= \vec{OB} + \vec{BP} = 2\mathbf{b} + \frac{2}{3}\vec{BA} = 2\mathbf{b} + \frac{2}{3}\mathbf{a} - \frac{4}{3}\mathbf{b} \\ &= \frac{2}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}. \end{aligned}$$

iv) The quadrilateral $OMQP$ is a trapezium because $\vec{OP} = 2\vec{MQ}$.

If one vector is a scalar multiple of another, then the two vectors are parallel.

Example (8):

$ACBY$ is a quadrilateral, with the diagonals AB and CY intersecting at point X .

The point X divides the line AB in the ratio $1 : 2$.

$$\overrightarrow{CA} = 3\mathbf{a}, \quad \overrightarrow{CB} = 6\mathbf{b} \quad \text{and} \quad \overrightarrow{BY} = 5\mathbf{a} - \mathbf{b}.$$

Prove that X divides the line CY in the ratio $2 : 3$.

$$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB} = -3\mathbf{a} + 6\mathbf{b}, \quad \text{or} \quad 6\mathbf{b} - 3\mathbf{a}.$$

Point X divides AB in the ratio $1 : 2$, so it lies one-third of the way along AB .

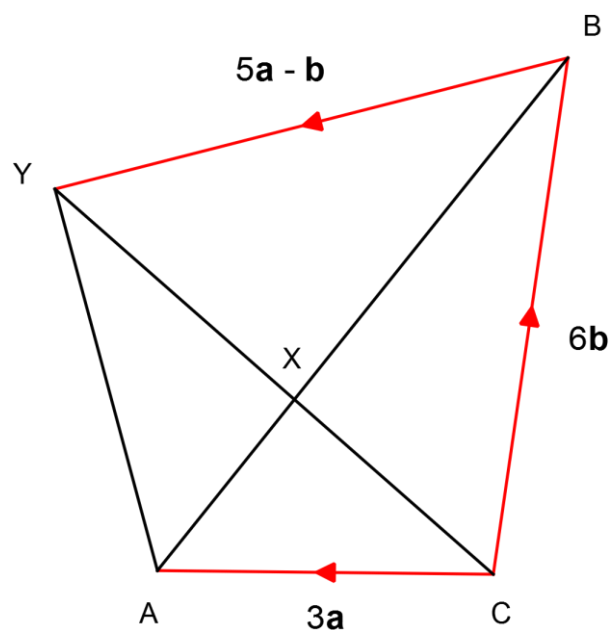
$$\text{Hence} \quad \overrightarrow{AX} = \frac{1}{3}\overrightarrow{AB} = 2\mathbf{b} - \mathbf{a}.$$

We then find vectors \overrightarrow{CX} and \overrightarrow{CY} :

$$\overrightarrow{CX} = \overrightarrow{CA} + \overrightarrow{AX} = 3\mathbf{a} + 2\mathbf{b} - \mathbf{a} = 2\mathbf{a} + 2\mathbf{b}.$$

$$\overrightarrow{CY} = \overrightarrow{CB} + \overrightarrow{BY} = 6\mathbf{b} + 5\mathbf{a} - \mathbf{b} = 5\mathbf{a} + 5\mathbf{b}.$$

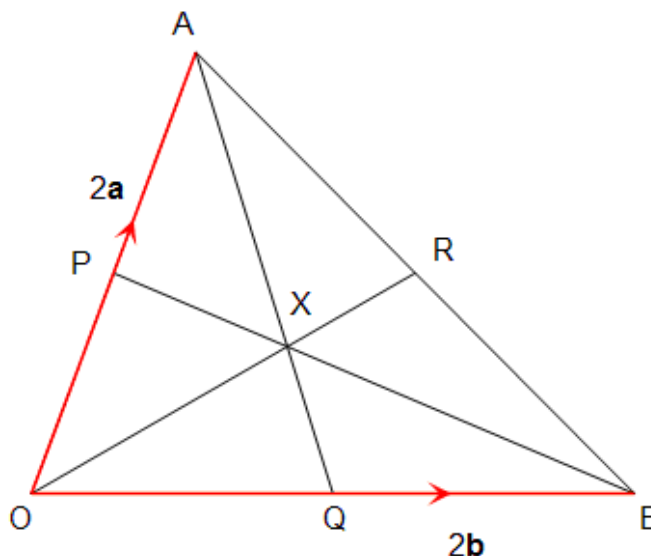
The length of CX is evidently two-fifths that of CY , so point X does indeed divide the diagonal CY in the ratio $2 : 3$.



Example (9): OAB is a triangle where P , Q and R are the midpoints of OA , OB and AB respectively, and X is the point at which all three intersect.

$$\vec{OA} = 2\mathbf{a}, \quad \vec{OB} = 2\mathbf{b}.$$

As a matter of interest, the lines AQ , BP and OR are known as the **medians** of the triangle, and point X is the **centroid** and also the triangle's centre of gravity..



Find expressions for the following in terms of \mathbf{a} and \mathbf{b} :

- i) \vec{AB} ; ii) \vec{OR} ; iii) \vec{AQ} ; iv) \vec{BP}

The ratios $OX : OR$, $AX : AQ$ and $BX : BP$ are all equal to $k : 1$ where k is a fractional constant.

- v) Express \vec{OX} in terms of a) \vec{OR} ; b) \mathbf{a} and \vec{AQ} ; c) \mathbf{b} and \vec{BP} .

vi) Hence solve the vector equations in v) and thus find the value of k .

i) $\vec{AB} = \vec{AO} + \vec{OB} = -2\mathbf{a} + 2\mathbf{b} = 2\mathbf{b} - 2\mathbf{a}.$

ii) $\vec{OR} = \vec{OA} + \vec{AR} = 2\mathbf{a} + \frac{1}{2}\vec{AB} = 2\mathbf{a} + \mathbf{b} - \mathbf{a} = \mathbf{a} + \mathbf{b}.$

iii) $\vec{AQ} = \vec{AO} + \vec{OQ} = -2\mathbf{a} + \mathbf{b} = \mathbf{b} - 2\mathbf{a}.$

iv) $\vec{BP} = \vec{BO} + \vec{OP} = -2\mathbf{b} + \mathbf{a} = \mathbf{a} - 2\mathbf{b}.$

v) a) $\vec{OX} = k\vec{OR} = k\mathbf{a} + k\mathbf{b}.$

v) b) $\vec{OX} = 2\mathbf{a} + k\vec{AQ} = 2\mathbf{a} + k(\mathbf{b} - 2\mathbf{a}) = 2\mathbf{a} + k\mathbf{b} - 2k\mathbf{a} = (2-2k)\mathbf{a} + k\mathbf{b}.$

v) c) $\vec{OX} = 2\mathbf{b} + k\vec{BP} = 2\mathbf{b} + k(\mathbf{a} - 2\mathbf{b}) = 2\mathbf{b} + k\mathbf{a} - 2k\mathbf{b} = (2-2k)\mathbf{b} + k\mathbf{a}.$

vi) Equating the results in v), we have $k = 2 - 2k \rightarrow 3k = 2$
 and thus $k = \frac{2}{3}$.

Hence the centroid X is two-thirds of the way along all three medians AQ , OR and BP .

Coordinate geometry with vectors.

This section complements the other examples on coordinate geometry, but with a greater emphasis on vectors, even though there will be a considerable amount of Pythagoras involved.

(The following examples use both \mathbf{i} - \mathbf{j} and column notations interchangeably for practice).

There are methods for finding angles between vectors, and testing for right angles, but they now come under the remit of Further Maths.

Vectors in Plane Figures.

Recall the following facts concerning side and diagonal lengths of triangles and quadrilaterals:
(Angle properties not stated)

Two vectors \mathbf{a} and \mathbf{b} are parallel if one is a scalar multiple of the other.

An equilateral triangle has all three sides equal.

An isosceles triangle has two sides of equal length, with the third one different.

A trapezium has one pair of parallel sides.

A kite has two pairs of adjacent sides of equal length.

A parallelogram has two pairs of parallel opposite sides, of equal length.

A rhombus has all of the properties of a parallelogram, plus having all sides equal in length.

A rectangle has all of the properties of a parallelogram, plus having its diagonals equal in length.

A square has all of the properties of a rectangle and rhombus combined.

Example (10): ABC is a triangle such that the position vector of A is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\overrightarrow{AB} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$.

Given that $\overrightarrow{BC} = \begin{pmatrix} -2 \\ -9 \end{pmatrix}$,

- i) find the coordinates of B and C ,
- ii) show that triangle ABC is isosceles.
- iii) show that $\cos ABC = \frac{4}{5}$.

iv) show that the area of the triangle is exactly 25.5 square units..

i) To find the coordinates of B , we add the position vector of A , i.e. $\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 7 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$

Hence $A = (0, 1)$ and $B = (7, 7)$.

The coordinates of C are found in the same way: $\begin{pmatrix} 7 \\ 7 \end{pmatrix} + \begin{pmatrix} -2 \\ -9 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$, so $C = (5, -2)$

ii) The length of AB , i.e. $|\overrightarrow{AB}| = \sqrt{7^2 + 6^2} = \sqrt{85}$.

In the same way, $|\overrightarrow{BC}| = \sqrt{(-2)^2 + (-9)^2} = \sqrt{85}$.

Hence the lengths AB and BC are equal.

The distance between $A (0, 1)$ and $C (5, -2)$ can be found similarly by Pythagoras:

$$|\overrightarrow{AC}| = \sqrt{(5-0)^2 + (-2-1)^2} = \sqrt{34}.$$

The two sides AB and BC are equal in length, but the length of AC is different. Hence triangle ABC is isosceles.

iii) By the cosine rule,

$$\cos ABC = \frac{(AB)^2 + (BC)^2 - (AC)^2}{2(AB)(BC)}.$$

$$= \frac{85 + 85 - 34}{2\sqrt{85}\sqrt{85}} = \frac{136}{170} = \frac{4}{5}.$$

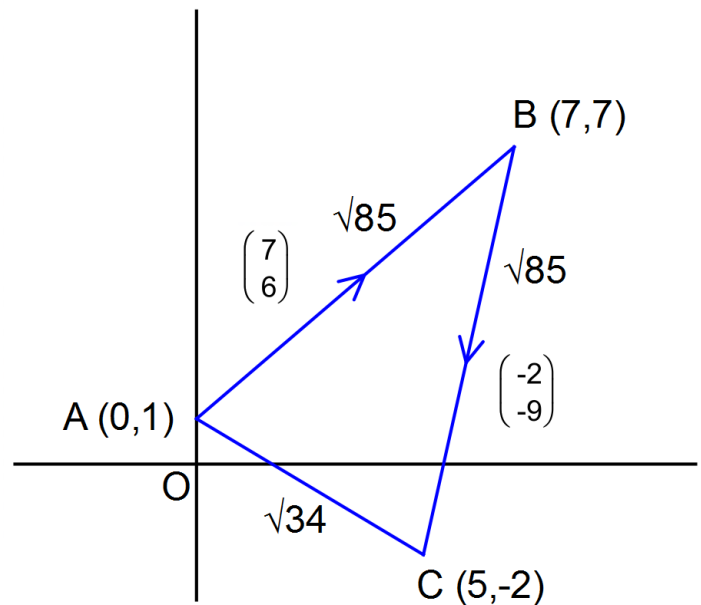
iv) If $\cos ABC = \frac{4}{5}$, then

$$\sin ABC = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}.$$

(using $\cos^2 ABC + \sin^2 ABC = 1$).

The area of the triangle ABC is therefore $\frac{1}{2} (AB)(BC) \sin ABC =$

$$\frac{1}{2} (\sqrt{85})(\sqrt{85}) \times \frac{3}{5} = 85 \times \frac{3}{10} = 25.5 \text{ square units.}$$



Example (11): The vertices of a quadrilateral $OABC$ have the following position vectors:

$$O = \mathbf{0}; A = 2\mathbf{i} + 9\mathbf{j}; B = 8\mathbf{i} + 2\mathbf{j}; C = 6\mathbf{i} - 7\mathbf{j}.$$

i) Show that $OABC$ is a rhombus. ii) Show that $OABC$ is not a square.

i) Because the position vector of O is the zero vector, we can immediately say that

$$\overrightarrow{OA} = 2\mathbf{i} + 9\mathbf{j} \text{ and } \overrightarrow{OC} = 6\mathbf{i} - 7\mathbf{j}. \text{ These vectors represent the adjacent sides of the quadrilateral.}$$

$$\text{The length of } OA, \text{ i.e. } |\overrightarrow{OA}|, = \sqrt{2^2 + 9^2} = \sqrt{85}; \text{ that of } OC, \text{ or. } |\overrightarrow{OC}|, = \sqrt{6^2 + (-7)^2} = \sqrt{85}.$$

Hence the adjacent sides OA and OC are equal.

To prove that $OABC$ is a rhombus, we could either calculate the lengths of OB and BC and show that all four sides are equal, or we could show that OA and CB are equal and parallel, as are OC and AB .

Since $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, its vector is $(8\mathbf{i} + 2\mathbf{j}) - (2\mathbf{i} + 9\mathbf{j}) = 6\mathbf{i} - 7\mathbf{j}$. Hence $\overrightarrow{AB} = \overrightarrow{OC}$, so its length is also $\sqrt{6^2 + (-7)^2} = \sqrt{85}$.

Also, $\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC}$, so its vector is

$$(8\mathbf{i} + 2\mathbf{j}) - (6\mathbf{i} - 7\mathbf{j}) = 2\mathbf{i} + 9\mathbf{j}.$$

Thus $\overrightarrow{CB} = \overrightarrow{OA}$, so its length is $\sqrt{2^2 + 9^2} = \sqrt{85}$.

All the sides of $OABC$ are equal, and both pairs of opposite sides are parallel, so $OABC$ is a rhombus.

ii) The diagonals of a square are equal in length, but those of a rhombus are not.

$$\text{The length of the diagonal } OB = \sqrt{8^2 + 2^2} = \sqrt{68}.$$

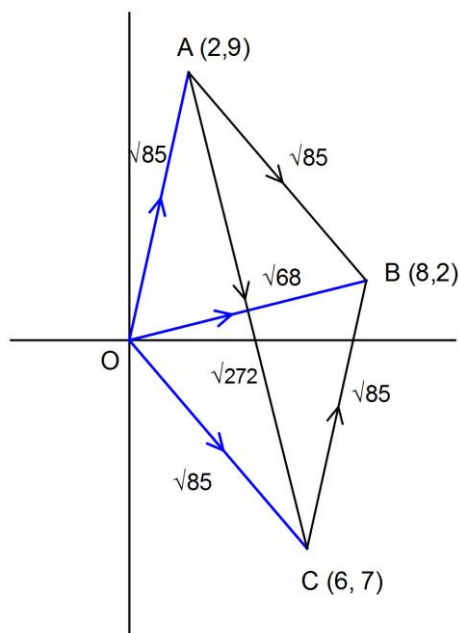
To find the length of the other diagonal AC , we reckon

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$

and its vector is $(6\mathbf{i} - 7\mathbf{j}) - (2\mathbf{i} + 9\mathbf{j}) = 4\mathbf{i} - 16\mathbf{j}$.

$$\text{The length is } \sqrt{4^2 + (-16)^2} = \sqrt{272}.$$

The diagonals OB and DC are unequal, so $OABC$ is not a square.



Example (12): The vertices of a quadrilateral $ABCD$ have the following position vectors:

$$A = \begin{pmatrix} 1 \\ -2 \end{pmatrix}; B = \begin{pmatrix} 0 \\ 2 \end{pmatrix}; C = \begin{pmatrix} 8 \\ 4 \end{pmatrix}; D = \begin{pmatrix} 9 \\ 0 \end{pmatrix}.$$

- i) Show that $ABCD$ is a rectangle, but not a square.
 ii) Find the area of the rectangle.

i) A sketch would show that AB and AD are two adjacent sides, and that AC is a diagonal.

Since point A has a position vector of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, we can also say that $\overrightarrow{OA} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, with the same applying to the other three points.

$$\text{So } \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}; \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix};$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} 9 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}.$$

We then work out BC and DC :

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}, \text{ so sides } BC \text{ and } AD \text{ are equal and parallel.}$$

$$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} - \begin{pmatrix} 9 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \text{ so sides } DC \text{ and } AB \text{ are equal and parallel.}$$

$ABCD$ is therefore at least a parallelogram, so we apply Pythagoras in reverse to show that the triangle ABD is right-angled, i.e.

$$|\overrightarrow{AC}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2.$$

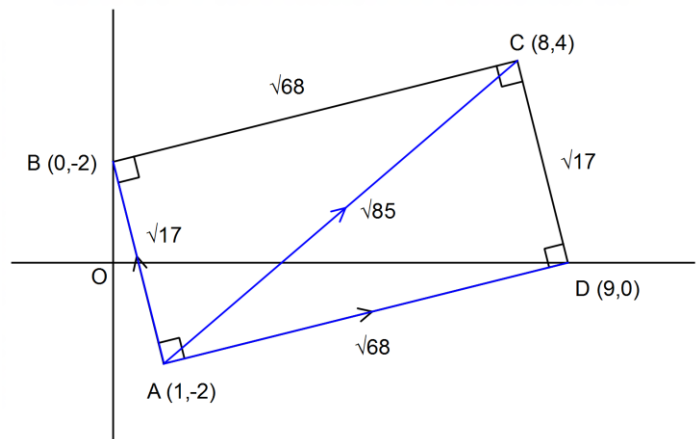
$$\text{Now } |\overrightarrow{AC}|^2 = 7^2 + 6^2 = 85,$$

$$|\overrightarrow{AB}|^2 = (-1)^2 + 4^2 = 17,$$

$$\text{and } |\overrightarrow{BC}|^2 = 8^2 + 2^2 = 68.$$

Hence $|\overrightarrow{AC}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2$,
 angle $ABC = 90^\circ$, and $ABCD$ is a rectangle.

Because the squares of the lengths of AB and BC are different, $ABCD$ cannot be a square.



- ii) The area of the rectangle is $\sqrt{17} \times \sqrt{68}$, or 34, square units.

Example (13): The diagram below shows a quadrilateral $ABCD$.

i) Show, using Pythagoras, that angle BAD is a right angle.

ii) Show, using Pythagoras, that $ABCD$ is a kite.

iii) Find the area of the kite.

iv) The point C is moved to C' such that $ABC'D$ is a square.

a) Find the position vector of C' .

b) Find the area of the resulting square.

v) The point A is moved to A' such that $A'BCD$ is a rhombus.

a) Find the position vector of A' . (C is the point $(14, 7)$ again).

b) Find the area of the rhombus.

i) Using Pythagoras,

$$(AB)^2 = (5-2)^2 + (9-4)^2 = 3^2 + 5^2 = 34$$

$$(AD)^2 = (7-2)^2 + (1-4)^2 = 5^2 + (-3)^2 = 34$$

$$(BD)^2 = (5-7)^2 + (9-1)^2 = (-2)^2 + 8^2 = 68$$

Since $(BD)^2 = (AB)^2 + (AD)^2$, the triangle BAD is right-angled.

ii) A kite has two adjacent pairs of sides equal, and from the last part, $AB = AD = \sqrt{34}$. We work out the lengths of BC and DC in the same way:

$$(BC)^2 = (14-5)^2 + (7-9)^2 = 9^2 + (-2)^2 = 85$$

$$(DC)^2 = (14-7)^2 + (7-1)^2 = 7^2 + 6^2 = 85$$

$BC = DC = \sqrt{85}$, so both pairs of adjacent sides of $ABCD$ are equal, therefore $ABCD$ is a kite.

The area of a kite (like that of a rhombus) is half the product of the diagonals, so we need to find the length of AC

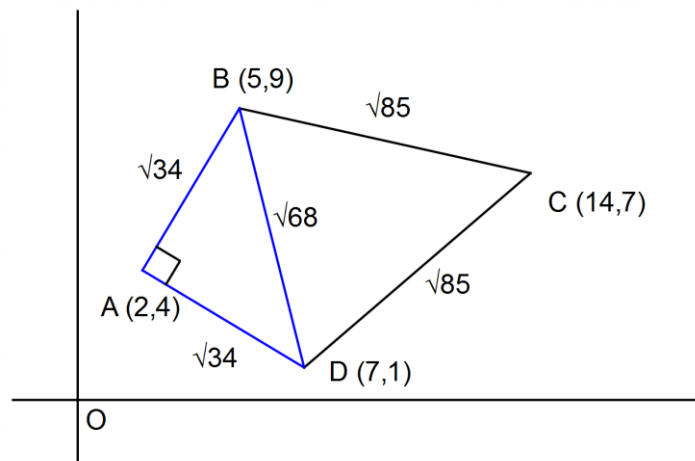
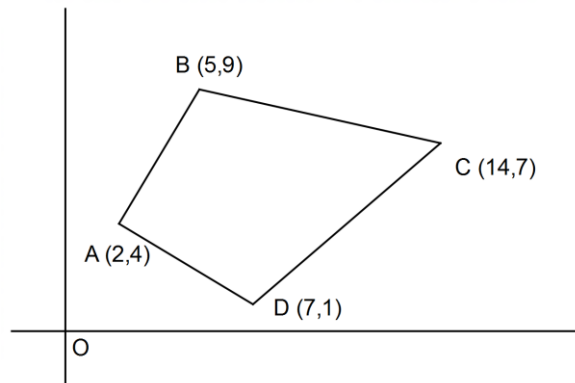
By Pythagoras, $(AC)^2 =$

$$(14-2)^2 + (7-4)^2 = 12^2 + 3^2 = 153, \text{ so } AC = \sqrt{153} \text{ or } 3\sqrt{17}.$$

As $(BD)^2 = 68$, $BD = \sqrt{68}$ or $2\sqrt{17}$.

The area of the kite is therefore

$$\frac{1}{2} (3\sqrt{17})(2\sqrt{17}) = 51 \text{ square units.}$$



iv) a) We have established that angle BAD is a right angle, and that lengths AB and AD are equal. Therefore for $ABC'D$ to be a square, the vectors BC' and AD must be equal and parallel, as must AB and DC' .

Let the position vector of C' be $\begin{pmatrix} x \\ y \end{pmatrix}$.

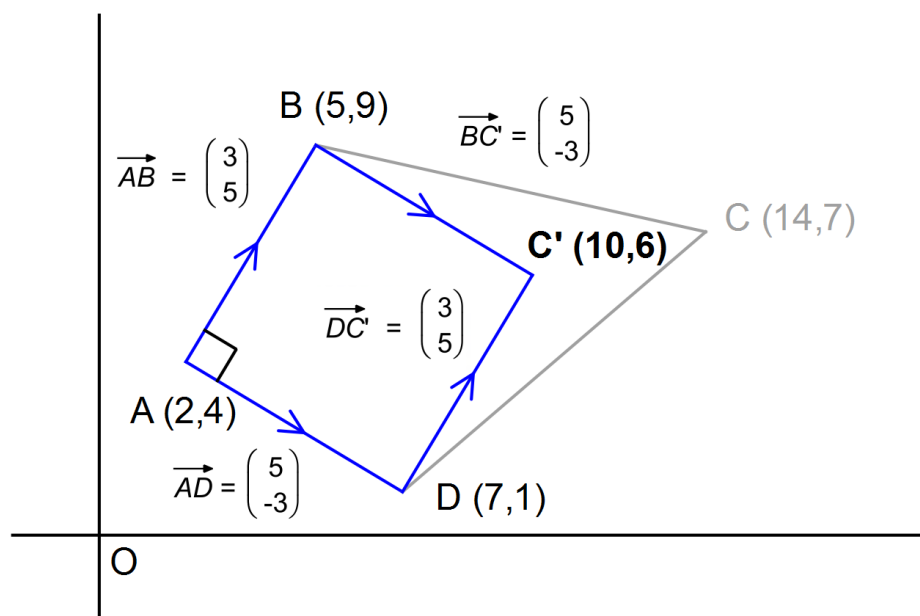
$$\text{Now } \vec{AD} = \vec{OD} - \vec{OA} = \begin{pmatrix} 7 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}, \text{ and } \vec{BC'} = \vec{OC'} - \vec{OB} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 5 \\ 9 \end{pmatrix} = \begin{pmatrix} x-5 \\ y-9 \end{pmatrix}.$$

Thus $\begin{pmatrix} x-5 \\ y-9 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$, hence the position vector of C' , $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 6 \end{pmatrix}$ for $ABC'D$ to be a square.

Using AB and DC' would lead to the same result, since if three sides of a quadrilateral are equal and two of them form a parallel pair, then the fourth side is equal to the other three, as well as parallel to the "unmatched" side.

$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 5 \\ 9 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \text{ and } \vec{DC'} = \vec{OC'} - \vec{OD} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 7 \\ 1 \end{pmatrix} = \begin{pmatrix} x-7 \\ y-1 \end{pmatrix}.$$

Thus $\begin{pmatrix} x-7 \\ y-1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, hence the position vector of C' , $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 6 \end{pmatrix}$ for $ABC'D$ to be a square.



b) We have worked out in part i) that $(AB)^2 = 34$, so the area of the square is 34 square units.

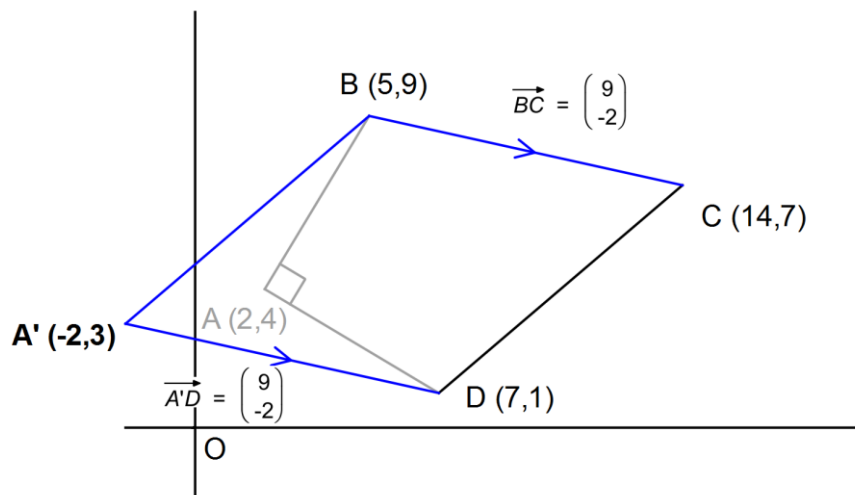
v) Unlike angle BAD , BCD is not a right angle, but the process of finding the direction vector of A' is identical.

The lengths of AB and AD are equal, so for $A'BCD$ to be a rhombus, the vectors BC and $A'D$ must be equal and parallel, as must $A'B$ and DC .

Again, let the position vector of A' be $\begin{pmatrix} x \\ y \end{pmatrix}$, and choose to work with vectors $\overrightarrow{A'D}$ and \overrightarrow{BC} .

$$\text{Now } \overrightarrow{A'D} = \overrightarrow{OD} - \overrightarrow{OA'} = \begin{pmatrix} 7 \\ 1 \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7-x \\ 1-y \end{pmatrix}, \text{ and } \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 14 \\ 7 \end{pmatrix} - \begin{pmatrix} 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \end{pmatrix}.$$

Thus $\begin{pmatrix} 7-x \\ 1-y \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \end{pmatrix}$, hence the position vector of A' , $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ for $A'BCD$ to be a rhombus.



The area of a rhombus is half the product of the diagonals, so we need to find the length of $A'C$

By Pythagoras, $(A'C)^2 = (14-2)^2 + (7-3)^2 = 16^2 + 4^2 = 272$, so $A'C = \sqrt{272}$ or $4\sqrt{17}$.

As $(BD)^2 = 68$, $BD = \sqrt{68}$ or $2\sqrt{17}$.

Hence the area of the rhombus is $\frac{1}{2}(4\sqrt{17})(2\sqrt{17}) = 68$ square units.