M.K. HOME TUITION

Mathematics Revision Guides
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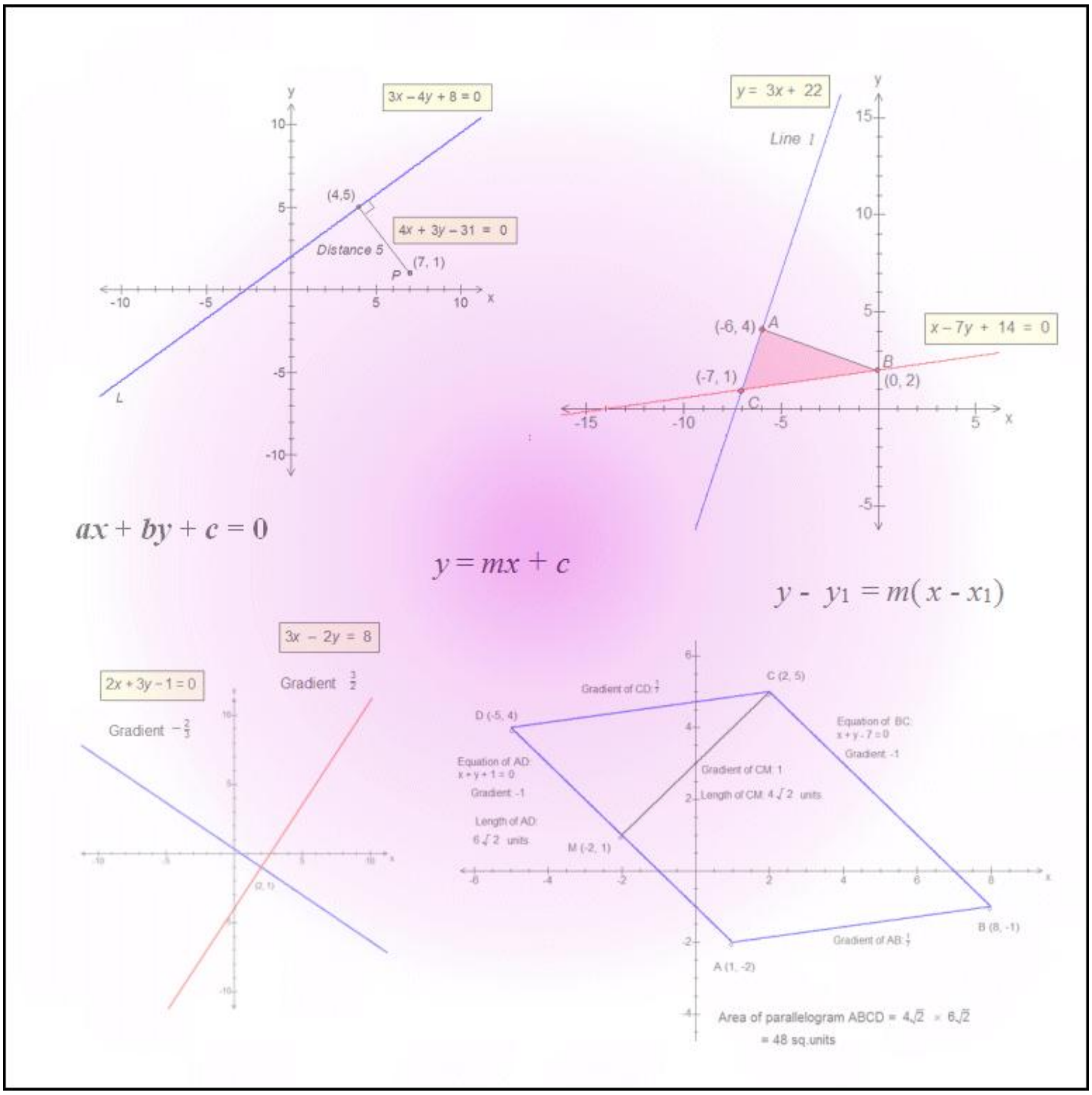
COORDINATE GEOMETRY- STRAIGHT LINES



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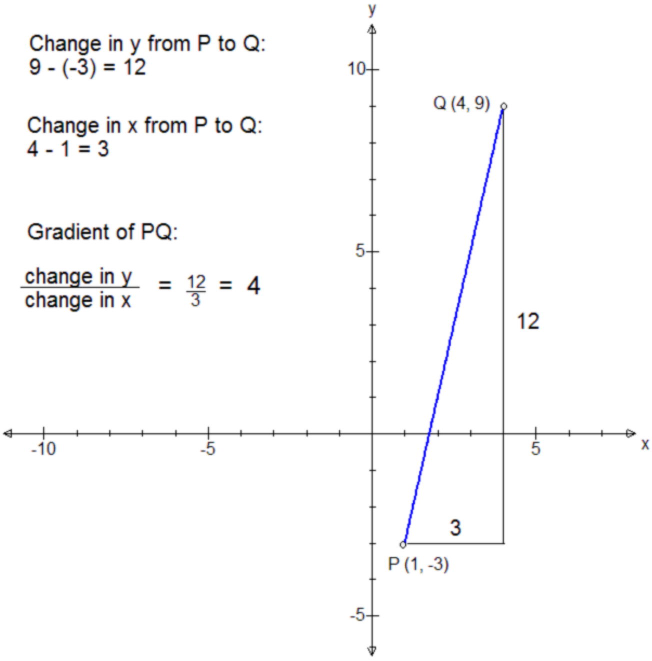
COORDINATE GEOMETRY - STRAIGHT LINES.

Gradient of a line.

The gradient of a line connecting two points (x_1, y_1) and (x_2, y_2) is given by the formula $\frac{y_2 - y_1}{x_2 - x_1}$ - it is the change in the value of *y* divided by the change in the value of *x*.

Example (1): Find the gradient of the line passing through the points *P*(1, -3) and *Q*(4, 9).



Taking (1, -3) as (x_1, y_1) and (4, 9) as (x_2, y_2), the gradient of the line above is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{9-(-3)}{4-1} = \frac{12}{3} = 4.$$

It does not matter which point is taken as (x_1, y_1) – the calculated gradients will be the same.

Example (2): Find the gradient of the line passing through the points (-1, 2) and (2, 10).

Taking (-1, 2) as (x_1, y_1) and (2, 10) as (x_2, y_2), the gradient of the line is $\frac{10-2}{2-(-1)} = \frac{8}{3}$.

Example (3): Find the gradient of the line passing through the points (1, 3) and (1, 7).

Here we run into trouble, since $x_1 = x_2 = 1$, and substituting into the formula would lead to division by zero, which is inadmissible, $\therefore$ the gradient is undefined. (The line is in fact parallel to the *y*-axis, and its equation is $x = 1$).

In general, if a line is parallel to the *y*-axis, its gradient is undefined.

Equation of a straight line.

A generalised straight line has the equation $\boldsymbol{ax + by + c = 0}$.

Any straight line has an equation that can be written in this form.

The line $2x + 3y = 7$ corresponds to $a = 2$, $b = 3$ and $c = -7$.

($2x + 3y = 7$ is equivalent to $2x + 3y - 7 = 0$.)

Provided that the straight line is not parallel to the y-axis (as in Example (3) earlier), its equation can also be written in the form $y = mx + c$.

This is known as the **gradient-intercept** equation because the gradient (m) and y-intercept (c) are clearly evident.

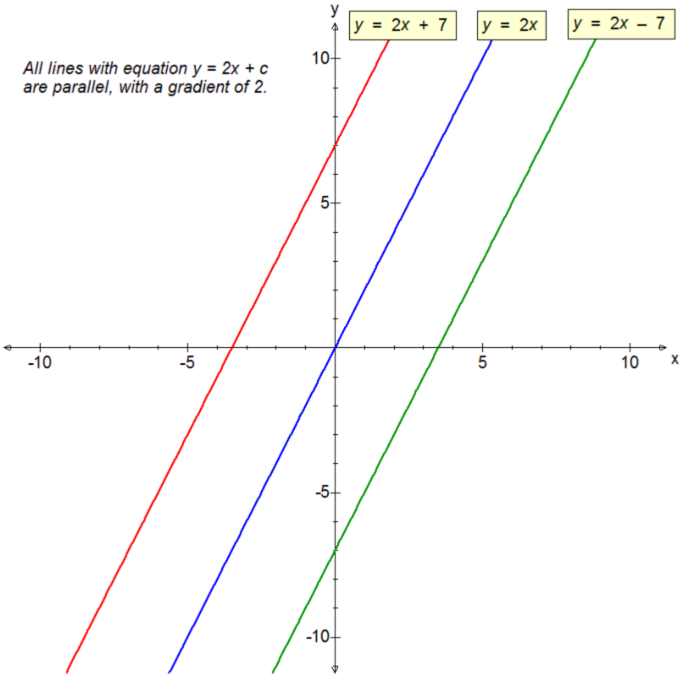
Examples (4).
(i) Find the equation of the line with gradient 4 passing through the point (0, -7).
The y-intercept is –7, and so the equation of the line is $y = 4x-7$.

(ii) Give the gradient and y-intercept of the line whose equation is $y = 3-2x$.
Rewriting the equation as $y = -2x + 3$, we can see that the gradient is –2 and the y-intercept is at (0,3).

Finding the equation of a line parallel to a given line, passing through a specified point.

Parallel lines all have the same gradient, as the graphs on the right show.

The graphs of $y = 2x + 7$, $y = 2x - 7$ and $y = 2x$ are all parallel, as are the graphs of $y = 2x + c$ where c is any constant.



Example (5): Find the equation of the straight line parallel to $y = 3x + 1$, and passing through the point (4,7).

The required line must have a gradient of 3, so its gradient-intercept form must be
$y = 3x + c$ or $c = y – 3x$.

Substituting $y = 7$ and $x = 4$ gives $c = 7 - 12$, $\Rightarrow c = -5$.
$\therefore$ the equation of the required line is $\boldsymbol{y = 3x - 5}$.

If the equation of the original line is given in the form $ax + by = c$, then finding the equation of the parallel line is particularly simple – you just need to substitute x and y to find the new value for c.

Example (6) : Find the equation of a line parallel to $4x + 3y = 11$, but passing through the point (5, 2).

Substituting $x = 5$, $y = 2$ and recalculating c gives the equation of the parallel line as $\boldsymbol{4x + 3y = 26}$ (or $\boldsymbol{4x + 3y - 26 = 0}$).

Finding the equation of a straight line given the gradient and one point on the line.

The equation of a straight line with gradient m and passing through the point (x_1, y_1) can be written as

$y - y_1 = m(x - x_1)$.

The resulting equation can then be re-expressed in either in the form '$ax + by + c = 0$' or in gradient-intercept form.

Either is acceptable unless the question asks for a particular style – the examples below give both for illustration.

Example (7): Find the equation of the straight line with gradient 2, passing through the point (3, 13).

Given the gradient $m = 2$, and $(x_1, y_1) = (3, 13)$, we obtain the equation $\boldsymbol{y - 13 = 2(x - 3)}$.

This can then be rearranged into either:

Gradient-intercept form ('*mx* + *c*'):
$y - 13 = 2(x - 3) \Rightarrow y - 13 = 2x - 6 \Rightarrow \boldsymbol{y = 2x + 7}$.

'*ax* + *by* + *c* = 0' form :
$y - 13 = 2(x - 3) \Rightarrow y - 13 = 2x - 6$
$\Rightarrow 2x - 6 - y + 13 = 0$
$\Rightarrow 2x - y + 7 = 0$
$\therefore$ the equation of the line is $\boldsymbol{2x - y + 7 = 0}$.

Also acceptable is $y - 13 = 2x - 6 \Rightarrow y - 13 - 2x + 6 = 0 \Rightarrow \boldsymbol{-2x + y - 7 = 0}$, which is the previous result multiplied by –1. (It is a minor matter of style to have the term in x positive).

When there are fractions involved, the working is the same, if a little harder:

Example (8): A straight line passes through the point (5, 2) and its gradient is $\frac{3}{4}$. Find its equation.
Given that $m = \frac{3}{4}$, and $(x_1, y_1) = (5, 2)$, the equation of the line is $\boldsymbol{y - 2 = \frac{3}{4}(x - 5)}$.

Rearranging the equation gives:

Gradient-intercept form ('*mx* + *c*'):
$y - 2 = \frac{3}{4}(x - 5) \Rightarrow y - 2 = \frac{3}{4}x - \frac{15}{4} \Rightarrow \boldsymbol{y = \frac{3}{4}x - \frac{7}{4}}$.

'*ax* + *by* + *c* = 0' form :
$y - 2 = \frac{3}{4}(x - 5) \Rightarrow 4y - 8 = 3(x - 5)$ (Multiply both sides by 4 to get rid of the awkward fractions).
$\Rightarrow 4y - 8 = 3x - 15$
$\Rightarrow 3x - 15 - 4y + 8 = 0$
$\Rightarrow 3x - 4y - 7 = 0$.
$\therefore$ the equation of the line is $\boldsymbol{3x - 4y - 7 = 0}$.

Examination hint: If a question merely asks for *an* equation of a straight line without specifying a particular form, then the $y - y_1 = m(x - x_1)$ form, such as $\boldsymbol{y - 2 = \frac{3}{4}(x - 5)}$ from the last example, is sufficient for a correct answer.

Finding the equation of a straight line given two points on the line.

The last two examples showed how to find the equation of a line given the gradient and one point on the line.

If we are given two points, then we can obtain the gradient of the line from their coordinates, and continue as before.

Example (9): Find the equation of the straight line passing through the points (-1, 2) and (4, 27). Give the answer in gradient-intercept form.

We need to find the gradient of the line, m , and then use the formula $y - y_1 = m(x - x_1)$.

Taking (-1, 2) as (x_1, y_1) and (4, 27) as (x, y) we obtain

$$m = \frac{y - y_1}{x - x_1} \Rightarrow m = \frac{27-2}{4-(-1)} \Rightarrow m = 5.$$

The equation of the line is therefore $y - 2 = 5(x + 1)$.

Rearranging into gradient-intercept form,

$y - 2 = 5(x + 1) \Rightarrow y - 2 = 5x + 5 \Rightarrow$ $\mathbf{y = 5x + 7}$.

Had we chosen (4, 27) as (x_1, y_1) and (-1, 2) as (x, y) we would still have obtained a gradient m of 5, and the final substitution would have given

$y - 27 = 5(x - 4) \Rightarrow y - 27 = 5x - 20 \Rightarrow$ $\mathbf{y = 5x + 7}$ as before**.**

Example (10): Find the equation of the straight line passing through the points (-2, 5) and (2, -4). Give the answer in '$ax + by + c = 0$' form.

We therefore find the gradient m and use $y - y_1 = m(x - x_1)$:

Taking (-2, 5) as (x_1, y_1) and (2, -4) as (x, y) we obtain

$$m = \frac{y - y_1}{x - x_1} \Rightarrow m = \frac{-4-5}{2-(-2)} \Rightarrow m = \frac{-9}{4}$$

The equation of the line is therefore $\mathbf{y - 5 = -\frac{9}{4}(x + 2)}$**.**

Rearranging into '$ax + by + c = 0$' form,

$y - 5 = -\frac{9}{4}(x + 2) \Rightarrow 4y - 20 = -9(x + 2)$ (get rid of the awkward fractions).

$\Rightarrow 4y - 20 = -9x - 18$

$\Rightarrow 4y - 20 + 9x + 18 = 0$

$\Rightarrow 4y + 9x - 2 = 0$

$\therefore$ equation of line is $\mathbf{9x + 4y - 2 = 0}$**.**

Converting between forms of straight-line equations.

Any equation of the form $ax + by + c = 0$ can be rearranged into gradient-intercept form as long as b is not zero.

$ax + by + c = 0$

$\Rightarrow by = -ax - c$

$\Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$

This shows that parallel lines can be produced by fixing a and b whilst allowing c to change.
Thus the lines $2x + 3y + 5 = 0$, $2x + 3y + 2 = 0$ and $4x + 6y + 9 = 0$ are all parallel.
($4x + 6y + 9 = 0$ is equivalent to $2x + 3y + 4.5 = 0$.)

It also follows that a line with equation $ax + by + c = 0$ has a gradient of $-\frac{a}{b}$.

Examples (11).

i) A straight line has an equation of $2x - 3y - 14 = 0$. Re-express it in gradient-intercept form.

Here, $a = 2$, $b = -3$, and $c = -14$, and so the equation of the line is $y = -\frac{2}{-3}x - \frac{-14}{-3}$ or

$y = \frac{2}{3}x - \frac{14}{3}$.

ii) A straight line has an equation of $y = \frac{3}{4}x - 5$. Re-express it in $ax + by + c = 0$ form.

The first step is to multiply by 4 throughout to get rid of the fraction: $4y = 3x - 20$.

This result can then be rearranged to $\mathbf{3x - 4y - 20 = 0}$.

Perpendicular straight lines.

When two straight lines are perpendicular, the product of their gradients is –1.

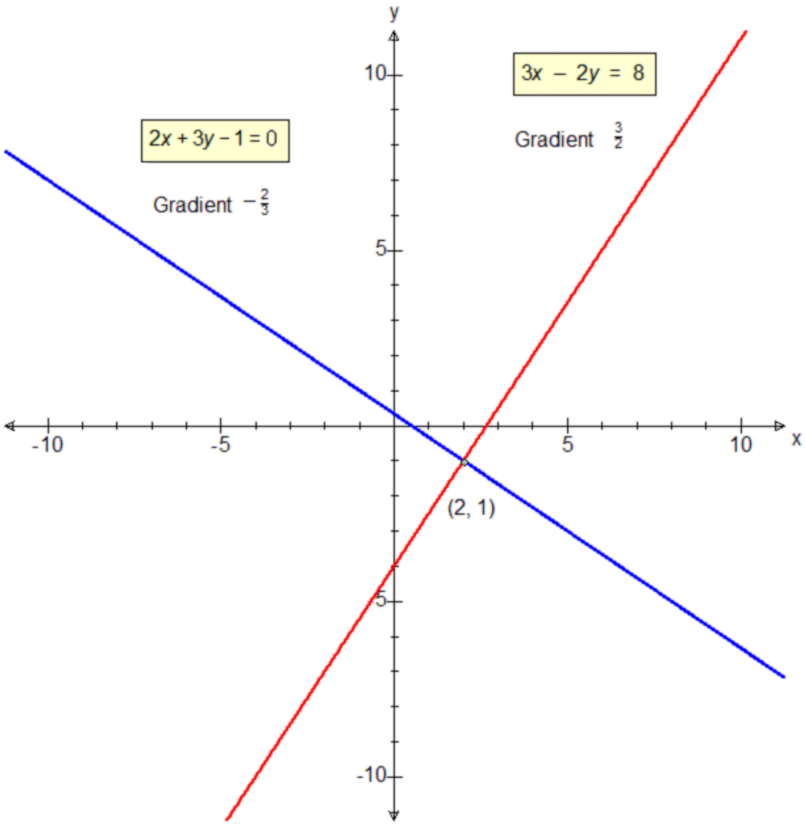
Also, any line perpendicular to $ax + by = c$ has the equation $bx – ay = c$ where c is any constant.
(The constants c need not be equal and can take any numerical values.)

Example (12): Find the equation of the straight line perpendicular to the line $3x – 2y = 8$ and passing through the point (2, -1). Give the result in '$ax + by + c = 0$' form.

We could convert the original into gradient-intercept form, but it is easier to use the above formula.

The equation of the straight line perpendicular to the line $3x – 2y = 8$, passing through (2, -1). is $2x + 3y = c$, and to find c we substitute $x = 2$, $y = -1 \Rightarrow c = 1$.

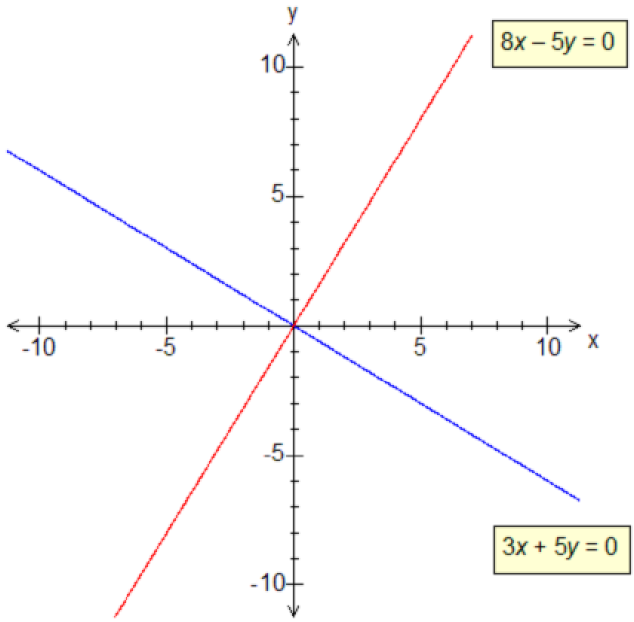
The equation of the perpendicular line is $2x + 3y = 1 \Rightarrow$ $\mathbf{2x + 3y - 1 = 0}$.



Example s (13):

i) Two straight lines have equations $8x - 5y = 0$ and $3x + 5y = 0$ respectively. Are they perpendicular or not ?

We need to find the gradient of each line and calculate their product.



A line with equation $ax + by + c = 0$ has a gradient of $-\frac{a}{b}$.

$\therefore$ The line with equation $8x - 5y = 0$ has gradient $-\frac{8}{-5}$ or $\frac{8}{5}$.

Similarly the line with equation $3x + 5y = 0$ has gradient $-\frac{3}{5}$.

The product of the gradients is

$\frac{8}{5} \times \left(-\frac{3}{5}\right) = -\frac{24}{25}$, which is not –1.

$\therefore$ The two lines are almost perpendicular, but not quite.

ii) Determine whether the following pairs of lines are parallel, perpendicular, or neither.
(Do not convert their equations in full.)

a) $y = 1 - 3x$ and $6x + 2y - 7 = 0$
b) $y = 5x - 2$ and $2x + 3y + 4 = 0$
c) $y = 5 - 4x$ and $x - 4y - 3 = 0$

a) The gradient of $y = 1 - 3x$ is -3; the gradient of $6x + 2y - 7 = 0$ is $-\frac{6}{2}$ or -3.

$\therefore$ The two lines are parallel as their gradients are equal.

b) The gradient of $y = 5x - 2$ is 5; the gradient of $2x + 3y + 4 = 0$ is $-\frac{2}{3}$.

$\therefore$ The two lines are neither parallel nor perpendicular.

c) The line $y = 5 - 4x$ has a gradient of -4; the line $x - 4y - 3 = 0$ has gradient $-\frac{1}{(-4)}$ or $\frac{1}{4}$.

$\therefore$ The two lines are perpendicular as their gradients have a product of -1.

Note: In each of the examples above, the x-and y-scales were **uniform** to illustrate the properties of the graphs of perpendicular lines. Using non-uniform scales would distort the picture, so be careful not to misinterpret such graphs !

Example (14): *OABC* is a square, with point *O* at the origin. Additionally, the points *A* and *B* lie on the straight line with equation $y = 20 - 2x$.

i) Find the coordinates of point *A*.

ii) Hence show, using congruent triangles, that the coordinates of point *C* are (-4, 8).

iii) Find the area of the square *OABC*.

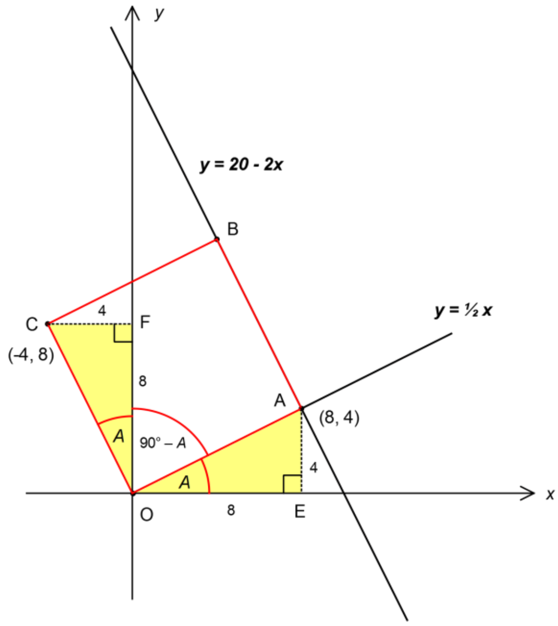
i) The gradient of the line *AB* is -2, and because *OAB* is a right angle, the gradient of *OA* = $\frac{1}{2}$.

Since *OA* also passes through the origin, its equation is
$y = \frac{1}{2}x$.

Point *A* lies on the intersection of *OA* and *AB*, so we solve the equations $y = 20 - 2x$ and $y = ½x$ simultaneously.

$20 - 2x = ½x \rightarrow 40 - 4x = x \rightarrow 5x = 40 \rightarrow x = 8.$

Substituting for $y = ½x$, we have $y = 4$, so point *A* has coordinates of (8, 4).



ii) Draw perpendicular lines from *A* to the *x*-axis at *E*, and from *C* to the *y*-axis at *F*, so that $\angle OEA = \angle OFC = 90°$.

Let *OE* = 8 units and *EA* = 4 units (from part (i))
Let $\angle EOA = A$.

Now $\angle EOF = 90°$ (angle between the *x*- and *y*– axes) .
Also $\angle AOC = 90°$ (angle of the square)

Thus $\angle AOF = 90° - A$ by subtraction, and
$\angle FOC = 90° - (90° - A) = A$.

Also, *OA* = *OC* (sides of a square).

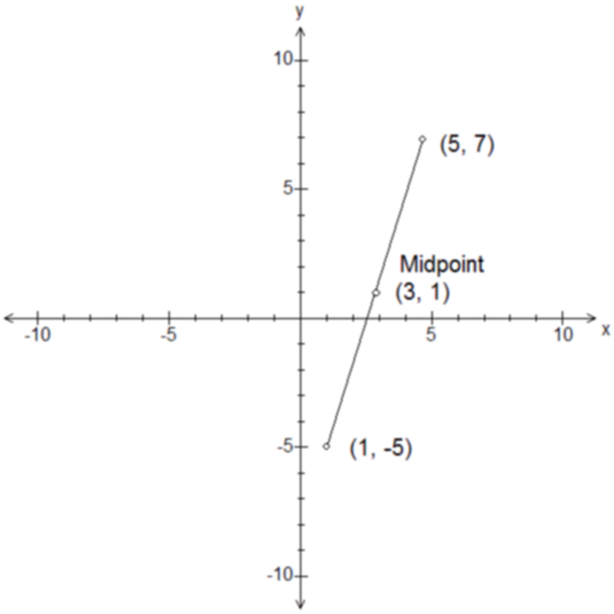
Hence $\angle OEA = \angle OFC$, $\angle EOA = \angle FOC$ and $OA = OC$.

Triangles *OEA* and *OFC* are congruent (RHS = RHS).

So *OF* = *OE* = 8 units, and *CF* = *AE* = 4 units.
Therefore the coordinates of *C* are (-4, 8). (The *y*-coordinate is negative as *C* is to the left of the *y*-axis).

iii) The side *OA* of the square is also the hypotenuse of the triangle *OEA*, and its area is $(OA)^2$.
By Pythagoras, $(OA)^2 = (OE)^2 + (EA)^2 = 8^2 + 4^2 =$ **80 units²**.

Midpoint of a line.



If a point *P* has coordinates (x_1, y_1) and a point *Q* has coordinates (x_2, y_2) , then the midpoint of the line *PQ* has the coordinates

$$(x_m, y_m) = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$$

In the above example, the midpoint of the line joining the points (1, -5) and (5, 7) is the point (3, 1).

Example (15): Find the midpoint of the line joining the points (-4, 5) and (6, 1).

The coordinates of the midpoint of the line are given as $\left(\frac{(-4)+6}{2}, \frac{5+1}{2}\right)$, simplifying to **(1,3)**.

Sometimes we might be given 'one end' and the midpoint, and be asked to find the 'other end'.

We rearrange the formula as $2(x_m, y_m) = (x_1 + x_2,\ y_1 + y_2)$
$\Rightarrow$ $(x_2, y_2) = 2(x_m, y_m) - (x_1,\ y_1)$.

The coordinates of the unknown end can therefore be found by subtracting those of the known end from double those of the midpoint.

Example (16): The midpoint *M* of the line *AB* has coordinates (2, 1). If point *A* is at (7, 5), find the coordinates of point *B*.

Here we are given $A = (x_1, y_1) = (7, 5)$ and the midpoint $M = (x_m, y_m) = (2, 1)$.

$\therefore$ $(x_2, y_2) = 2(x_m, y_m) - (x_1,\ y_1)$
$\Rightarrow$ $(x_2, y_2) = (4, 2) - (7, 5)$

$\Rightarrow$ $(x_2, y_2) = (-3, -3)$.

This answer can also be visualised as follows: point *M* is a translation of *A* by (2 - 7), or -5 units, in *x* and by (1 - 5), or – 4 units, in *y*.

Point *B* is therefore an equivalent translation of *M* by the same amount, so its coordinates are (2 - 5, 1 - 4) or (-3, -3).

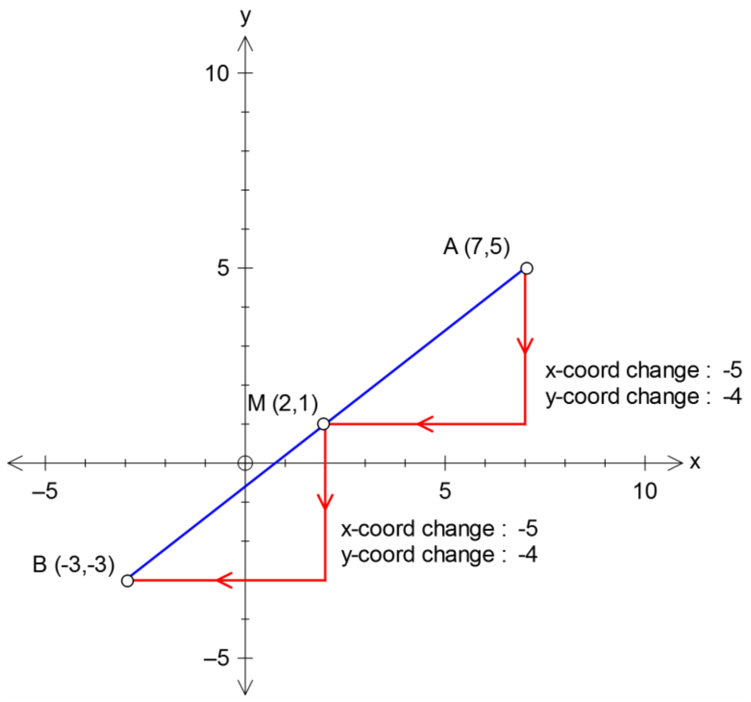
An alternative approach to solving the last problem is to use a "step" method as shown below.

When we get from *A* to *M*, the *x*-coordinate changes from 7 to 2 – a decrease of 5.

The *y*-coordinate changes from 5 to 1 – a decrease of 4.

To get from *M* to *B*, we repeat the changes.

We therefore decrease the *x*- and *y*-coordinates of *M* by 5 and 4 respectively to give those of *B*, namely (-3, -3).



Lines divided in a given ratio.

This is an extension of the method used to find the midpoint of a line, where the division of the line was in the ratio of 1:1.

If the point *R* divides the line *AB* in the ratio *p*:*q* , then we have to use proportionate division methods.

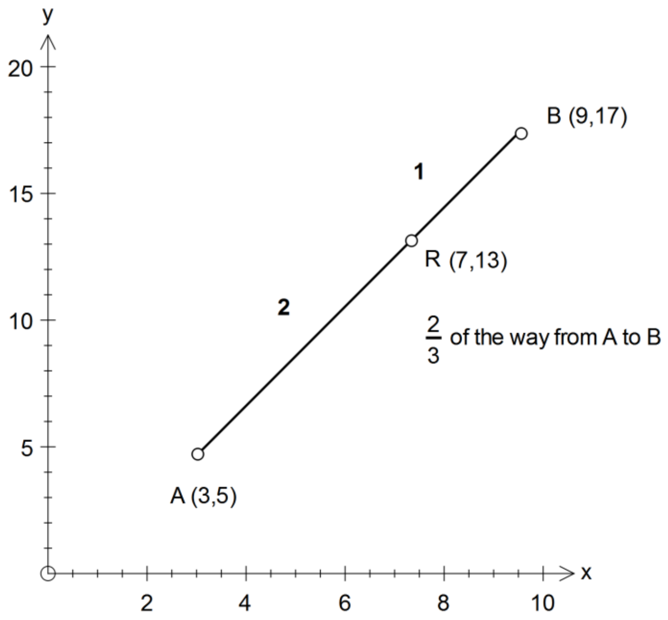
Example (17): The coordinates of points *A* and *B* are (3, 5) and (9, 17).
The point *R* divides *AB* in the ratio 2:1. Find the coordinates of *R*.

Adding the proportional parts together we have 2 + 1 = 3, so *R* is two-thirds of the way along the line joining *A* to *B*.

The difference between the *x*-coordinates of *A* and *B* is 9 - 3 or 6, and two-thirds of that difference is 4. The *x*-coordinate of *R* is therefore 3 + 4 or 7. (7 is two-thirds of the way from 3 to 9).

Similarly, the difference between the *y*-coordinates of *A* and *B* is 12, and two-thirds of 12 is 8. Hence the *y*-coordinate of *R* is 5 + 8 or 13.

$\therefore$ The coordinates of *R* are (7, 13)



Let point *R* divide line *AB* in the ratio *p* : *q*.
If *A* has coordinates (x_1, y_1) and *B* has coordinates (x_2, y_2) , then the coordinates of *R* are given by the formula

$$(x_r, y_r) = \left(x_1 + \frac{p}{p+q}(x_2 - x_1),\ y_1 + \frac{p}{p+q}(y_2 - y_1) \right)$$

Again, we might have the case of being given 'one end' and the point of division, and being asked to find the 'other end'.

Example (18): The point *R* divides the line *AB* in the ratio 2 : 3, and the coordinates of points *A* and *R* are (-4, -5) and (6, 3). Find the coordinates of *B.*

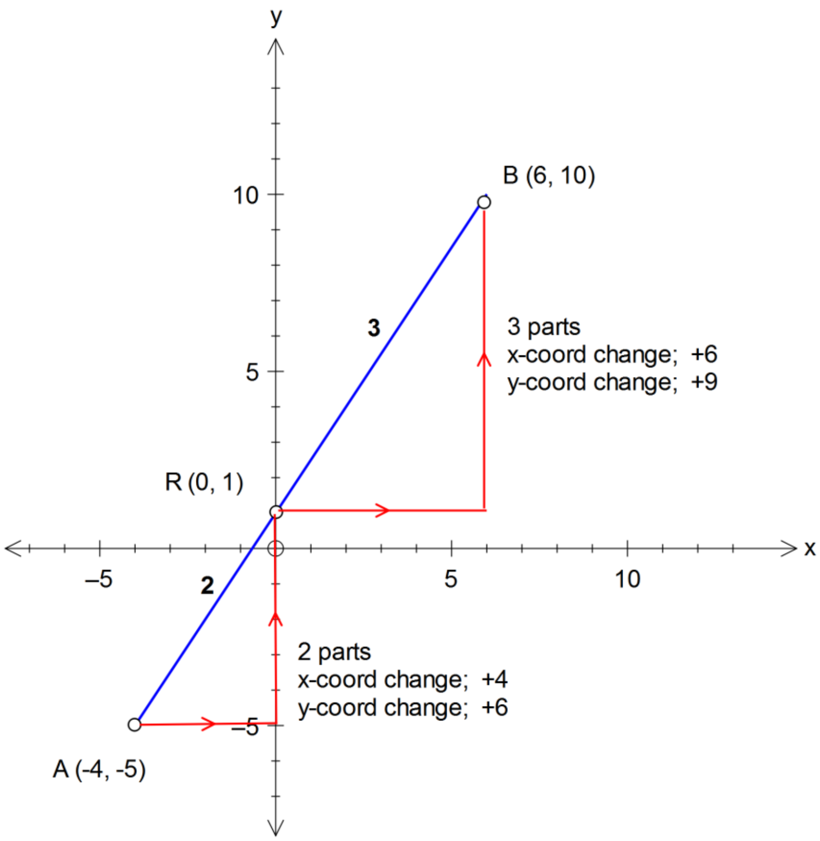
The "step" method is the easier one to use here. The distance from *A* to *R* is 2 proportional parts, and that from *R* to *B* is 3 parts. The difference between the *x*-coordinates of *A* and *R* is 4, and the difference between the *y*-coordinates is 6.

The coordinate differences corresponding to 2 proportional parts are 4 (for *x*) and 6 (for *y*).
From the above, 1 part corresponds to an *x*-difference of 2 and a *y*-difference of 3.

The distance from *R* to *B* amounts to 3 proportional parts, i.e. an *x*-difference of 3 × 2 or 6, and a *y*-difference of 3 × 3, or 9.

We need to add 6 and 9 to the *x*- and *y*-coordinates of *R* respectively, to obtain the corresponding coordinates for *B*, which work out as (0 + 6, 1 + 9), or (6, 10).

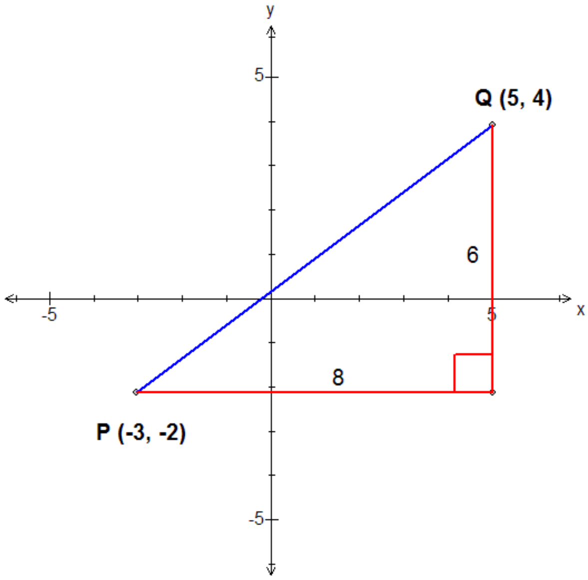
∴ **The coordinates of *B* are (6, 10).**



The distance between two points.

The distance between two points can be found by applying Pythagoras' theorem.

Example (19): Find the length of the line joining the points *P* (-3, -2) and *Q* (5, 4).



The line joining the two points can be visualised as the hypotenuse of a right-angled triangle whose other two sides run parallel with the axes and whose right angle is at the point (5,-2).

The lengths of the two sides are therefore:

8 units for the one parallel to the *x*-axis, obtained by subtracting the *x*-coordinate of the point (-3,-2) from that of (5,4) - i.e. 5 - (-3) = 8.

6 units for the one parallel to the *y*-axis, obtained by subtracting the *y*-coordinate of the point (-3,-2) from that of (5,4) - i.e. 4 – (-2) = 6.

The length of the hypotenuse, and therefore the distance *PQ*, is $\sqrt{8^2 + 6^2} = \sqrt{100}$ units, or **10 units.**

In general, the length of a line joining two points (x_1, y_1) and (x_2, y_2) on the plane is expressed as

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example (20): Find the distance between the points (-3, -7) and (5, 8).

Taking (x_1, y_1) as (-3, -7) and (x_2, y_2) as (5, 8), the length of the line joining the two points is

$\sqrt{(5-(-3))^2 + (8-(-7))^2} = \sqrt{8^2 + 15^2} = \sqrt{289}$ or 17 units.

The intersection of two straight lines.

Two non-parallel straight lines have one point of intersection. To find this intersection, we use simultaneous equations.

Example (21): Find the point of intersection between the following pairs of lines:

i) $y = 5x - 8$ and $y = 1 - 4x$
ii) $x + 2y - 7 = 0$ and $5x - 4y + 35 = 0$.
iii) $3x - y - 5 = 0$ and $y = 2x - 1$.

i) The two lines intersect when $5x - 8 = 1 - 4x$.

Solving the linear equation in x, $5x - 8 = 1 - 4x \Rightarrow 9x = 9 \Rightarrow x = 1$.

Substituting $x = 1$ in either original equation gives $y = -3$, so the two lines intersect at **(1, -3)**.

An alternative starting strategy is to rearrange the equations as $5x - y = 8$ and $4x + y = 1$, then eliminate y by adding the two to give $9x = 9$ and hence $x = 1$ as before.

The first method is quicker, and less prone to accident !

ii) We solve the equations simultaneously by elimination:

$x + 2y - 7 = 0$ $\quad A$
$5x - 4y + 35 = 0$ $\quad B$

$2x + 4y - 14 = 0$ $\quad 2A$
$5x - 4y + 35 = 0$ $\quad B$

$7x + 21 = 0$ $\quad 2A + B$

Eliminating y, we have $x = -3$.
Substituting in equation A, we than have $(-3) + 2y - 7 = 0 \Rightarrow 2y = 10 \Rightarrow y = 5$.

$\therefore$ The two lines $x + 2y - 7 = 0$ and $5x - 4y + 35 = 0$ intersect at **(-3, 5).**

iii) This time, the substitution method is easier to use – we substitute for y in the first equation.

$3x - y - 5 = 0 \Rightarrow 3x - (2x - 1) - 5 = 0$
$\Rightarrow 3x - 2x + 1 - 5 = 0$
$\Rightarrow x - 4 = 0$ and thus $x = 4$.

Substituting $x = 4$ into $y = 2x - 1$ gives $y = 7$.

$\therefore$ The two lines $3x - y - 5 = 0$ and $y = 2x - 1$ intersect at **(4, 7).**

Example (22): Point *P* has coordinates of (7, 1). What is its perpendicular distance from the line *L* whose equation is $3x - 4y + 8 = 0$?

A line with equation $ax + by + c = 0$ has a gradient of $-\frac{a}{b}$, and therefore the gradient of *L* is $\frac{3}{4}$.

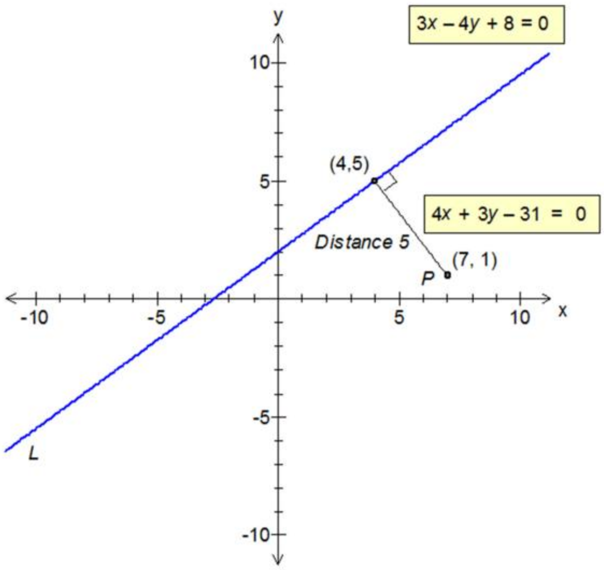
Or: $3x - 4y = -8 \Rightarrow 3x = 4y - 8 \Rightarrow y = \frac{3}{4}x + 2$.

The perpendicular from *P* to *L* will thus have a gradient of $-\frac{4}{3}$, and its equation can be found out in the usual way; it is $y - 1 = -\frac{4}{3}(x - 7)$

$\Rightarrow y = -\frac{4}{3}x + \frac{31}{3}$

$\Rightarrow 3y = -4x + 31$

$\Rightarrow 4x + 3y - 31 = 0$.



We then find the point of intersection of the two lines by solving the equations simultaneously:

$3x - 4y + 8 = 0$ *A*
$4x + 3y - 31 = 0$ *B*

$9x - 12y + 24 = 0$ *3A*
$16x + 12y - 124 = 0$ *4B*

$25x - 100 = 0$ *3A + 4B*

This gives $x = 4$, and substituting into $16 + 3y - 31 = 0$ gives $y = 5$.

The point of intersection is thus (4, 5), and we use Pythagoras to work out its distance from (7, 1);

it is $\sqrt{(7-4)^2 + (1-5)^2} = \sqrt{25} = 5$ units.

N.B. There is a general formula to compute the perpendicular distance *l* of a point (x_1, y_1) from a line $ax + by + c = 0$, but it is outside the scope of most A-level syllabuses.

The distance is given by the formula $l = \frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}$.

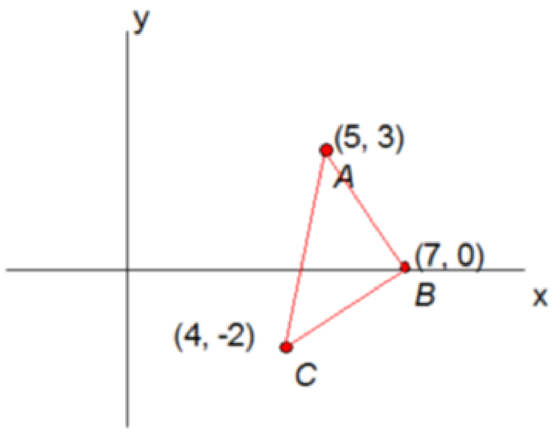
Given $(x_1, y_1) = (7, 1)$ and $3x - 4y + 8 = 0$, the distance works out as

$$\frac{3(7) - 4(1) + 8}{\sqrt{(3^2 + 4^2)}} = \frac{25}{5} = 5 \text{ units.}$$

Miscellaneous Geometry Problems.

Many examination questions on Coordinate Geometry involve properties of shapes, usually triangles and quadrilaterals. Such questions bring together combinations of ideas learnt earlier in this section.

Example (23) : Points A , B and C have coordinates of (5, 3), (7, 0) and (4, -2) respectively.



Show that the triangle *ABC* is right-angled.

There are two methods of verifying a right-angled triangle.

i) Find the gradients of the lines making up the three sides, and look for a pair whose product is –1.

In this example, the gradient of *AB* is $\frac{7-5}{0-3} = -\frac{2}{3}$ and that of BC is $\frac{4-7}{(-2)-0} = \frac{3}{2}$.

Since the product of the gradients is $-\frac{2}{3} \times \frac{3}{2} = -1$, lines *AB* and *BC* are perpendicular and hence triangle *ABC* is right-angled. (Here we have shown the result after calculating two gradients – sometimes we might have to work out all three).

ii) Find the lengths of the sides and check if they obey Pythagoras' theorem.

The length of *AB* is $\sqrt{(7-5)^2 + (0-3)^2}$, or $\sqrt{2^2 + (-3)^2} = \sqrt{13}$ units.

The length of *BC* is $\sqrt{(4-7)^2 + ((-2)-0)^2}$, or $\sqrt{(-3)^2 + (-2)^2} = \sqrt{13}$ units.

The length of *CA* is $\sqrt{(5-4)^2 + (3-(-2))^2}$, or $\sqrt{1^2 + 5^2} = \sqrt{26}$ units.

The square of the length of *CA* is equal to the sum of the squares of the lengths of *AB* and *BC*, therefore the triangle *ABC* is right-angled. (In this example, it is also isosceles.)

Example (23a) : Find the area of the triangle *ABC* from Example (23).

The area of any triangle is ½(base × height). If the triangle is right-angled, the base and the height are the two sides containing the right angle.

Here we need to use the lengths of *AB* and *BC*, as these are the sides containing the right angle.

The area of *ABC* is thus ½(length of *AB* × length of *BC*) or ½ × √13 × √13, or 6.5 square units.

Example (23b) : Using the triangle *ABC* from Example (22a), find the coordinates of point *D* such that the quadrilateral *ABCD* is a square.

The diagonals of a square bisect each other – a property shared by parallelograms, rhombuses and rectangles.

One diagonal of the square is *AC* and the other is *BD*. Since the diagonals bisect each other, they will meet at their common midpoint *M*.

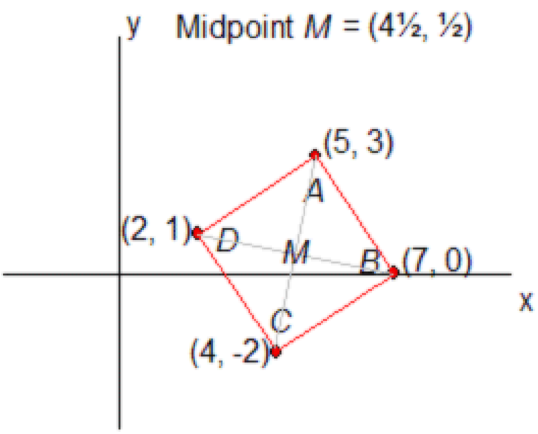
Using the coordinates of *A* and *C*, the midpoint *M* has coordinates of $\left(\frac{5+4}{2}, \frac{3-2}{2}\right)$, or $\left(4\frac{1}{2}, \frac{1}{2}\right)$.

To find the co-ordinates of *D*, we subtract those of *B* from double those of the midpoint *M*:
Here we are given $B = (x_1, y_1) = (7, 0)$ and the midpoint $M = (x_m, y_m) = (4½, ½)$.

$\therefore$ Coordinates of $D = 2(x_m, y_m) - (x_1, y_1)$
$\Rightarrow$ $D = (9, 1) - (7, 0)$ $\Rightarrow$ ***D* = (2, 1).**

Alternatively, the midpoint *M* is a translation of *B* by (4½ - 7), or –2½ units, in *x* and by (½-0), or ½ unit, in *y*.

Point *D* is therefore an equivalent translation of *M* by the same amount, so its coordinates are (4½ - 2½, ½+ ½) or **(2, 1)**.



Example (24):

Three straight lines have the following equations:

L_1: $x - 2y = 2$
L_2: $2x - 11y = 25$
L_3: $3x + y = 20$

The three lines enclose a triangular region, intersecting as follows:

i) The lines L_1 and L_2 meet at point P. Verify that the coordinates of P are (-4, -3).

ii) Lines L_2 and L_3 meet at point Q, and lines L_1 and L_3 meet at point R.
Find the coordinates of Q and R

iii) Sketch the three lines on a graph, and show that the triangle PQR is isosceles.

i) Substituting $(x, y) = (-4, -3)$ into the equation for L_1 gives $x - 2y = (-4) - 2(-3) = 2$, so equation holds.
Similarly, substituting in L_2 gives $2x - 11y = 2(-4) - 11(-3) = 25$, so that equation also holds.
$\therefore$ Coordinates of P are (-4, -3).

ii) To find the coordinates of Q, we solve the equations for L_2 and L_3 simultaneously:

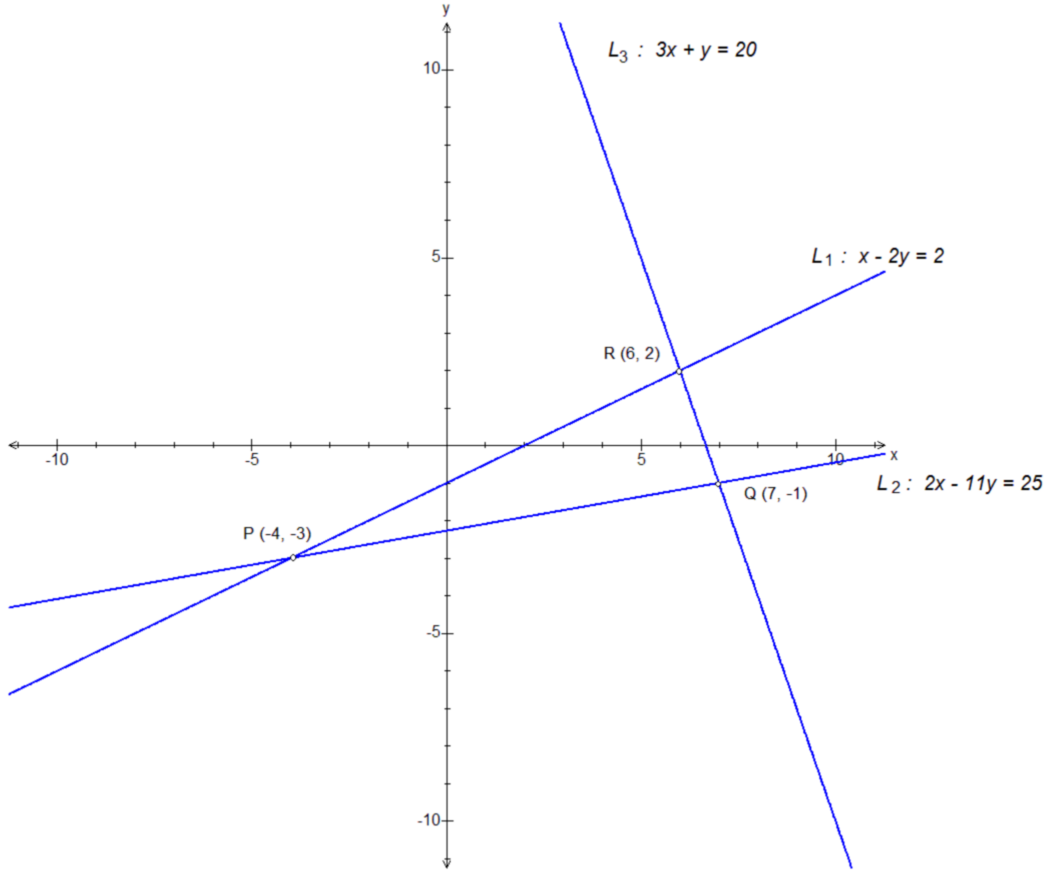
$2x - 11y = 25$	L_2
$3x + y = 20$	L_3
$2x - 11y = 25$	L_2
$33x + 11y = 220$	$11L_3$
$35x = 245$	$L_2 + 11L_3$

The x-coordinate of Q is 7, and substituting into $3x + y = 20$ (equation of L_3) gives $y = -1$.
$\therefore$ Coordinates of $Q = (7, -1)$.

iii) Again we solve the equations for L_1 and L_3 simultaneously to find the coordinates of R:

$x - 2y = 2$	L_1
$3x + y = 20$	L_3
$x - 2y = 2$	L_1
$6x + 2y = 40$	$2L_3$
$7x = 42$	$L_1 + 2L_3$

The x-coordinate of R is 6, and substituting into $x - 2y = 2$ (equation of L_1) gives $y = 2$.
$\therefore$ Coordinates of $R = (6, 2)$.



The three lines are shown above, with their points of intersection. Even a rough sketch would show that *PR* and *PQ* are both longer than *QR*, and therefore the more likely to be equal in length.

The length of *PR* is $\sqrt{(6-(-4))^2+(2-(-3))^2}$, or $\sqrt{10^2+5^2}=\sqrt{125}$ units.

The length of *PQ* is $\sqrt{(7-(-4))^2+((-1)-(-3))^2}$, or $\sqrt{11^2+2^2}=\sqrt{125}$ units.

Sides *PQ* and *PR* are equal in length, and therefore triangle *PQR* is isosceles.

Example (25): A quadrilateral *ABCD* has vertices *A*(1, -2), *B*(8, -1), *C*(2, 5), *D* (-5, 4).

i) Verify that the side *AD* lies on the line $x + y + 1 = 0$, and that side *AB* lies on a line with gradient $\frac{1}{7}$.

ii) Find the equation of the line containing the side *BC*, and the gradient of the line containing side *CD*. Hence show that *ABCD* is a parallelogram.

iii) Find the midpoint *M* of side *AD*, and the gradient and length (in surd form) of the line *CM*.

iv) Find the length of side *AD* in surd form, and hence the area of the parallelogram, using the results from part iii).

i) Substituting the coordinates of *A*, namely $(x, y) = (1, -2)$ into the equation $x + y + 1 = 0$ gives $1 - 2 + 1 = 0$, which holds true.

Also substituting the coordinates of *D*, i.e. $(x, y) = (-5, 4)$ into the same equation gives $-5 + 4 + 1 = 0$, which again holds true.

The gradient of the line containing *AB* is $\frac{-1-(-2)}{8-1} = \frac{1}{7}$.

ii) Taking *B*(8, -1) as (x_1, y_1) and *C*(2, 5) as (x, y) we obtain

$$m = \frac{y - y_1}{x - x_1} \Rightarrow m = \frac{5-(-1)}{2-8} \Rightarrow m = \frac{6}{-6} = -1.$$

The equation of the line is therefore $y + 1 = -(x - 8)$ or $y + 1 = 8 - x$.

Rearranging into '$ax + by + c = 0$' form, $y + 1 - 8 + x = 0 \rightarrow x + y - 7 = 0$.

$\therefore$ equation of line containing side *BC* is $\boldsymbol{x + y - 7 = 0}$.

The gradient of the line containing *CD* is $\frac{5-4}{2-(-5)} = \frac{1}{7}$.

The lines containing *AB* and *CD* have the same gradient, so sides *AB* and *CD* are parallel.

The equation of the line containing *AD* is $x + y + 1 = 0$.
The equation of the line containing *BC* is $x + y - 7 = 0$.
Each line has a gradient of -1, so sides *AD* and *BC* are also parallel.

$\therefore$ Quadrilateral *ABCD* is a parallelogram.

iii) The midpoint, *M*, of side *AD*, has coordinates of $\left(\frac{1-5}{2}, \frac{-2+4}{2}\right)$, or $(-2, 1)$.

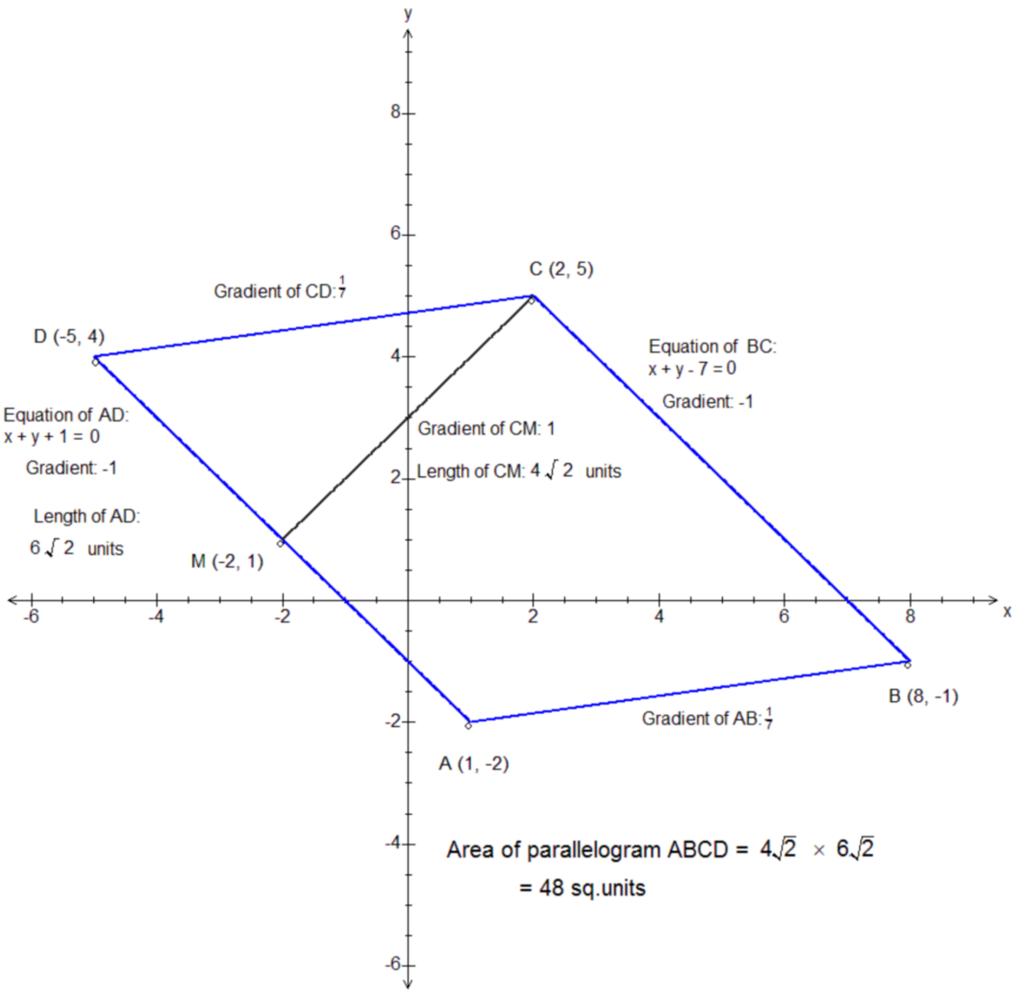
The gradient of *CM* is therefore $\frac{5-1}{2-(-2)} = \frac{4}{4} = 1$.

The length *CM* is $\sqrt{(2-(-2))^2 + (5-1)^2}$, or $\sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$ units.

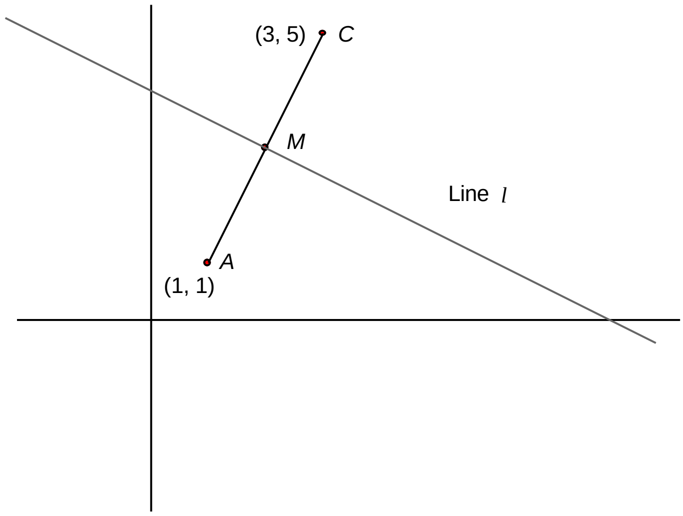
iv) The length *AD* is $\sqrt{(1-(-5))^2+((-2)-4)^2}$, or $\sqrt{6^2+6^2}=\sqrt{72}=6\sqrt{2}$ units.

Looking at the results from parts ii) and iii), the gradient of *CM* = 1 and that of *AD* = -1.
Hence *CM* is perpendicular to *AD* and its length is the perpendicular height of the parallelogram *ABCD.*

The area of the parallelogram is therefore the product of the base *AD* and the height *CM*,
or $4\sqrt{2}\times 6\sqrt{2}=24\sqrt{2}\sqrt{2}=48$ square units.



Example (26):
Point *A* has coordinates (1, 1) and *C* has coordinates (3, 5). *M* is the midpoint of *AC*, and the line *l* is perpendicular to *AC*.



i) Find the coordinates of *M* and hence the equation of *l* in '$ax + by + c = 0$' form.
ii) Point *B* has coordinates (-2, 5). Show that it lies on *l*.
iii) Find the coordinates of the point *D* such that *ABCD* is a rhombus.
iv) Find the lengths *MC* and *MB* , and hence calculate the area of the rhombus *ABCD*.

i) Using the coordinates of *A* and *C*, the midpoint *M* has coordinates of $\left(\frac{3+1}{2}, \frac{1+5}{2}\right)$, or **(2, 3)**.

The gradient of *AC* is $\frac{5-1}{3-1} = 2$, and hence the gradient of *l* must be –½.
(Product of the gradients of perpendicular lines is –1).

Since the point *M* lies on *l*, the equation of *l* is
$y - 3 = -\frac{1}{2}(x - 2) \Rightarrow 2y - 6 = -x + 2$
$\Rightarrow 2y - 6 + x - 2 = 0$
$\Rightarrow \mathbf{x + 2y - 8 = 0.}$

ii) If point *B* has coordinates (-2, 5), then substituting for *x* and *y* in the previous equation gives $-2 + 10 - 8 = 0$, which is consistent $\therefore$ *B* lies on *l*.

iii) Because the diagonals of a rhombus bisect each other, the point *M* must also be the midpoint of *BD*. To find the co-ordinates of *D*, we subtract those of *B* from double those of the midpoint *M*:

$\therefore$ Coordinates of *D* = 2(2, 3) - (-2, 5)
$\Rightarrow$ *D* = (4, 6) - (-2, 5) $\Rightarrow$ ***D* = (6, 1).**

Alternatively *M* is at (2, 3) and is therefore a translation of *B* by (2–(-2)),or 4 units, in *x* and by (3-5), or -2 units, in *y*.

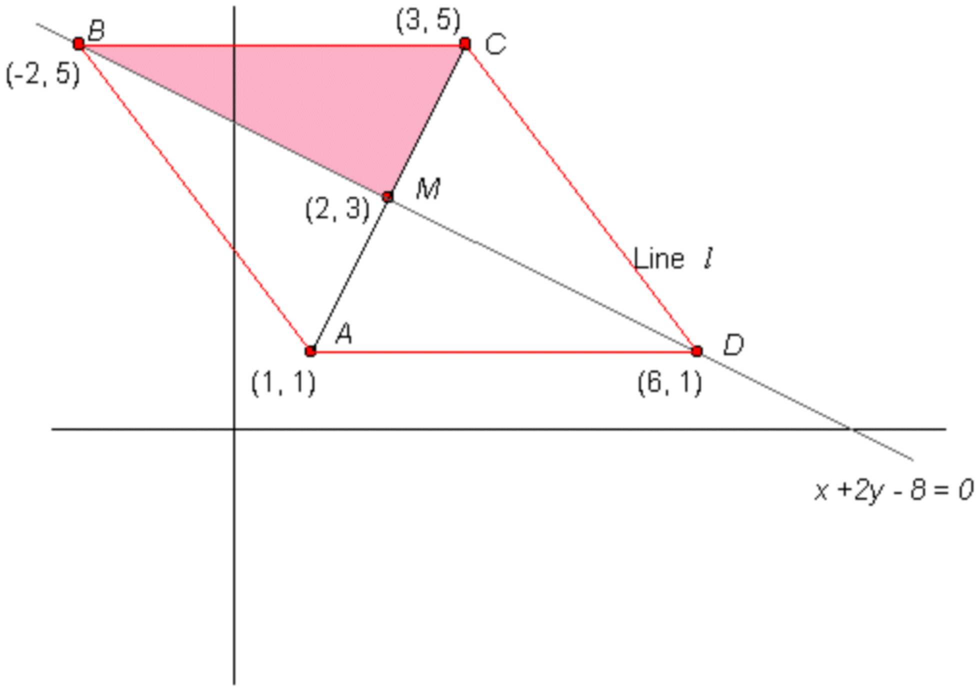
The point *D* must therefore be a further translation of *M*, again by 4 units in *x* and –2 units in *y*, therefore its coordinates are (2 + 4, 3 – 2), or **(6, 1)**.

iv) The length of *MC* can be found using Pythagoras: $\sqrt{(3-2)^2+(5-3)^2}$, or
$\sqrt{1^2+2^2}=\sqrt{5}$ units.

Similarly with the length of *MB* ; that is $\sqrt{((-2)-2)^2+(5-3)^2}$, or $\sqrt{4^2+2^2}=\sqrt{20}$ units.

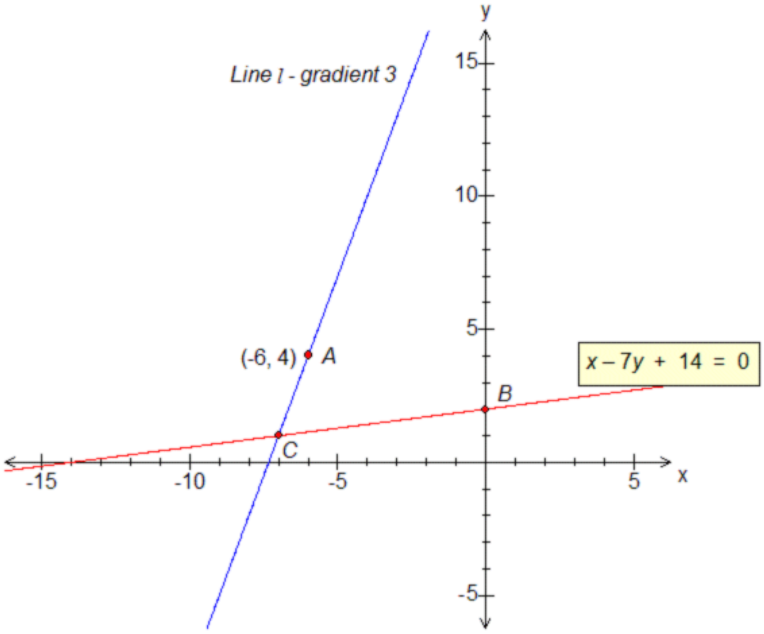
$\therefore$ the area of the triangle *CMB* is ½ × √5 × √20, or ½√100, or 5 sq.units.

But, by symmetry, the rhombus *ABCD* is made up of four triangles congruent to *CMB*, and so its area is **20 sq. units.**



Example (27):
i) A line *l* has a gradient of 3 and passes through the point *A* with coordinates (-6, 4). Find its equation.

ii) A second line has the equation $x - 7y + 14 = 0$. It meets the *y*-axis at point *B* and meets *l* at point *C*. Find the coordinates of *B* and *C*.



iii) Show that angle *BAC* = 90°.
iv) Find the area of the triangle *ABC*.

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i) Taking (-6, 4) as (x_1, y_1), we substitute $m = 3$, $x_1 = 6$ and $y_1 = 4$ into $y - y_1 = m(x - x_1)$.

This gives $y - 4 = 3(x + 6) \Rightarrow y - 4 = 3x + 18 \Rightarrow$ $\mathbf{y = 3x + 22}$ in gradient-intercept form.

In '$ax + by + c = 0$' form, $y - 4 = 3x + 18$
$\Rightarrow 3x + 18 = y - 4$
$\Rightarrow 3x + 18 - y + 4 = 0$
$\Rightarrow \mathbf{3x - y + 22 = 0.}$

ii) Substitute 0 for *x* in the equation of the line to give $-7y + 14 = 0$. This gives $y = 2$ and therefore the coordinates of point *B* are **(0, 2)**.
To find point *C* where the two lines intersect, we solve the equations $y = 3x+22$ and $x - 7y + 14 = 0$ simultaneously.

This gives $x - 7(3x+22)+14 = 0$
$\Rightarrow x - 21x - 154 + 14 = 0$
$\Rightarrow -20x - 140 = 0$
$\Rightarrow x = -7$

Substituting $x = -7$ into the equation (in either form) from i) gives $y = 1$, and hence point *C* = **(-7,1).**

iii) Points *A* and *C* are on the same line, namely $y = 3x + 22$. The gradient of that line is 3.

Point *A* is at (-6, 4) and point *B* is at (0, 2). The gradient of the line joining them is $\frac{2-4}{0-(-6)}$ or $-\frac{1}{3}$.

The product of the gradients of *AB* and *AC* is –1, therefore they are perpendicular.
$\therefore$ *CAB* is a right angle.

iv) To find the area of the triangle *ABC*, we multiply the base by the height and halve it. We can use either *AC* or *AB* as the base here, but not *BC*, as that is the hypotenuse!

The distance between points *A* and *B* is obtained as follows:
Taking (x_1, y_1) as point *A* (0, 2) and (x_2, y_2) as point *B* (-6, 4), the length of *AB* is
$\sqrt{(-6)^2 + (4-2)^2}$ or
$\sqrt{40}$ units.

Similarly the distance between points *A* and *C* is obtained by taking (x_1, y_1) as point *A* (-6, 4) and (x_2, y_2) as point *C* (-7, 1). This works out as
$\sqrt{(-7-(-6))^2 + (1-4)^2}$ or
$\sqrt{10}$ units.

The area of the triangle is therefore
$\frac{1}{2} \times \sqrt{40 \times 10}$ or 10 square units.