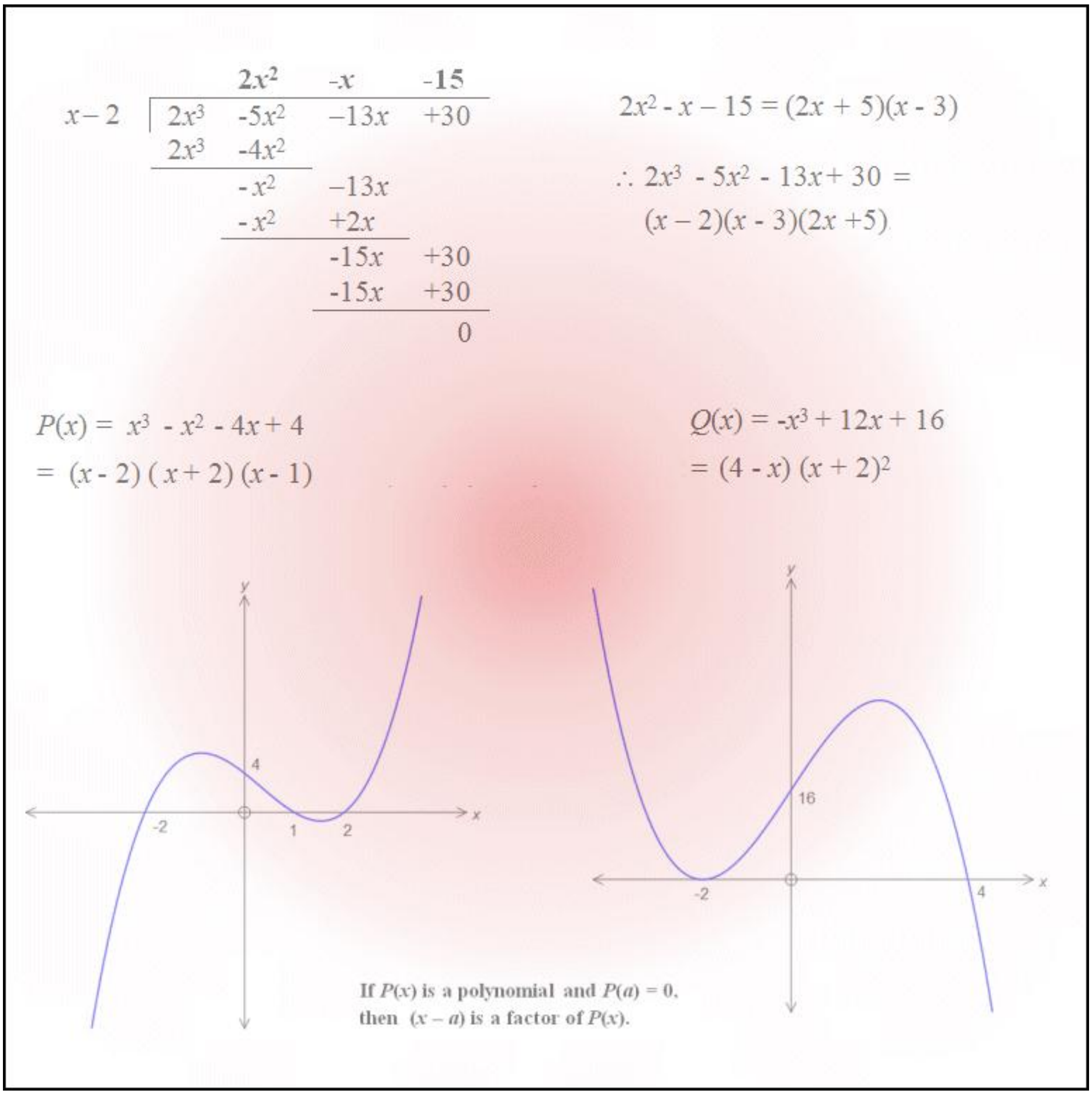
M.K. HOME TUITION

Mathematics Revision Guides
Level: AS / A Level

AQA : **C1** Edexcel: **C2** OCR: **C2** OCR MEI: **C1**

POLYNOMIALS



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POLYNOMIALS

A **polynomial** expression is one that takes the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots\dots\dots + a_0$$

where $a_n, a_{n-1} \dots\dots a_0$ are constants and n is a positive integer.

For example, $3x^3 - 5x + 6$ is a polynomial, where $a_3 = 3$, $a_2 = 0$, $a_1 = -5$ and $a_0 = 6$.

The **degree** of a polynomial is the highest power of x in it, the degree of $3x^3 - 5x + 6$ is 3.
A quadratic thus has a degree of 2 and a linear expression a degree of 1. (A constant can be said to have degree of 0).

Algebraic division.

Division of polynomials is analogous to that of integers. Thus if you work out 38 ÷ 5, you obtain a **quotient** of 7 and a **remainder** of 3. This relationship can be shown as (7 × 5) + 3 = 38. Also 38 is the **dividend** and 5 the **divisor**.

The long division format is the most common method used at AS level, and so will be featured here .

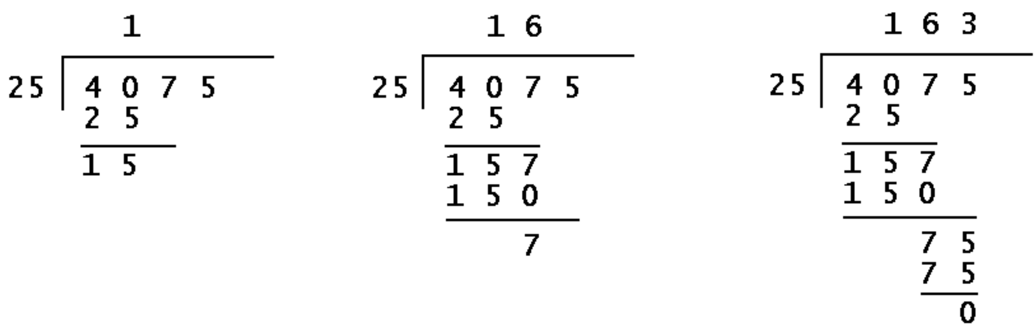
Pre- example (1): Find the value of 4075 ÷ 25.

Since 25 does not go into 4, we leave the space above the 4 blank. We can divide 25 into 40, so we put the answer, 1, above the 0, write the value of 1 × 25 below the 40, and subtract to find the remainder, 15. (First diagram)

Next, we bring down the next digit, 7, in the dividend and proceed to divide 25 into 157. The largest multiple of 25 below 157 is 25 × 6 or 150, so we write 150 below the 157, and subtract 150 from 157 to get 7. (Second diagram).

Then we bring down the next digit, 5, and proceed to divide 75 by 25. Now 75 is exactly 25 × 3, so we write 75 under the 75, with the final subtraction leaving a remainder of zero.

∴ 4075 ÷ 25 = 163.



Example (1): Divide $2x^2 + 9x - 5$ by $x + 5$.

Although this quadratic can be factorised quite easily, the method will be shown for illustration.

$$\begin{array}{r|lll} x+5 & 2x^2 & +9x & -5 \end{array}$$

We first look at the terms in the highest power of x in the dividend and the divisor. They are $2x^2$ in the dividend and x in the divisor. Dividing $2x^2$ by x gives $2x$.

We therefore put $2x$ in the quotient, and the product $(2x)(x + 5)$, namely $2x^2 + 10x$, underneath the terms $2x^2 + 9x$.

Note how all identical powers of x are in the same column each time – important ! .

$$\begin{array}{r|lll} & & \mathbf{2x} & \\ \hline x+5 & 2x^2 & +9x & -5 \\ & \mathbf{2x^2} & \mathbf{+10x} & \end{array}$$

Next, as in ordinary long division, we subtract and bring down the next term.

$$\begin{array}{r|lll} & & \mathbf{2x} & \\ \hline x+5 & 2x^2 & +9x & -5 \\ & 2x^2 & +10x & \\ \hline & & \mathbf{-x} & \mathbf{-5} \end{array}$$

Now we have to divide x into $-x$ to obtain a result of -1. We thus bring down $(-1)(x + 5)$, or $(-x - 5)$ and put -1 in the quotient.

$$\begin{array}{r|lll} & & \mathbf{2x} & \mathbf{-1} \\ \hline x+5 & 2x^2 & +9x & -5 \\ & 2x^2 & +10x & \\ \hline & & -x & -5 \\ & & \mathbf{-x} & \mathbf{-5} \\ \hline & & & \mathbf{0} \end{array}$$

Subtraction leaves no remainder here, and thus the quotient obtained by dividing $(2x^2 + 9x - 5)$ by $(x + 5)$ is $(2x - 1)$ exactly.

Note also how the dividend $(2x^2 + 9x - 5)$ has degree 2, and the divisor $(x + 5)$ and quotient $(2x - 1)$ have degree 1.

The degree of the dividend is always equal to the sum of the degrees of the divisor and the quotient.

Example(2): Divide $x^3 - 4x^2 - 9x + 36$ by $x + 3$.

$x + 3$	x^3	$-4x^2$	$-9x$	$+36$

Continue as in the previous example:

Dividing x^3 by x gives x^2, and $x^2 (x + 3) = x^3 + 3x^2$, so we put that below the dividend and x^2 in the quotient.

		$\mathbf{x^2}$		
$x + 3$	x^3	$-4x^2$	$-9x$	$+36$
	$\mathbf{x^3}$	$\mathbf{+ 3x^2}$		

We then subtract to obtain a remainder of $-7x^2$ and bring down the next term, $-9x$.
Dividing $-7x^2$ by x gives $-7x$, and as $(-7x)(x + 3) = -7x^2 - 21x$, we put that below the dividend and $-7x$ in the quotient.

		$\mathbf{x^2}$	$\mathbf{-7x}$	
$x + 3$	x^3	$-4x^2$	$-9x$	$+36$
	x^3	$+ 3x^2$		
		$\mathbf{- 7x^2}$	$\mathbf{-9x}$	
		$\mathbf{- 7x^2}$	$\mathbf{-21x}$	

Subtracting again, we have a remainder of $12x$ and bring down the last term, $+36$.
Dividing $-2x$ by x gives 12, and $(12)(x + 3) = 12x + 36$, so we put that below the dividend and 12 in the quotient.

		$\mathbf{x^2}$	$\mathbf{-7x}$	$\mathbf{+12}$
$x + 3$	x^3	$-4x^2$	$-9x$	$+36$
	x^3	$+ 3x^2$		
		$\mathbf{- 7x^2}$	$\mathbf{-9x}$	
		$\mathbf{- 7x^2}$	$\mathbf{-21x}$	
			$\mathbf{12x}$	$\mathbf{+36}$
			$\mathbf{12x}$	$\mathbf{+36}$
				0

The final subtraction leaves no remainder, i.e. the division comes out exact, and so the quotient obtained by dividing $(x^3 - 4x^2 - 9x + 36)$ by $(x + 3)$ is $(x^2 - 7x + 12)$.

Missing powers in the dividend.

Example(3): Divide $x^3 - 5x - 2$ by $x + 2$.

The long division method requires a little care, because the term in x^2 is zero, but its place must still be included in the layout.

$x + 2$	x^3	$+ 0x^2$	$–5x$	-2

Continue as in the previous example:

Dividing x^3 by x gives x^2, and $x^2 (x + 2) = x^3 + 2x^2$, so we put that below the dividend and x^2 in the quotient.

		$\mathbf{x^2}$		
$x + 2$	x^3	$+ 0x^2$	$–5x$	-2
	$\mathbf{x^3}$	$\mathbf{+ 2x^2}$		

We then subtract to obtain a remainder of $-2x^2$ and bring down the next term, $-5x$.
Dividing $-2x^2$ by x gives $-2x$, and multiplying $(-2x)(x + 2) = -2x^2 - 4x$, so we put that below the dividend and $-2x$ in the quotient.

		$\mathbf{x^2}$	$\mathbf{-2x}$	
$x + 2$	x^3	$+ 0x^2$	$–5x$	-2
	x^3	$+ 2x^2$		
		$\mathbf{- 2x^2}$	$\mathbf{–5x}$	
		$\mathbf{- 2x^2}$	$\mathbf{–4x}$	

Subtracting again, we have a remainder of $-x$ and bring down the last term, -2.
Dividing $-x$ by x gives -1, and $(-1)(x + 2) = -x - 2$, so we put that below the dividend and -1 in the quotient.

		$\mathbf{x^2}$	$\mathbf{-2x}$	$\mathbf{-1}$
$x + 2$	x^3	$+ 0x^2$	$–5x$	-2
	x^3	$+ 2x^2$		
		$- 2x^2$	$–5x$	
		$- 2x^2$	$–4x$	
			$\mathbf{-x}$	$\mathbf{-2}$
			$\mathbf{-x}$	$\mathbf{-2}$
				$\mathbf{0}$

The quotient obtained by dividing $(x^3 - 5x - 2)$ by $(x + 2)$ is $(x^2 - 2x – 1)$.

Missing powers in the quotient.

Example(4): Divide x^3 $-3x^2 - 5x + 15$ by $x - 3$.

$$\begin{array}{r|llll} x-3 & x^3 & -3x^2 & -5x & +15 \end{array}$$

(Dividing x^3 by x gives x^2)

$$\begin{array}{r|llll} & & \mathbf{x^2} & & \\ \hline x-3 & x^3 & -3x^2 & -5x & +15 \\ & \mathbf{x^3} & \mathbf{-3x^2} & & \end{array}$$

Subtraction will leave us with a zero remainder in the x^2 term, so dividing $0x^2$ by x gives us $0x$.
The term in x in the quotient is zero, but we still place it in the quotient.

$$\begin{array}{r|llll} & & \mathbf{x^2} & \mathbf{+\,0x} & \\ \hline x-3 & x^3 & -3x^2 & -5x & +15 \\ & x^3 & -3x^2 & & \\ \cline{2-3} & & \mathbf{0\,x^2} & & \end{array}$$

Because of the zero remainder in the last division, we finish by bringing down the next *two* terms to correspond with the two terms in the divisor.

(Dividing $-5x$ by x gives -5)

$$\begin{array}{r|llll} & & \mathbf{x^2} & \mathbf{+\,0x} & \mathbf{-5} \\ \hline x-3 & x^3 & -3x^2 & -5x & +15 \\ & x^3 & -3x^2 & & \\ \cline{2-3} & & 0\,x^2 & & \\ \cline{3-4} & & & \mathbf{-5x} & \mathbf{+15} \\ & & & \mathbf{-5x} & \mathbf{+15} \\ \cline{4-5} & & & & \mathbf{0} \end{array}$$

Dividing (x^3 $-3x^2 - 5x + 15$) by ($x - 3$) gives a quotient of ($x^2 - 5$) .

Missing powers in the divisor.

Example(5): Divide $x^4 - 2x^3 - 7x^2 + 8x + 12$ by $x^2 - 4$.

Here we have a missing power of x in the divisor, but again its place must be included in the layout.

Notice that the dividend is of the 4th degree and the divisor a quadratic. The quotient will thus be of degree (4 – 2) or 2, i.e. a quadratic.

$x^2 + 0x - 4$	x^4	$-2x^3$	$-7x^2$	$+8x$	$+12$

Dividing x^4 by x^2 gives x^2, and $x^2(x^2 - 4) = x^4 - 4x^2$, so we put that below the dividend and x^2 in the quotient, making sure that the missing powers of x still have their places included in the working.

			$\mathbf{x^2}$		
$x^2 + 0x - 4$	x^4	$-2x^3$	$-7x^2$	$+8x$	$+12$
	$\mathbf{x^4}$	$\mathbf{-0x^3}$	$\mathbf{-4x^2}$		

We then subtract to obtain a remainder of $-2x^3 - 3x^2$ and bring down the next term, $8x$.

			$\mathbf{x^2}$		
$x^2 + 0x - 4$	x^4	$-2x^3$	$-7x^2$	$+8x$	$+12$
	$\mathbf{x^4}$	$\mathbf{-0x^3}$	$\mathbf{-4x^2}$		
		$\mathbf{-2x^3}$	$\mathbf{-3x^2}$	$\mathbf{+8x}$	

Dividing $-2x^3$ by x^2 gives $-2x$, and as $(-2x)(x^2 - 4) = -2x^3 + 8x$, we put that below the dividend and $-2x$ in the quotient.

			$\mathbf{x^2}$	$\mathbf{-2x}$	
$x^2 + 0x - 4$	x^4	$-2x^3$	$-7x^2$	$+8x$	$+12$
	x^4	$-0x^3$	$-4x^2$		
		$-2x^3$	$-3x^2$	$+8x$	
		$\mathbf{-2x^3}$	$\mathbf{+0x^2}$	$\mathbf{+8x}$	
			$\mathbf{-3x^2}$	$\mathbf{+0x}$	$\mathbf{+12}$

Subtracting again, we have a remainder of $-3x^2$ and bring down the last term, +12.
Dividing $-3x^2$ by x^2 gives -3, and $(-3)(x^2 - 4) = -3x^2 + 12$, so we put that below the dividend and -3 in the quotient

			$\mathbf{x^2}$	$\mathbf{-2x}$	$\mathbf{-3}$
$x^2 + 0x - 4$	x^4	$-2x^3$	$-7x^2$	$+8x$	$+12$
	x^4	$-0x^3$	$-4x^2$		
		$-2x^3$	$-3x^2$	$+8x$	
		$-2x^3$	$+0x^2$	$+8x$	
			$-3x^2$	$+0x$	$+12$
			$\mathbf{-3x^2}$	$\mathbf{+0x}$	$\mathbf{+12}$
					$\mathbf{0}$

Dividing $(x^4 - 2x^3 - 7x^2 + 8x + 12)$ by $(x^2 - 4)$ gives a quotient of $(x^2 - 2x - 3)$.

Example (6): Find the quotient and the remainder when dividing $x^3 - 7x^2 + 6x - 1$ by $x - 3$.

		$\mathbf{x^2}$		
$x - 3$	x^3	$-7x^2$	$+6x$	-1
	$\mathbf{x^3}$	$\mathbf{-3x^2}$		

		$\mathbf{x^2}$	$\mathbf{-4x}$	
$x - 3$	x^3	$-7x^2$	$+6x$	-1
	x^3	$-3x^2$		
		$\mathbf{-4x^2}$	$\mathbf{+6x}$	
		$\mathbf{-4x^2}$	$\mathbf{+12x}$	

		$\mathbf{x^2}$	$\mathbf{-4x}$	$\mathbf{-6}$
$x - 3$	x^3	$-7x^2$	$+6x$	-1
	x^3	$-3x^2$		
		$-4x^2$	$+6x$	
		$-4x^2$	$+12x$	
			$\mathbf{-6x}$	$\mathbf{-1}$
			$\mathbf{-6x}$	$\mathbf{+18}$
				$\mathbf{-19}$

This time, there is a final remainder, namely -19.

So $x^3 - 7x^2 + 6x - 1 = (x - 3)(x^2 - 4x - 6) - 19$.

The Remainder Theorem.

In Example (4) above we divided $x^3 - 7x^2 + 6x - 1$ by $x - 3$ to give a quotient of $x^2 - 4x - 6$ and a remainder of -19.

Another way to find out the remainder is to substitute certain values for x.
Writing down $x^3 - 7x^2 + 6x - 1 = (x - 3)(Ax^2 + Bx + C) + D$, we can see that substituting $x = 3$, the right-hand side of the expression simplifies to D because the product of the brackets is zero.
This gives $3^3 - 7(3^2) + (6 \times 3) - 1 = 27 - 63 + 18 - 1 = -19$ as before.

Therefore, when a polynomial $P(x)$ is divided by $(x - a)$, the remainder is $P(a)$.

Example (7): Find the remainder when the polynomial $P(x) = x^3 - 7x^2 + 6x - 1$ is divided by

(a) $x + 1$; (b) $x - 2$; (c) $2x + 1$

In (a) the remainder is $P(-1) = -1 - 7 - 6 - 1 = -15$.
In (b) the remainder is $P(2) = 8 - 28 + 12 - 1 = -9$.
In (c) the remainder is $P(-0.5) = -0.125 - 1.75 - 3 - 1 = -5.875$.

For (c) the theorem can be generalised as:
when a polynomial $P(x)$ is divided by $(ax - b)$, the remainder is $P\left(\frac{b}{a}\right)$.

The Factor Theorem.

This is a special case of the remainder theorem when the remainder is zero. It states that:

If $P(x)$ is a polynomial and $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

Example (8):Show that (x-2) is a factor of $P(x) = x^3 - x^2 - 4x + 4$, and hence solve $P(x) = 0$.

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Substituting $x = 2$, we find that $P(2) = 2^3 - 2^2 - (4\times2) + 4$ or $8 - 4 - 8 + 4 = 0$.
$\therefore$ (x-2) is a factor of $P(x)$.

We then factorise the expression completely:

$$\begin{array}{l|llll} & & \mathbf{x^2} & & \\ \cline{2-5} x-2 & x^3 & -x^2 & -4x & +4 \\ & \mathbf{x^3} & \mathbf{-2x^2} & & \end{array}$$

$$\begin{array}{l|llll} & & \mathbf{x^2} & \mathbf{+x} & \\ \cline{2-5} x-2 & x^3 & -x^2 & -4x & +4 \\ & x^3 & -2x^2 & & \\ \cline{2-3} & & \mathbf{x^2} & \mathbf{-4x} & \\ & & \mathbf{x^2} & \mathbf{-2x} & \end{array}$$

$$\begin{array}{l|llll} & & \mathbf{x^2} & \mathbf{+x} & \mathbf{-2} \\ \cline{2-5} x-2 & x^3 & -x^2 & -4x & +4 \\ & x^3 & -2x^2 & & \\ \cline{2-3} & & x^2 & -4x & \\ & & x^2 & -2x & \\ \cline{3-4} & & & \mathbf{-2x} & \mathbf{+4} \\ & & & \mathbf{-2x} & \mathbf{+4} \\ \cline{4-5} \end{array}$$

The quotient is therefore $x^2 + x - 2$.

The next step is to factorise the quotient, giving $(x + 2)(x - 1)$.

$x^3 - x^2 - 4x + 4 = (x - 2)(x + 2)(x - 1)$.

$\therefore$ The solutions of $P(x) = 0$ are $x = 1$, 2 and –2.

Again, a more generalised form of the Factor Theorem states :

If $P(x)$ is a polynomial and $P\left(\frac{b}{a}\right) = 0$, then $(ax - b)$ is a factor of $P(x)$.

Example (9): A polynomial is given by $Q(x) = 2x^3 - 5x^2 - 13x + 30$.

a) Find the value of $Q(-2)$ and $Q(2)$, and state one factor of $Q(x)$.
b) Factorise $Q(x)$ completely.

a) $Q(-2) = 2(-2)^3 - 5(-2)^2 - (13 \times (-2)) + 30 = -16 - 20 + 26 + 30 = 20$.
$Q(2) = 2(2)^3 - 5(2)^2 - (13 \times (2)) + 60 = 16 - 20 - 26 + 30 = 0$.

$\therefore$ One factor of $2x^3 - 5x^2 - 13x + 30$ is $(x - 2)$.

b) We then divide $(x - 2)$ into $2x^3 - 5x^2 - 13x + 30$ to obtain a quadratic quotient:

		$\mathbf{2x^2}$	$\mathbf{-x}$	**-15**
$x - 2$	$2x^3$	$-5x^2$	$-13x$	$+30$
	$2x^3$	$-4x^2$		
		$-x^2$	$-13x$	
		$-x^2$	$+2x$	
			$-15x$	$+30$
			$-15x$	$+30$

The quotient is therefore $2x^2 - x - 15$.

Trial inspection and factorising gives $2x^2 - x - 15 = (2x + 5)(x - 3)$.

$\therefore$ $2x^3 - 5x^2 - 13x + 30$ factorises fully to $(x - 2)(x - 3)(2x + 5)$.

Example (10): A polynomial is given by $P(x) = x^3 - 2x^2 - 4x + k$ where k is an integer constant.

Find the values of k satisfying the following conditions:

i) the graph of $y = P(x)$ passes through the origin.

ii) the graph of $y = P(x)$ intersects the y-axis at the point (0,5).

iii) $(x - 3)$ is a factor of $P(x)$.

iv) the remainder when $P(x)$ is divided by $(x + 1)$ is 5.

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In i), $P(0) = 0$ when the graph of $P(x)$ passes through the origin, therefore $0^3 - 2(0)^2 - 4(0) + k = 0$ and thus $k = 0$.

In ii), $P(0) = 5$, therefore $0^3 - 2(0)^2 - 4(0) + k = 5$ and thus $k = 5$.

In iii), $(x - 3)$ is a factor of $P(x)$ if $P(3) = 0$ by the Factor Theorem.
$\therefore\ 3^3 - 2(3)^2 - 4(3) + k = 0$
$\Rightarrow 27 - 18 - 12 + k = 0$
$\Rightarrow k = 3$.

In iv), $P(-1) = 5$ by the Remainder Theorem.

$\therefore\ (-1)^3 - 2(-1)^2 - 4(-1) + k = 5$
$\Rightarrow -1 - 2 + 4 + k = 5$
$\Rightarrow k = 4$.

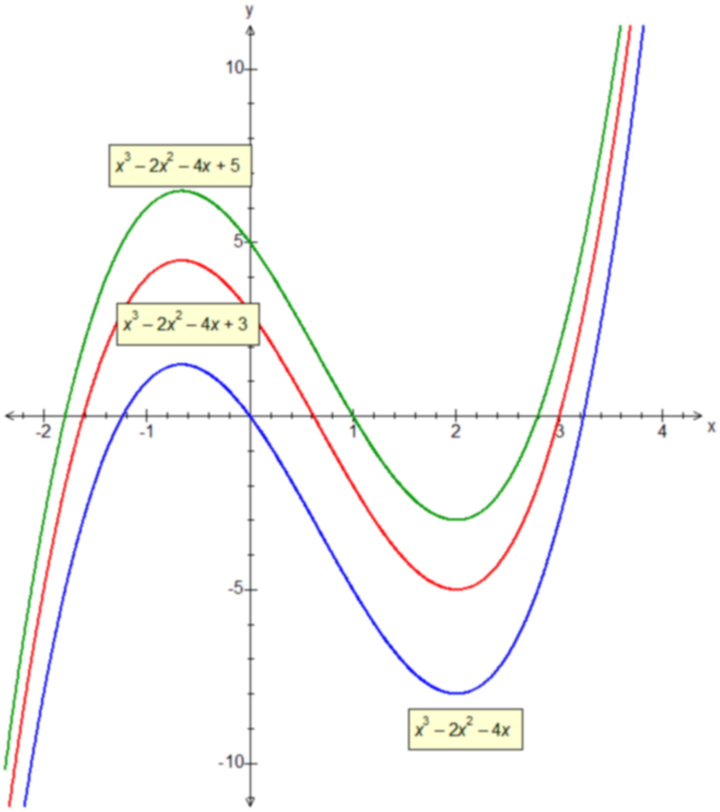
The solutions to parts i), ii) and iii) are shown graphically on the right.

Notice the following:

i) The graph of $x^3 - 2x^2 - 4x$ passing through the origin.

ii) the graph of $x^3 - 2x^2 - 4x + 5$ passing through the point (0,5). It also appears to pass through (1,0) – confirmed by substituting $x = 1$ in the expression, $\therefore$ $(x-1)$ is a factor as well.

iii) The graph of $x^3 - 2x^2 - 4x + 3$ passing through (3, 0).



Example(11): The polynomial $Q(x) = 3x^3 + 2x^2 - bx + a$ where a and b are integer constants.

It is given that $(x - 1)$ is a factor of $Q(x)$, and that division of $Q(x)$ by $(x + 1)$ gives a remainder of 10. Find the values of a and b.

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If $(x - 1)$ is a factor of $Q(x)$, then $Q(1) = 0$ by the Factor Theorem.
Substituting $x = 1$, we have :

$3 + 2 - b + a = 0$
$\Rightarrow 5 - b + a = 0$
$\Rightarrow a - b = -5$

If $(x + 1)$ leaves a remainder of 10 when divided into $Q(x)$, then $Q(-1) = 10$ by the Remainder Theorem.

Substituting $x = -1$, we have:

$-3 + 2 + b + a = 10$
$\Rightarrow -1 + a + b = 10$
$\Rightarrow a + b = 11$

This leaves us with two linear simultaneous equations:

$a - b = -5$ $\quad A$
$a + b = 11$ $\quad B$

$2a = 6$ $\quad A+B$

Substituting $a = 3$ into equation B gives $b = 8$.

Hence $Q(x) = 3x^3 + 2x^2 - 8x + 3$.

(Question does not ask for the expression to be formally factorised.)

Example(12): The polynomial $P(x) = 6x^3 - 23x^2 + ax + b$ where a and b are integer constants.

It is given that division of $P(x)$ by $(x - 3)$ gives a remainder of 11, and that division of $P(x)$ by $(x + 1)$ gives a remainder of -21.

Find the values of a and b and hence factorise the expression.

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If $(x - 3)$ leaves a remainder of 11 when divided into $P(x)$, then $P(3) = 11$ by the Remainder Theorem.

Substituting $x = 3$, we have $162 - 207 + 3a + b = 11$
$\Rightarrow -45 + 3a + b = 11$
$\Rightarrow 3a + b = 56$

Similarly, if $(x + 1)$ leaves a remainder of -21 when divided into $P(x)$, then $P(-1) = -21$ by the Remainder Theorem.

Substituting $x = -1$, we have $-6 - 23 - a + b = -21$
$\Rightarrow -29 - a + b = -21$
$\Rightarrow -a + b = 8$

This leaves us with two linear simultaneous equations:

$3a + b = 56$ *A*
$-a + b = 8$ *B*

$4a = 48$ *A-B*

Substituting $a = 12$ into equation *A* gives $b = 20$.

$\therefore P(x) = 6x^3 - 23x^2 + 12x + 20$.

To factorise the equation, try substituting various values of x to find one that gives zero;
$P(1) = 6 - 23 + 12 + 20 = 15$, so $(x – 1)$ is not a factor by the Remainder Theorem.
$P(2) = 48 – 92 + 24 + 20 = 0$, so $(x - 2)$ is a factor.

We can then proceed to factorise the cubic:

		$\mathbf{6x^2}$	$\mathbf{-11x}$	$\mathbf{-10}$
$x - 2$	$6x^3$	$-23x^2$	$+12x$	$+20$
	$6x^3$	$-12x^2$		
		$-11x^2$	$+12x$	
		$-11x^2$	$+22x$	
			$-10x$	$+20$
			$-10x$	$+20$

$\therefore P(x) = (x - 2)(6x^2 - 11x - 10)$.

The quadratic quotient in turn factorises to $6x^2 - 11x - 10 = (3x + 2)(2x - 5)$

$\therefore P(x) = (x - 2)(3x + 2)(2x - 5)$.

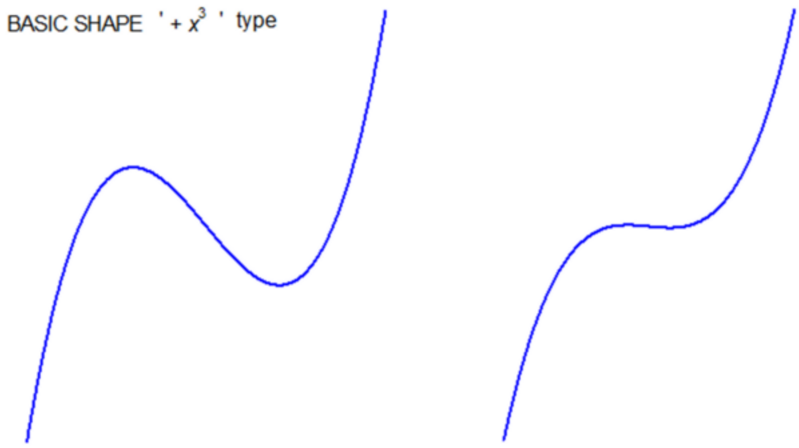
Sketching cubic graphs.

Examination questions might also ask for a sketch of a polynomial graph, usually no more complex than a cubic.

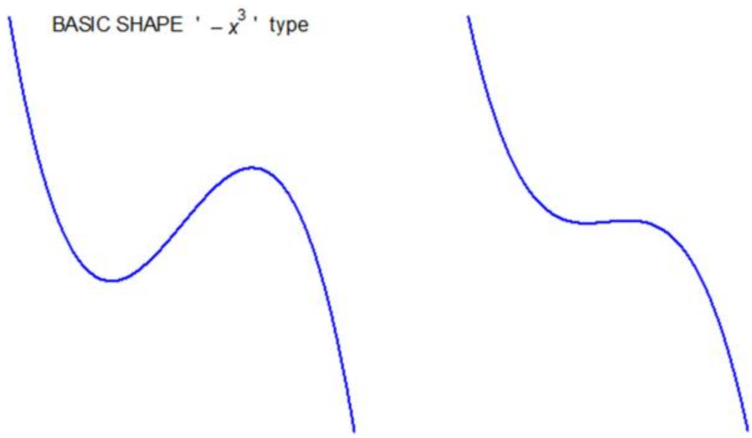
The main criteria for sketching a cubic graph are a) obtaining the correct general shape, b) finding the intercepts and c) , locating and finding any turning points if asked to do so.

The examples below will not require any work on finding turning points.

The basic shape of a cubic graph features a 'double bend' of varying severity. If the coefficient of x^3 is positive, then the curve follows a general lower left to upper right direction.



If the coefficient of x^3 is negative, then the curve follows a general upper left to lower right direction.



Finally, if the cubic has repeated factors, the x-intercept at that particular root is a tangent to the x-axis.

Example (13): The polynomial $P(x) = x^3 - x^2 - 4x + 4$ in Example (8) was factorised to

$P(x) = (x - 2)(x + 2)(x - 1)$.

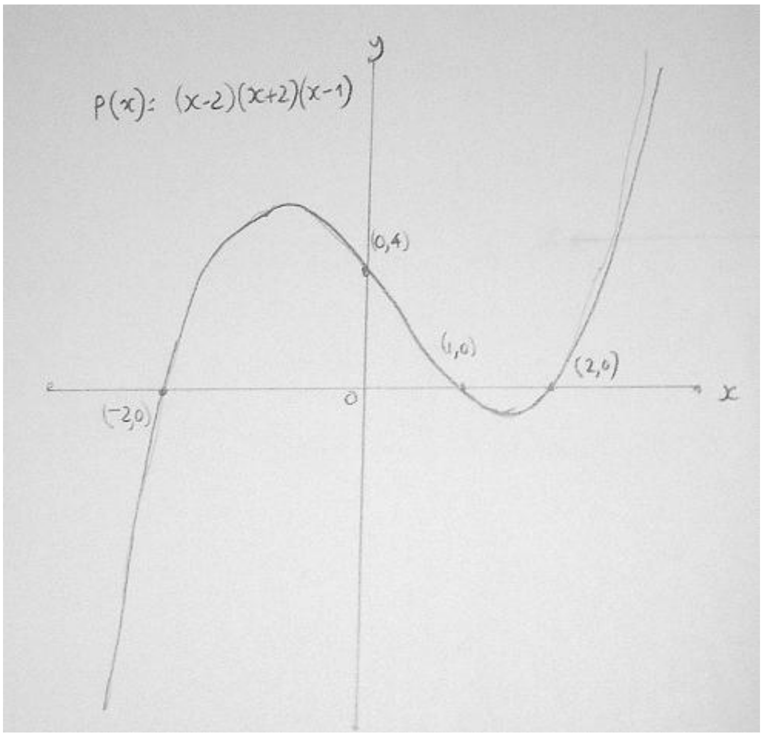
Sketch the graph of $P(x)$.

Since the coefficient of x^3 is positive, the general shape of the graph is an increasing one from lower left to upper right, namely of the basic '+ x^3' type.

The x-intercepts correspond to the roots at x = -2, 1 and 2, and so we plot the points (-2, 0), (1, 0) and (2, 0).

When x = 0, y = 4, so we plot the y-intercept at (0, 4).

Finally, we connect the points with a basic '+ x^3' curve, with a local maximum at about x = -1 and a local minimum at about x = 1.5.



Example (14): Show by the Factor Theorem that (x + 4) is a factor of $P(x) = x^3 - 2x^2 - 15x + 36$. Hence factorise $P(x)$ completely and sketch its graph.

Substituting $x = -4$ gives $P(-4) = -64 - 32 + 60 + 36 = 0$, $\therefore$ (x + 4) is a factor of $P(x)$.

		$\mathbf{x^2}$	$\mathbf{-6x}$	**+9**
$x + 4$	x^3	$-2x^2$	$-15x$	$+36$
	x^3	$+4x^2$		
		$-6x^2$	$-15x$	
		$-6x^2$	$-24x$	
			$9x$	$+36$
			$9x$	$+36$

The quotient, $x^2 - 6x + 9$, can be factorised to $(x-3)^2$.

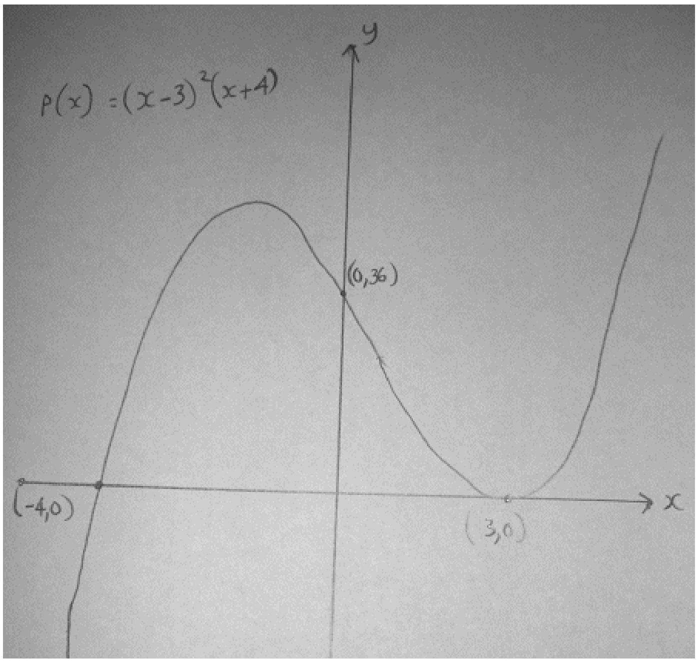
$\therefore$ $P(x) = x^3 - 2x^2 - 15x + 36$ factorises fully to $(x + 4)(x - 3)^2$.

From the above facts, we can deduce that the graph meets the x–axis at (-4, 0) and (3, 0).

Because (x – 3) is a repeated factor, the x-intercept (3, 0) is also a tangent to the x-axis.

When $x = 0$, $y = 36$, so the graph meets the y-axis at (0, 36).

Finally, we connect the points with a basic '+ x^3' curve, with a local maximum at about $x = -1$ and a tangent to the x-axis (actually a local minimum) at $x = 3$.



Example (15): Sketch the graph of $Q(x) = -x^3 + 12x + 16$.
We are given that $(4 – x)$ is a factor.
Note: $4-x$ is the same as $–x + 4$.

The first thing to notice here is that the coefficient of x^3 is negative, and therefore the graph will be of the '$- x^3$' type.

Firstly we factorise $Q(x)$ completely:

		x^2	**$+4x$**	**$+4$**
$-x + 4$	$-x^3$	$+0x^2$	$+12x$	$+16$
	$-x^3$	$+4x^2$		
		$- 4x^2$	$+12x$	
		$- 4x^2$	$+16x$	
			$-4x$	$+16$
			$-4x$	$+16$

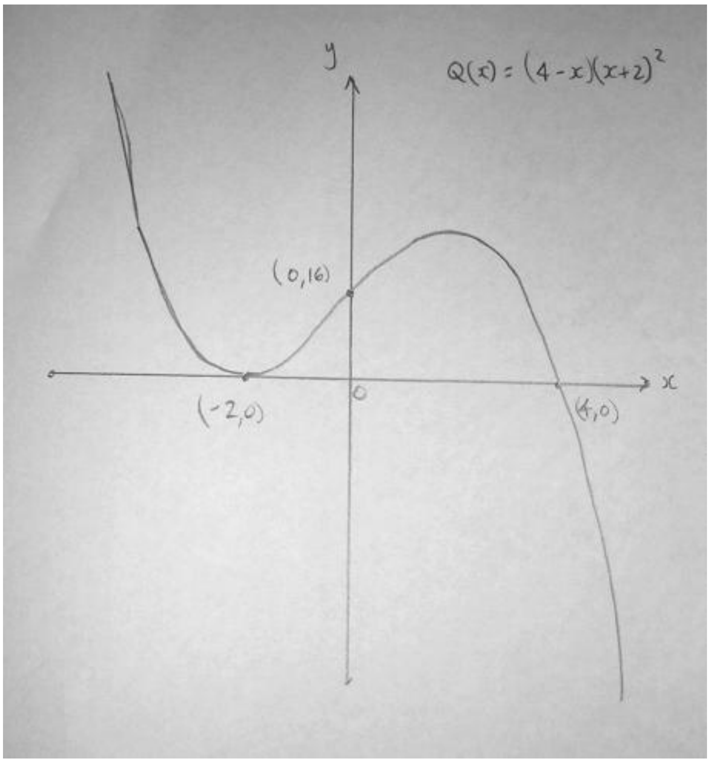
The quotient, $x^2 + 4x + 4$, can be factorised to $(x+2)^2$.

$\therefore\ Q(x) = -x^3 + 12x + 16$ factorises fully to $(4 - x)(x + 2)^2$.

From the above, the graph meets the x–axis at (4, 0) and (-2, 0).
We have a repeated factor of $(x+2)$, and so the x-intercept (-2, 0) is also a tangent to the x-axis.

When $x = 0$, $y = 16$, so the graph meets the y-axis at (0, 16).

Finally, we connect the points with a basic '$- x^3$' curve, with a tangent to the x-axis (actually a local minimum) at $x = -2$ and a local maximum at about $x = 2$.



Alternative method of dividing / factorising polynomials.

Although the long division method is the most commonly used one for dividing and factorising polynomials, there is another method which can sometimes prove easier to use – called the method of equating coefficients.

Example (16):The polynomial $P(x) = 2x^3 + 3x^2 - 23x - 12$ has two linear factors of $(x – 3)$ and $(x + 4)$. Find the third linear factor.

Since $P(x)$ is a cubic, its factorised form is $(x – 3)(x + 4)(Ax + B)$ where A and B are constants. The only way to obtain the term of $2x^3$ in $P(x)$ is to multiply x, x and Ax together from the factors. By equating the x^3 terms, $A = 2$.

Similarly, the only way to obtain the term of -12 in $P(x)$ is to multiply -3, 4 and B together. Equating the constants gives $B = 1$.

Hence the third linear factor of $P(x)$ is $2x + 1$.

Example (17): Find the quotient and the remainder when the polynomial $P(x) = 6x^3 - 13x^2 + 16x - 3$ is divided by the polynomial $Q(x) = 2x^2 - 3x + 5$.

The degree of the divisor is 2, and so the quotient will be of degree 1.

Therefore $6x^3 - 13x^2 + 16x - 3 = (2x^2 - 3x + 5)(Ax + B) + (Cx + D)$.

Expanding, we have $6x^3 - 13x^2 + 16x - 3 = (2Ax^3 - 3Ax^2 + 5Ax) + (2Bx^2 - 3Bx + 5B) + Cx + D$.

Equating the x^3 terms, we have $2A = 6$, so $A = 3$.

Equating the x^2 terms, we have $2B – 3A = -13$, or $2B – 9 = -13$, or $2B = -4$, so $B = -2$.

Equating the x terms, we have $5A – 3B + C = 16$, or $15 + 6 + C = 16$, or $21 + C = 16$, so $C = -5$.

Equating the constants, we have $5B + D = -3$, or $-10 + D = -3$, so $D = 7$.

$\therefore$ The quotient is $Ax + B$ or $3x - 2$, and the remainder is $Cx + D$ or $-5x + 7$, or $7 – 5x$.

$\therefore$ $6x^3 - 13x^2 + 16x - 3 = (2x^2 - 3x + 5)(3x - 2) + (7 – 5x)$.

Corresponding long division method:

			3*x*	**-2**
$2x^2 - 3x + 5$	$6x^3$	$-13x^2$	$+16x$	-3
	$6x^3$	$-9x^2$	$+15x$	
		$-4x^2$	x	-3
		$-4x^2$	$+6x$	-10
			$-5x$	$+7$

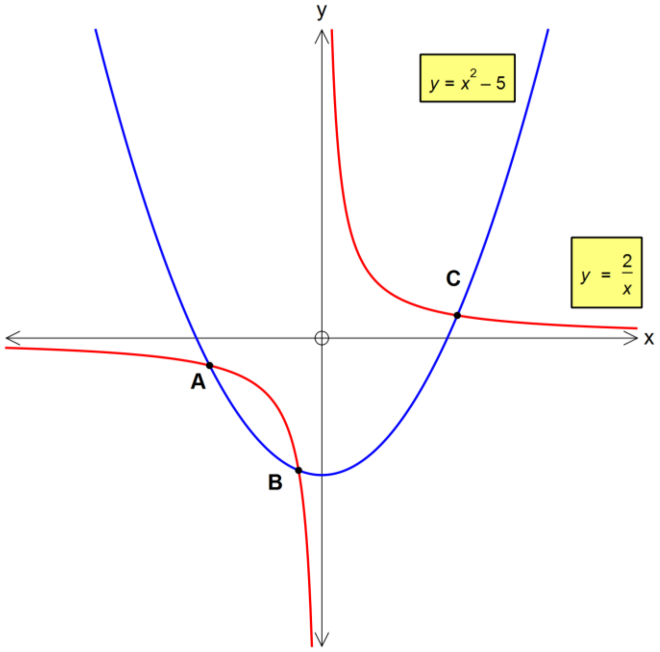
Example (18). The graphs of $y = x^2 - 5$ and $y = \frac{2}{x}$ are shown below. The two curves intersect at the points **A**. **B** and **C**.

i) Show algebraically that the x-coordinates of points **A**. **B** and **C** are the roots of the equation $x^3 - 5x - 2 = 0$.

ii) Point **A** has integer coordinates. Find them using the Factor Theorem.

iii) Hence find the coordinates of points **B** and **C**, giving your values in the form

$a + b\sqrt{2}$ where a and b are integers.



i) Starting with $x^2 - 5 = \frac{2}{x}$, we multiply both sides by x:

$x^3 - 5x = 2$, and so $x^3 - 5x - 2 = 0$. (We have turned the equation into a polynomial.)

ii) Substituting $x = -2$ into the resulting cubic gives $(-2)^3 - 5(-2) - 2 = -8 + 10 - 2 = 0$.
Hence the x-coordinate of **A** is -2 and the y-coordinate is -1 by substituting in either $x^2 - 5$ or $\frac{2}{x}$.

iii) From ii), we know that $(x+2)$ is a factor. Dividing $(x^3 - 5x - 2)$ by $(x + 2)$ gives us the quadratic quotient of $x^2 - 2x - 1$. (Full working in Example (3)).

The equation $x^2 - 2x - 1 = 0$ can be solved by completing the square (used here) or the general formula.

$x^2 - 2x - 1 = 0 \Rightarrow (x - 1)^2 - 1 - 1 = 0 \Rightarrow (x - 1)^2 - 2 = 0 \Rightarrow (x - 1)^2 = 2 \Rightarrow (x - 1) = \pm\sqrt{2}$

Hence $\mathbf{x = 1 \pm \sqrt{2}}$.

The x-coordinate of **B** is the negative one, i.e. $1 - \sqrt{2}$, and that of **C** is the positive one, i.e. $1 + \sqrt{2}$

Substituting in $y = \frac{2}{x}$ gives the y-coordinate of **B** as $\frac{2}{1 - \sqrt{2}}$ which can be rationalised to

$$\frac{2}{1-\sqrt{2}} \times \frac{1+\sqrt{2}}{1+\sqrt{2}} = \frac{2+2\sqrt{2}}{-1} = -2 - 2\sqrt{2}\,.$$

Similarly the y-coordinate of **C** is $\frac{2}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{2-2\sqrt{2}}{-1} = -2 + 2\sqrt{2}$

$\therefore$ The curves intersect at **A** (-2 ,-1) , **B** $(1-\sqrt{2}, -2-2\sqrt{2})$ and **C** $(1+\sqrt{2}, -2+2\sqrt{2})$.

APPENDIX Fourth-degree (quartic) graphs. MEI only

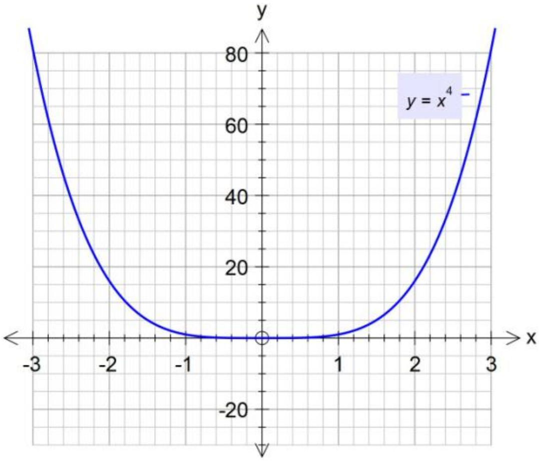
Sketching of polynomial graphs in examination questions is restricted to quadratics and cubics, but sometimes higher powers crop up in schoolwork. We shall have a brief look at fourth-degree or quartic graphs.

The basic quartic graph of $y = x^4$ resembles that of $y = x^2$ but is shallower for $-1 < x < 1$ and steeper for other values of *x*.

It also has only one minimum point at the origin.

Other quartic graphs are a little more complicated, and we shall restrict ourselves to those where the coefficient of x^4 is positive.

The graph of $y = -x^4$ is a reflection of $y = x^4$ in the *x*-axis.



Quartic graphs can have up to three turning points and intersect the *x*-axis at up to four points (one more than cubics !), but like quadratics, they can also not intersect the *x*-axis at all. The following examples do not cover all cases, but give an idea of what to look for in sketching selected quartic graphs.

Four distinct roots.

The graph on the right is typical of a general quartic, with a distinctive 'W' shape.

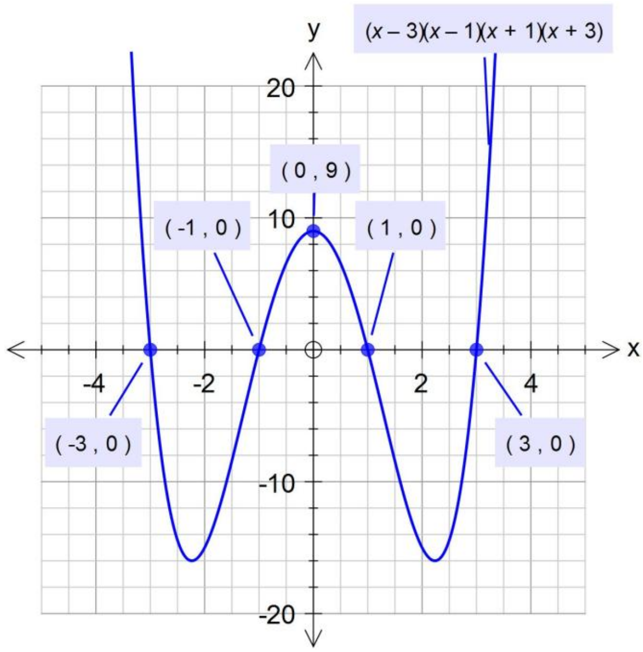
It intersects the *x*-axis at four points and has a local maximum when $x = 0$.

It also has two local minima, but students are not expected to sketch them accurately.

As *x* becomes large and positive, so does *y* .

As *x* becomes large and negative, *y* becomes large and positive.

The graph of $y = (x-3)(1-x)(x+1)(x+3)$ is obtained from the one shown by reflecting in the *x*-axis, and has one local minimum and three local maxima.

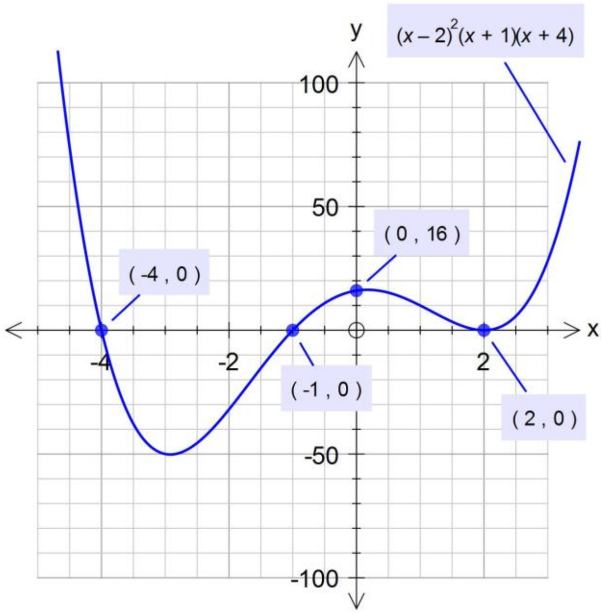


One twice-repeated root.

The 'W' is less symmetrical, but there are still two minima and one maximum.

The graph is a tangent to the x-axis when $x = 2$, corresponding to the repeated root and one of the minima.

At the distinct roots of $x = -1$ and $x = -4$, the graph still intersects the x-axis.



Two twice-repeated roots.

Again, we have two minima and one maximum.

The graph is a tangent to the x-axis when $x = 2$ and when $x = -2$, corresponding to the repeated roots and both of the minima.

As there are no longer any distinct roots, the graph no longer intersects the x-axis.